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Posterior integration of independent stochastic estimates

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Contents

Abstract

We develop a unified approach to posterior integration of prior stochastic estimates (probability distributions) provided by independent statistically inaccurate observation methods. Our departure point is the posterior event formed in the product of the probability spaces associated with the prior stochastic estimates. The Bayesian probability conditioned to the posterior event has identical projections onto the coordinate spaces; its common projection is defined to be the posterior integrated stochastic estimate. We view integration as a binary operation on the set of all probabilities on a given finite set of elementary events and analyze its algebraic properties. We show how integration changes the information quality of the integrated probabilities and study integral convergence properties of infinite sequences of probabilities.

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Posterior integration of independent stochastic estimates

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Introduction

Studies of complex systems are non-separable from analyses of partial and imprecise information received from alternative sources. Sometimes, pieces of information received from different sources seem to disagree with each other. In particular applications, researchers facing such phenomena employ specific features of the systems under investigation to reconcile alternative pieces of information and generate integrated knowledge (see, e.g., Nilsson, *et al*, 2007).

We propose a unified approach to integrating pieces of information provided by alternative sources. We restrict our study to the case where information has a probabilistic character. We suppose that several independent methods are used to observe a deterministic element (for example, the true value of a parameter of a complex system) and each method represents the latter as a probability distribution. Thus, we deal with a family of probability distributions providing alternative descriptions to the same object. Our principle assumption is that we have no ground for giving a preference to any distribution in the family.

The proposed approach to treating families of distibutions differs from the traditional ones. For example, Wald's theory of statistical decisions (Wald, 1949) focuses on the optimal (risk minimizing) decisions made under uncertain distributions of the 'states of nature'. As opposed to Wald's theory, we do not consider the issue of optimization; our approach does not involve objective functions and decisions. We intend to simply reconcile alternative probabilistic models – we call them 'prior stochastic estimates', to construct an integrated model – a 'posterior integrated stochastic estimate' – that incopropates the features of all the prior ones.

Non-specified distributions are dealt with in theory of comparison of experiments (Blackwell, 1951, 1953; Sulganik and Zilcha, 1997); the theory establishes rules for comparison of the information values of experiments viewed as sets of probability distributions. Our approach bases, in contrast, on the assumption that the prior stochastic estimates are not ordered in information values (the underlying experiments are informationally equivalent).

Statistics generated by different information sources, each operating with a certain probability, are analyzed using models with mixed distributions (see, e.g., Fruhwirth-

Schnatter, 2006). This setting implies that the information sources complement each other, whereas we view those as alternative ones.

In our study we generally follow the Bayesian approach (see, e.g., Berger, 1985). Our departure point is the posterior event in the product of the probability spaces associated with the prior stochastic estimates. The posterior event reflects the fact that all the prior stochastic estimates represent the same deterministic element. An immediate implication is that an elementary event in the product probability space is, posteriorly, admissible if and only if its components are identical. The collection of all admissible elementary events forms the posterior event. The latter is, therefore, the 'diagonal' in the product probability space. The Bayesian probability conditioned to the posterior event has identical projections onto the coordinate spaces. Its common projection is defined to be the posterior integrated stochastic estimate.

To avoid technical complcations, we consider the simplest case where the prior stochastic estimates are defined on a finite set of elementary events, Z.

In Section 1 we introduce basic notations and provide an informal motivation for our study; we justify our definition of integration as a transformation of a finite number of probabilities on Z (prior stochastic estimates) into a probability on Z (the integrated posterior one).

In Section 2 we study integration as a binary algebraic operation.

In Subsection 2.1 we notice that integration as a binary operation is associative and commutative and can therefore be interpreted as multiplication; in this interpretation, the uniform probability acts as the unit and every concentrated probability plays the role of a zero.

In Subsection 2.2 we keep viewing integration as multiplication and introduce integral powers of probabilities. We prove that the n-th integral power of a probability preserves the initial probabilistic priorities on Z and makes these priorities sharper as n grows. Thus, integration of several identical independent observation methods sharpens the prior stochastic estimate provided by each of those methods and the sharpness grows as the number of the observation methods grows. Moreover, if we let n go to infinity, the n-th integral power of a given probability, π, converges to the probability concentrated at (and uniform on) the set of all elementary events most likely with respect to π. Following a standard algebraic definition, we introduce integral roots of probabilities and state that the n-th integral root of a probability π converges to the probability uniform on the set of all elementary events likely to appear under π.

In Subsection 2.3 we define disintegration as the binary operation inverse to integration. We state that the result of desintegration of a probability over another one is unique if both probabilities provide non-zero likelihoods to all elementary events. An implication is that the set of all probabilities providig non-zero likelihoods to all elementary events forms a topological abelian group with respect to integration.

In Section 3 we study how integration changes the quality of probabilities as models for the observed elementary event.

In Subsection 3.1 we analyze how the initial likelihoods of selected elementary events change if an initial observation method represented by probability π_1 is integrated with another observation method represented by probability π_2. We show that, generically, integration raises the initial likelihood of every elementary event most likely with respect to π_2 and reduces that of every elementary event least likely with respect to π_2.

In Subsection 3.2 we introduce measures of concentration – continuous real-valued functions of probabilities, which reach their highest values at the concentrated probabilities. If the result of integration of probabilities π_1 and π_2 has a higher measure of concentration than each of those probabilities does, the observation methods underlying π_1 and π_2 enhance each other and provide, together, more information than each of them does, being taken separately. In this situation, we call π_1 and π_2 consistent (with respect to the given measure of concentration). Here, we restrict our analysis to the max measure of concentration, which measures each probability via its maximum value. We state that, generically, π_1 and π_2 are consistent with respect to the max measure of concentration (max consistent) if both provide the maximum likelihoods to a same elementary event. Based on this observation, we give a few statements on the structures of sets of probabilities, invariant with respect to integration.

In Subsection 3.3 we introduce the marginal measure for the probabilities, which measures every probability via its minimum non-zero value. The marginal measure is in a sense dual to the max measure of concentration. The smaller is the marginal measure of a probability, the 'less uniform' is the latter. We call probabilities π_1 and π_2 maginally consistent if the result of integration of π_1 and π_2 has a smaller marginal measure than π_1 and π_2 do. We show that, generically, π_1 and π_2 are marginally consistent if both provide the minimum likelihoods to a same elementary event.

In Subsection 3.4 we define a max raiser for probabilities $\pi_1, \ldots, \pi_n$ as a probability that, being integrated with any π_i, raises its max measure of concentration. The observation method lying behind a max raiser for $\pi_1, \ldots, \pi_n$ improves, through integration, each of the observation methods lying behind $\pi_1, \ldots, \pi_n$. We state that any probability giving a sufficiently high priority to an elementary event having non-zero likelihoods with respect to $\pi_1, \ldots, \pi_n$ is, generically, a max raiser for $\pi_1, \ldots, \pi_n$. We show that a non-uniform probability is a max raiser for any $\pi_1, \ldots, \pi_n$ sufficiently close to the uniform probability. Finally, we prove that if n is smaller than the number of the elementary events, then, generically, for any $\pi_1, \ldots, \pi_n$ one can find a max raiser sufficiently close to the uniform probability.

In Section 4 we study the asymptotic behavior of the results of integration of the first n elements, $\pi_1, \ldots, \pi_n$, of a given sequence of probabilies, $(\pi_i)_{i=1}^{\infty}$, as $n \to \infty$.

Subsection 4.1 starts with definitions. If the results of integration of $\pi_1, \ldots, \pi_n$ converge to some probability, π, we call $(\pi_i)_{i=1}^{\infty}$ integrally convergent and call π its integral limit; otherwise, we call $(\pi_i)_{i=1}^{\infty}$ integrally divergent. If $(\pi_i)_{i=1}^{\infty}$ is integrally convergent and its integral limit is concentrated, we call $(\pi_i)_{i=1}^{\infty}$ integrally concentrated. If $(\pi_i)_{i=1}^{\infty}$ is integrally concentrated, observation methods $1, 2 \ldots$ lying, respectively, behind $\pi_1, \pi_2, \ldots$ improve each other, through integration, to a degree, at which we get complete information on the observed elementary event. If $(\pi_i)_{i=1}^{\infty}$ is integrally convergent and not integrally concentrated, methods $1, 2 \ldots$, being integrated, 'find a compromise' and provide a definite though incomplete information on the observed element. If $(\pi_i)_{i=1}^{\infty}$ is integrally divergent, methods $1, 2 \ldots$ disagree. We prove a few statements on integral convergence. In particular, we state that if $\pi_1, \pi_2, \ldots$ give visibly the highest values to a same element, $(\pi_i)_{i=1}^{\infty}$ is integrally concentrated at that element; an intepretation is that observation methods that unambiguously agree on prioritizing a certain element enhance their common probabilistic priority through integration so that the common priority turns into certainty. Also, we show that $(\pi_i)_{i=1}^{\infty}$ is integrally convergent if $(\pi_i)_{i=1}^{\infty}$ prioritize the elementary events in the same order.

In Subsection 4.2 we note that finite permutations in sequences of probabilities do not change the results of integration, whereas infinite ones can possess the opposite property.

1 Basic notations and motivation

1.1 Basic notations

In what follows, Z is a non-empty finite set whose number of elements is bigger than one. As usual, a probability on Z is defined to be a non-negative function on Z such that the sum of its values is one. A pair (Z, π) where π is a probability on Z is understood as a probability space. We write $|E|$ for the number of the elements of a finite set E. We denote Π the set of all probabilities on Z; Π^+ denotes the set of all positive-valued $\pi \in \Pi$; and $\bar{\pi}$ denotes the constant, or *uniform*, probability on Z, whose value is $1/|Z|$. A $\pi \in \Pi$ will be said to be *concentrated* if $\pi(z) = 1$ for some $z \in Z$ (implying $\pi(z') = 0$ for all other $z' \in Z$); z will then be said to be the *concentration point* for π (π will be said to be *concentrated at* z). For every $\pi \in \Pi$ we denote $Z^+(\pi) = \{z \in Z : \pi(z) > 0\}$.

We consider Π as a metric space equipped with the mean square metric $(\pi_1, \pi_2) \mapsto [\sum_{z\in Z} |\pi_1(z) - \pi_2(z)|^2]^{1/2}$. Clearly, Π is a compactum. For every natural k, Π^k will be viewed as a the product of k copies of the metric space Π; and every subset of Π^k will be considered as its metric subspace. In the sense of these metric spaces we understand the continuity of functions defined on subsets of Π^k and taking values in R^1 or in Π.

1.2 Motivation

Suppose an unknown element $z^0 \in Z$ is observed using alternative independent observation methods $1, \ldots, n$. Each method is, generally, inaccurate in a statistical sense, namely, it represents z^0 as a probability π_i on Z. The probabilities $\pi_1, \ldots, \pi_n$ serve as *prior stochastic estimates* for z^0.

We consider a *posterior* situation that occurs after the use of methods $1, \ldots, n$. Our goal is to show a way to integrating alternative pieces of knowledge provided by these methods.

Our approach is based on a trivial observation that in the posterior situation elements $z_1, \ldots, z_n \in Z$ resulting from n independent random tests from methods $1, \ldots, n$ are true if and only if $z^0 = z_1 = \ldots = z_n$. Since z^0 is unknown, $z_1 = \ldots = z_n$ is a necessary *posterior consistency condition* in the product probability space $(Z^n, P) = (Z, \pi_1)\times, \ldots \times (Z, \pi_n)$; here $P = \pi_1 \times \ldots \times \pi_n$. The posterior consistency condition determines a *posterior event*

$$E_* = \{(z_1, \ldots, z_n) \in Z^n : z_1 = \ldots = z_n\} = \{(z, \ldots, z) : z \in Z\},$$

which is necessarily realized in (Z^n, P) in the posterior situation. We have

$$P(E_*) = \sum_{z\in Z} \pi_1(z) \ldots \pi_n(z).$$

If $P(E_*) = 0$, methods $1, \ldots, n$ are *in contradiction* in the sense that for every $z \in Z$ there is a method, i, which evaluates the observed element z^0 as z with a zero probability, $\pi_i(z) = 0$.

Suppose $P(E_*) > 0$, implying that methods $1, \ldots, n$ are *not in contradiction* in the sense that there exists a $z \in Z$ such that all the methods give non-zero probabilities for the fact that $z^0 = z$. Then the Bayesian conditional probability $P(\cdot|E_*)$ is defined on E_*:

$$P((z,\dots,z)|E_*) = \frac{\pi_1(z)\dots\pi_n(z)}{P(E_*)}$$

for every $z \in Z$. Set

$$(\pi_1 \cdot \ldots \cdot \pi_n)(z) = P((z,\dots,z)|E_*)$$

for every $z \in Z$ Clearly, $\pi_1 \cdot \ldots \cdot \pi_n$ is a probability on Z.

In the probability space $(Z, \pi_1 \cdot \ldots \cdot \pi_n)$, for every $z \in Z$ the probability of $z^0 = z$ is proportional to $\pi_1(z)\dots\pi_n(z)$ – the probability of the fact that all the methods admit that $z^0 = z$. The latter probability is a measure of a 'consensus' of methods $1,\dots,n$ in conjecturing that $z^0 = z$. All the methods contribute to the value of the 'consensus measure' $\pi_1(z)\dots\pi_n(z)$ equally, and each method, i, has a 'power of veto' in the sense that the 'consensus measure' of z vanishes if $\pi_i(z) = 0$. Thus, probability $\pi_1 \cdot \ldots \cdot \pi_n$ provides an integrated knowledge on z^0, which results from a posterior analysis of the use of methods $1,\dots,n$. We understand $\pi_1 \cdot \ldots \cdot \pi_n$ as a *posterior stochastic estimate* resulting from the prior stochastic estimates $\pi_1,\dots,\pi_n$. The transformation of $\pi_1,\dots,\pi_n$ into $\pi_1 \cdot \ldots \cdot \pi_n$ will be called *integration* of $\pi_1,\dots,\pi_n$.

2 Algebraic properties of integration

2.1 Integration as multiplication

Now we study integration of probabilities (prior stochastic estimates) systematically.

Coming back to a definition given in the previous section, we say that $\pi_1,\dots,\pi_n \in \Pi$ are *in contradiction* if $\pi_1(z)\dots\pi_n(z) = 0$ for every $z \in Z$; otherwise, $\pi_1,\dots,\pi_n$ will be said to be *not in contradiction.* For every natural n, we write $\Pi^{(n)}$ for the set of all $(\pi_1,\dots,\pi_n) \in \Pi^n$, such that $\pi_1,\dots,\pi_n$ are not in contradiction.

Remark 1 The following statements hold evidently:

(i) $\Pi^{(1)} = \Pi$;

(ii) $(\pi_1, \pi_2) \in \Pi^{(2)}$ for every $\pi_1 \in \Pi$ and $\pi_2 \in \Pi^+$;

(iii) if $(\pi_1,\dots,\pi_n) \in \Pi^{(n)}$, then $(\pi_{i_1},\dots,\pi_{i_n}) \in \Pi^{(n)}$ for any permutation, $(i_1,\dots,i_n)$, in $(1,\dots,n)$ and every natural n;

(iv) $(\Pi^+)^n \subset \Pi^{(n)}$ for every natural n;

(v) $(\pi,\dots,\pi) \in \Pi^{(n)}$ for every $\pi \in \Pi$ and every natural n.

Following the preliminary definitions given in the previous section (in which we set $n = 2$), introduce a map $(\pi_1, \pi_2) \mapsto \pi_1 \cdot \pi_2 : \Pi^{(2)} \mapsto \Pi$ such that for every $(\pi_1, \pi_2) \in \Pi^{(2)}$

$$(\pi_1 \cdot \pi_2)(z) = \frac{\pi_1(z)\pi_2(z)}{\sum_{z' \in Z} \pi_1(z')\pi_2(z')} \quad (z \in Z);$$

we call it *integration* (of non-contradicting probabilities). For every $(\pi_1, \pi_2) \in \Pi^{(2)}$, the probability $\pi_1 \cdot \pi_2$ will be said to be the *result of integration* of π_1 and π_2.

Remark 2 It is easily seen that integration is continuous.

The next theorem shows that integration possesses the algebraic properties of multiplication – commutativity and associativity, the uniform probability, $\bar{\pi}$, plays the role of a unit, and every concentrated probability acts as a zero.

Remark 3 It is clear that if $(\pi_1, \pi_2, \pi_3) \in \Pi^{(3)}$, then $((\pi_1 \cdot \pi_2), \pi_3), (\pi_1, (\pi_2, \pi_3)) \in \Pi^{(2)}$.

Theorem 1 *The following statements are true.*
1) Integration is commutative, i.e., $\pi_1 \cdot \pi_2 = \pi_2 \cdot \pi_1$ *for all* $(\pi_1, \pi_2) \in \Pi^{(2)}$.
2) Integration is associative, i.e., $(\pi_1 \cdot \pi_2) \cdot \pi_3 = \pi_1 \cdot (\pi_2 \cdot \pi_3)$ *for all* $(\pi_1, \pi_2, \pi_3) \in \Pi^{(3)}$.
3) For every $\pi \in \Pi$ *it holds that* $(\pi, \bar{\pi}) \in \Pi^{(2)}$ *and* $\pi \cdot \bar{\pi} = \pi$.
4) For every $\pi, \pi_* \in \Pi^{(2)}$ *such that* π_* *is concentrated, it holds that* $\pi \cdot \pi_* = \pi_*$.

Proof. Statement 1 is obvious. Prove statement 2. Let $(\pi_1, \pi_2, \pi_3) \in \Pi^{(3)}$. Take an arbitrary $z \in Z$. By definition

$$(\pi_1 \cdot \pi_2)(z) = \pi_1(z)\pi_2(z)c_{12}$$

where

$$c_{12} = \frac{1}{\sum_{z' \in Z} \pi_1(z')\pi_2(z')},$$

and

$$((\pi_1 \cdot \pi_2) \cdot \pi_3)(z) = (\pi_1 \cdot \pi_2)(z)\pi_3(z)c_{(12)3} = \pi_1(z)\pi_2(z)\pi_3(z)c_{12}c_{(12)3}$$

where

$$c_{(12)3} = \frac{1}{\sum_{z' \in Z}(\pi_1 \cdot \pi_2)(z')\pi_3(z')} = \frac{1}{\sum_{z' \in Z} \pi_1(z')\pi_2(z')\pi_3(z')c_{12}}.$$

Therefore,

$$((\pi_1 \cdot \pi_2) \cdot \pi_3)(z) = \frac{\pi_1(z)\pi_2(z)\pi_3(z)}{\sum_{z' \in Z} \pi_1(z')\pi_2(z')\pi_3(z')}.$$

Similarly, we state that the ratio given on the right hand side equals $(\pi_1 \cdot (\pi_2 \cdot \pi_3))(z)$. Thanks to the arbitrary choice of $z \in Z$, we have $(\pi_1 \cdot \pi_2) \cdot \pi_3 = \pi_1 \cdot (\pi_2 \cdot \pi_3)$. Statement 2 is proved.

Prove statement 3. Let $\pi \in \Pi$. Recall that the uniform probability, $\bar{\pi}$, takes the single value $1/|Z|$. Then by definition, for every $z \in Z$ we have

$$(\pi \cdot \bar{\pi})(z) = \frac{\pi(z)1/|Z|}{\sum_{z' \in Z} \pi(z')1/|Z|} = \pi(z).$$

Statement 4 is obvious. The proof is complete.

Following a preliminary definition given in the previous section, for every natural $n \geq 2$ we define *n-tuple integration* of to be the map $(\pi_1, \ldots, \pi_n) \mapsto \pi_1 \cdot \ldots \cdot \pi_n : \Pi^{(n)} \mapsto \Pi$ where

$$(\pi_1 \cdot \ldots \cdot \pi_n)(z) = \frac{\pi_1(z) \ldots \pi_n(z)}{\sum_{z' \in Z} \pi_1(z') \ldots \pi_n(z')} \quad (z \in Z); \tag{1}$$

we call $\pi_1 \cdot \ldots \cdot \pi_n$ the *result of integration* of $\pi_1, \ldots, \pi_n$.

Remark 4 It is easily seen that n-tuple integration is continuous for every natural $n \geq 2$.

Theorem 1 implies the following.

Corollary 1 *For every natural n and every $(\pi_1, \dots, \pi_n) \in \Pi^{(n)}$ the integration result $\pi_1 \cdot \dots \cdot \pi_n$ does not change if integration is carried out in any order and in any number of steps. Namely,*

$$\pi_1 \cdot \dots \cdot \pi_n = (\pi_{i_1} \cdot \dots \cdot \pi_{i_{k_1}}) \cdot (\pi_{i_{k_1}+1} \cdot \dots \cdot \pi_{i_{k_2}}) \dots \cdot (\pi_{i_{k_m}} \cdot \dots \cdot \pi_{i_n})$$

for any permutation $(i_1, \dots, i_n)$ in $(1, \dots, n)$ and any increasing sequence $(k_j)_1^m$ in $\{2, \dots, n-1\}$.

2.2 Integral powers

Based on Corollary 1 (see also Remark 1, (v)), for every $\pi \in \Pi$ and every natural n we denote π^n the result of integration of n copies of π; we call π^n the n-th *integral power* of π.

The next theorem, which follows straightforwardly from the definition of the result of integration of n copies of a probability, shows that an n-th integral power of a probability preserves the initial probabilistic priorities and makes these priorities sharper as n grows. An interpretation is that integration of several identical independent observation methods sharpens the stochastic estimate provided be each of those methods and the sharpness grows as the number of the observation methods grows.

Recall that $Z^+(\pi) = \{z \in Z : \pi(z) > 0\}$ for every $\pi \in \Pi$ (see notations in Secition 1).

Theorem 2 *Let $\pi \in \Pi$. For every natural n the following statements hold true.*

1) For every $z \in Z \setminus Z^+(\pi)$ it holds that $\pi^n(z) = 0$.

2) For every $z_1, z_2 \in Z^+(\pi)$ it holds that

$$\frac{\pi^n(z_1)}{\pi^n(z_2)} = \left(\frac{\pi(z_1)}{\pi(z_2)}\right)^n.$$

A straightforward implication from Theorem 2 is the following.

Corollary 2 *Let $\pi \in \Pi$, Z_* be the set of all maximizers of π in Z and a $\pi_* \in \Pi$ be uniform on Z_*, i.e., $\pi_*(z) = 0$ for all $z \in Z \setminus Z^+(\pi)$ and $\pi_*(z) = 1/|Z_*|$ for all $z \in Z_*$. It holds that $\pi^n \to \pi_*$ in Π as $n \to \infty$.*

Given a $\pi \in \Pi$ and a natural n, we call a $\pi_* \in \Pi$ an n-th *integral root* of π if $\pi_*^n = \pi$. In accordance with Theorem 2, an n-th integral root of a probability π sets the same – though less sharp – priorities to the elements of Z; in this context, the n-th integral root of π can be interpreted as a prototype observation method lying behind the one represented by π.

Theorem 3 *For every $\pi \in \Pi$ and every natural n there exists the unique n-th integral root of π.*

Proof. Take a $\pi \in \Pi$. By definition a $\pi_* \in \Pi$ is an n-th root of π if and only if $\pi_*^n = \pi$ or

$$\frac{\pi_*^n(z)}{\sum_{z' \in Z} \pi_*^n(z')} = \pi(z) \quad (z \in Z).$$

For simplicity we order the elements of Z, i.e., set $Z = \{z_1, \ldots, z_N\}$ where $N = |Z|$. Then $\pi_* \in \Pi$ is an n-th integral root of π if and only if $(\pi_*^n(z_1), \ldots, \pi_*^n(z_N), \sum_{i=1}^N \pi_*^n(z_i))$ solves the system of algebraic equations

$$\begin{aligned} & x_1 - \pi(z_1)x_{N+1} = 0, \\ & \ldots \\ & x_N - \pi(z_N)x_{N+1} = 0, \\ & x_1 + \ldots + x_N - x_{N+1} = 0 \end{aligned} \tag{2}$$

under the additional constraints

$$x_1, \ldots, x_N \geq 0, \quad x_1^{1/n} + \ldots + x_N^{1/n} = 1. \tag{3}$$

Denote A the matrix of system (2). We have

$$A = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & 0 & -\pi(z_1) \\ 0 & 1 & 0 & \ldots & 0 & 0 & -\pi(z_2) \\ & & & \ldots & & & \\ 0 & 0 & 0 & \ldots & 0 & 1 & -\pi(z_N) \\ 1 & 1 & 1 & \ldots & 1 & 1 & -1 \end{pmatrix}.$$

The sum of the N first rows of A equals its $N+1$-th row and its left-upper $N \times N$ submatrix is nondegenerate. Therefore, the $(N+1) \times (N+1)$-matrix A has rank N. Consequently, the set of all solutions to (2) is a one-dimensional subspace of R^{N+1}.

Let $(y_1, \ldots, y_{N+1})$ be a non-zero solution to (2). Taking into account that $\pi(z_1), \ldots, \pi(z_N) \geq 0$ and not all of them are zero, and looking at (2), we see that $y_{N+1} \neq 0$, not all of y_i $(i \in \{1, \ldots, N\})$ are zero and the signs of all non-zero ones coincide with the sign of y_{N+1}. With no loss of generality, set $y_{N+1} > 0$ (otherwise multiply $y_1, \ldots, y_{N+1}$ by -1). Then $y_1, \ldots, y_{N+1} \geq 0$.

Let

$$\lambda = \left(\frac{1}{y_1^{1/n} + \ldots + y_N^{1/n}} \right)^n \tag{4}$$

and

$$x_i = \lambda y_i \quad (i \in \{1, \ldots, N+1\}). \tag{5}$$

Obviously, $x_1, \ldots, x_N$ satisfy the former constraint given in (3). Furthermore,

$$x_1^{1/n} + \ldots + x_N^{1/n} = \lambda^{1/n}(y_1^{1/n} + \ldots + y_N^{1/n}) = 1; \tag{6}$$

we see that the latter constraint given in (3) is satisfied too. Thus, π_*, an n-th integral root of π exists and is given by

$$\pi_*(z_i) = x_i^{1/n} \quad (i \in \{1, \ldots, N\}). \tag{7}$$

If π_* is an n-th integral root of π and $x_1, \ldots, x_N$ are defined by (7) and $x_{N+1} = x_1 + \ldots + x_N$, then, as stated above, $x_1, \ldots, x_{N+1}$ solve (2) and satisfy (3). Since the subspace of all solutions to (2) is one-dimensional, (5) holds with some real λ. Then (3) implies (6); hence, λ is given by (30). Thus, the n-th integral root of π is unique. The proof is complete.

Based on Theorem 3 stating, for every natural n, the existence and uniqueness of the n-th integral root of any $\pi \in \Pi$, we denote it $\pi^{1/n}$.

Remark 5 From Theorem 2 it follows that given a $\pi \in \Pi$, for every $z \in Z \setminus Z^+(\pi)$ we have $\pi^{1/n}(z) = 0$ for all natural n, and for every $z_1, z_2 \in Z^+(\pi)$ we have

$$\frac{\pi^{1/n}(z_1)}{\pi^{1/n}(z_2)} = \left(\frac{\pi(z_1)}{\pi(z_2)}\right)^{1/n} \to 1 \quad \text{as} \quad n \to \infty.$$

The latter implies that $\pi^{1/n} \to \pi_*$ in Π where π_* is uniform on $Z^+(\pi)$, i.e., $\pi_*(z) = 0$ for all $z \in Z \setminus Z^+(\pi)$ and $\pi_*(z) = 1/|Z^+(\pi)|$ for all $z \in Z^+(\pi)$.

Referring to Theorem 3, we introduce rational integral powers of probabilities. For every $\pi \in \Pi$ and every natural n and m, we set $\pi^{m/n} = (\pi^m)^{1/n}$ and call it the *m/n-th integral power* of π.

Remark 6 For rational integral powers of probabilities, standard arithmetic relations are valid. More specifically, for every $\pi \in \Pi$ and every natural n and m, in the definition of $\pi^{m/n}$ the multiplication and division in m/n can follow in an arbitrary order, namely, $\pi^{m/n} = (\pi^m)^{1/n}$ can also be defined as $\pi^{m/n} = (\pi^{1/n})^m$. Indeed, we have

$$((\pi^{1/n})^m)^n = (\pi^{1/n})^{mn} = (\pi^{1/n})^{nm} = ((\pi^{1/n})^n)^m = \pi^m.$$

Hence, referring to the definition of the n-th integral root of a π^m, we find that $\pi^{m/n} = (\pi^{1/n})^m$.

2.3 Disintegration

From the definition of integration, it follows that $Z^+(\pi \cdot \pi_1) = Z^+(\pi) \cap Z^+(\pi_1)$ for every $(\pi, \pi_1) \in \Pi^{(2)}$. We use this observation in the following definition.

Given $\pi_1, \pi_2 \in \Pi$ such that $Z^+(\pi_2) \subset Z^+(\pi_1)$, a $\pi \in \Pi$ will be said to be a *result of disintegration of π_2 over π_1* if $Z^+(\pi_2) = Z^+(\pi) \cap Z^+(\pi_1)$ (implying $(\pi, \pi_1) \in \Pi^{(2)}$) and $\pi \cdot \pi_1 = \pi_2$.

Theorem 4 *Let $\pi_1, \pi_2 \in \Pi$ and $Z^+(\pi_2) \subset Z^+(\pi_1)$. The following statements hold true.*

1) There exists a result of disintegration of π_2 over π_1.

2) If π is a result of disintegration of π_2 over π_1, then a $\pi' \in \Pi$ is a result of disintegration of π_2 over π_1 if and only if $\pi'|_{Z^+(\pi_1)} = \mu\pi|_{Z^+(\pi_1)}$ with some $\mu > 0$, where $\pi|_{Z^+(\pi_1)}$ and $\pi'|_{Z^+(\pi_1)}$ are the restrictions to $Z^+(\pi_1)$ of π and π', respectively.

Proof. For simplicity we order the elements of Z, namely, set $Z = \{z_1, \dots, z_N\}$ where $N = |Z|$, so that $Z^+(\pi_2) = \{z_1, \dots, z_k\}$ and $Z^+(\pi_1) = \{z_1, \dots, z_m\}$ with some $k, m \in \{1, \dots, N\}$, $m \geq k$. Then

$$\pi_1(z_i), \pi_2(z_i) > 0 \qquad (i \in \{1, \dots, k\}), \tag{8}$$

$$\pi_1(z_i) > 0,\ \pi_2(z_i) = 0 \quad (i \in \{k+1, \dots, m\}), \tag{9}$$

$$\pi_1(z_i), \pi_2(z_i) = 0 \quad (i \in \{m+1, \dots, N\}). \tag{10}$$

By definition a $\pi \in \Pi$ is a result of disintegration of π_2 over π_1 if $Z^+(\pi_2) = Z^+(\pi) \cap Z^+(\pi_1)$ and

$$\frac{\pi(z)\pi_1(z)}{\sum_{z' \in Z} \pi(z')\pi_1(z')} = \pi_2(z) \quad (z \in Z),$$

or, equivalently, $(\pi(z_1), \ldots, \pi(z_N), \sum_{i=1}^{N} \pi(z_i)\pi_1(z_i))$ is a solution to the system of algebraic equations

$$\begin{array}{l} \pi_1(z_1)x_1 - \pi_2(z_1)x_{N+1} = 0, \\ \ldots \\ \pi_1(z_N)x_N - \pi_2(z_N)x_{N+1} = 0, \\ \pi_1(z_1)x_1 + \ldots + \pi_1(z_N)x_N - x_{N+1} = 0 \end{array} \tag{11}$$

under constraints

$$x_1, \ldots, x_k > 0, \quad x_{k+1}, \ldots, x_m = 0, \quad x_{m+1}, \ldots, x_N \geq 0, \quad x_1 + \ldots + x_N = 1 \tag{12}$$

(in cases $k = m$ and $m = N$ the second and third constraints are, respectively, omitted).

In case $k < m$, we set

$$x_{k+1}, \ldots, x_m = 0; \tag{13}$$

then, in view of (9), $x_{k+1}, \ldots, x_m$ satisfy the equations given in lines $k+1, \ldots, m$ of system (11). Looking at (10), we see that if $m < N$, the equations given in lines $m+1, \ldots, N$ of system (11) are satisfied trivially by any $x_{m+1}, \ldots, x_N$.

Let us now consider the rest of system (11) – its subsystem composed of its equations given in rows $1, \ldots, k$ and $N+1$,

$$\begin{array}{l} \pi_1(z_1)x_1 - \pi_2(z_1)x_{N+1} = 0, \\ \ldots \\ \pi_1(z_k)x_k - \pi_2(z_k)x_{N+1} = 0, \\ \pi_1(z_1)x_1 + \ldots + \pi_1(z_k)x_k - x_{N+1} = 0; \end{array} \tag{14}$$

the latter equation in (14) is equivalent to the $(N+1)$-th one in (11) thanks to (13) and (10). Denote A the matrix of system (14). We have

$$A = \begin{pmatrix} \pi_1(z_1) & 0 & 0 & \ldots & 0 & 0 & -\pi_2(z_1) \\ 0 & \pi_1(z_2) & 0 & \ldots & 0 & 0 & -\pi_2(z_2) \\ & & & \ldots & & & \\ 0 & 0 & 0 & \ldots & 0 & \pi_1(z_k) & -\pi_2(z_k) \\ \pi_1(z_1) & \pi_1(z_2) & \pi_1(z_3) & \ldots & \pi_1(z_{k-2}) & \pi_1(z_{k-2}) & -1 \end{pmatrix}.$$

The sum of the k first rows of A equals its $k+1$-th row and its left-upper $k \times k$ submatrix is nondegenerate in view of (8). Therefore, the $(k+1) \times (k+1)$-matrix A has rank k. Consequently, the set of all solutions to (14) is a one-dimensional subspace of R^{k+1}. Let $(y_1, \ldots, y_k, y_{N+1})$ be a non-zero solution to (14). With no loss of generality, set $y_{N+1} > 0$ (otherwise multiply $y_1, \ldots, y_k, y_{N+1})$ by -1). Then $y_1, \ldots, y_k, y_{N+1} > 0$. Take a $c \in (0, 1]$ and set

$$\lambda = \frac{c}{y_1 + \ldots + y_k},$$

$$x_i = \lambda y_i \quad (i \in \{1, \ldots, k, N+1\}), \tag{15}$$

implying

$$x_1, \ldots, x_k > 0, \quad x_1 + \ldots + x_k = c.$$

Combining with (13), we see that the first and second constraints given in (12) are satisfied by $x_1, \dots, x_m$.

If $m = N$, then, letting $c = 1$ and recalling (13) (provided $m > k$), we get that $x_1, \dots, x_N, x_{N+1} \geq 0$ solves system (11) and satisfies (12) (where the third constraint is omitted and the second one is omitted in case $m = k$). If $m < N$, take arbitrary $x_{m+1}, \dots, x_N \geq 0$ whose sum equals $1 - c$. Then $x_1, \dots, x_N, x_{N+1}$ solve system (11) and satisfy (12) (where the second constraint is omitted if $m = k$). Thus, a $\pi \in \Pi$ given by

$$\pi(z_i) = x_i \quad (i \in \{1, \dots, N\}) \tag{16}$$

is a result of disintegration of π_2 over π_1. Statement 1 is proved.

Prove statement 2. Let a $\pi' \in \Pi$ be such that $\pi'|_{Z^+(\pi_1)} = \mu\pi|_{Z^+(\pi_1)}$ with some $\mu > 0$; equivalently, for

$$x'_i = \pi'(z_i) \quad (i \in \{1, \dots, N\}) \tag{17}$$

it holds that

$$x'_i = \mu x_i \quad (i \in \{1, \dots, m\});$$

the latter, in view of (13) (in case $m > k$), implies

$$x'_{k+1}, \dots, x'_m = 0. \tag{18}$$

Let

$$x'_{N+1} = \pi_1(z_1)x'_1 + \ldots + \pi_1(z_N)x'_N. \tag{19}$$

Obviously, $x'_{N+1} = \mu x_{N+1}$. Since $(x_1, \dots, x_m, x''_{m+1}, \dots, x''_N, x_{N+1})$ with arbitrary $x''_{m+1}, \dots, x''_N \geq 0$ (provided $m < N$) solves system (11) under constraints (12), we get that $(\pi'(z_1), \dots, \pi'(z_N), x'_{N+1}) = (x'_1, \dots, x'_{N+1})$ possesses the same property, which is equivalent to the fact that π is a result of disintegration of π_2 over π_1.

Conversly, let π' be a result of disintegration of π_2 over π_1. Then $(x'_1, \dots, x'_{N+1})$ defined by (17) and (19) solves system (11) and satisfies (12); in particular (18) holds if $m > k$. Then necessarily

$$x'_i = \lambda' y_i \quad (i \in \{1, \dots, k, N+1\})$$

for some $\lambda' > 0$. Therefore, by (15)

$$x'_i = \mu x_i \quad (i \in \{1, \dots, k, N+1\})$$

where $\mu = \lambda'/\lambda$. Then, in view of (16), (17), (13) and (18) (the two latter relations holding in case $m > k$) we have

$$\pi'(z_i) = \mu\pi(z_i) \quad (i \in \{1, \dots, m\})$$

implying $\pi'|_{Z^+(\pi_1)} = \mu\pi|_{Z^+(\pi_1)}$. The theorem is proved.

Given a $\pi_1 \in \Pi$ and a $\pi_2 \in \Pi$ such that $Z^+(\pi_2) \subset Z^+(\pi_1)$, we denote $[\pi_2/\pi_1]$ the set of all results of disintegration of π_2 over π_1. The multi-valued map $(\pi_1, \pi_2) \mapsto [\pi_2/\pi_1]$ defined on the set of all $(\pi_1, \pi_2) \in \Pi^2$ such that $Z^+(\pi_2) \subset Z^+(\pi_1)$, will be called *disintegration.*

Corollary 3 *Let $\pi_1, \pi_2 \in \Pi$ and $Z^+(\pi_2) \subset Z^+(\pi_1)$ The following statements hold true.*

1) If $Z^+(\pi_1) = Z$, then $[\pi_2/\pi_1]$ is one-element.

2) If $Z^+(\pi_1) \neq Z$ and $\pi \in [\pi_2/\pi_1]$ then

$$[\pi_2/\pi_1] = \left\{\pi' \in \Pi : \pi'|_{Z^+(\pi_1)} = \mu\pi|_{Z^+(\pi_1)},\ 0 < \mu \leq \frac{1}{\sum_{z \in Z^+(\pi_1)} \pi(z)}\right\}.$$

Proof. Let $Z^+(\pi_1) = Z$. If $\pi' \in [\pi_2/\pi_1]$ then by statement 2 of Theorem 4 $\pi' = \pi'|_{Z^+(\pi_1)} = \mu\pi|_{Z^+(\pi_1)} = \mu\pi$ with some $\mu > 0$. Since $\pi, \pi' \in \Pi$, necessarily $\mu = 1$. Statement 1 is proved.

Let $Z^+(\pi_1) \neq Z$. By statement 2 of Theorem 4 $\pi' \in [\pi_2/\pi_1]$ if and only if $\pi'|_{Z^+(\pi_1)} = \mu\pi|_{Z^+(\pi_1)}$ with some $\mu > 0$. The latter is feasible if and only if $\mu \sum_{z\in Z^+(\pi_1)} \pi(z) \in (0, 1]$, which completes the proof of statement 2.

Remark 7 Under assumptions of Corollary 3, there is a unique $\pi' \in [\pi_2/\pi_1]$ such that $Z^+(\pi') = Z^+(\pi_2)$. Indeed, let a $\pi' \in [\pi_2/\pi_1]$ be such that

$$\pi'|_{Z^+(\pi_1)} = \frac{1}{\sum_{z\in Z^+(\pi_1)} \pi(z)} \pi|_{Z^+(\pi_1)}. \tag{20}$$

Then $\sum_{z\in Z^+(\pi_1)} \pi'(z) = 1$, implying $Z^+(\pi') \subset Z^+(\pi_1)$. Furthermore, the fact that $\pi' \cdot \pi_1 = \pi_2$ implies that $\pi'(z) = 0$ for every $z \in Z^+(\pi_1) \setminus Z^+(\pi_2)$. and $\pi'(z) > 0$ for every $z \in Z^+(\pi_2)$. Consequently, $Z^+(\pi') = Z^+(\pi_2)$. Conversely, if a $\pi' \in [\pi_2/\pi_1]$ is such that $Z^+(\pi') = Z^+(\pi_2)$, (20) holds necessarily.

Based on Remark 7, for every $\pi_1, \pi_2 \in \Pi$ such that $Z^+(\pi_2) \subset Z^+(\pi_1)$, we denote π_2/π_1 the single element $\pi \in [\pi_2/\pi_1]$ such that $Z^+(\pi) = Z^+(\pi_2)$.

Remark 8 As follows from Remark 7, π_2/π_1 is the element of $[\pi_2/\pi_1]$, which provides the maximum probability values to all $z \in Z^+(\pi_2)$ and zero probability values to all $z \notin Z^+(\pi_2)$.

Remark 9 For every $\pi \in \Pi^+$ (see notations in Section 1) we have $Z^+(\pi) = Z$. Therefore, the restriction of disintegration to $\Pi^+ \times \Pi^+$ is defined correctly. By statement 1 of Corollary 3, the restriction of disintegration to $\Pi^+ \times \Pi^+$ is single-valued. Clearly, it takes values in Π^+ implying that Π^+ is invariant with respect to both integration and disintegration. One can easily show that disintegration as a function on $\Pi^+ \times \Pi^+$ with values in Π^+ is continuous. Therefore, in view of the continuity of integration (see Remark 4) and the properties of integration given in Theorem 1, Π^+ equipped with integration (understood as multiplication) is an abelian topological group, in which the uniform probability, $\bar{\pi}$, is the unit.

With respect to integration, disintegration plays the same role division does with respect to multiplication in arithmetic. Constraint $Z^+(\pi_2) \subset Z^+(\pi_1)$ in the definition of $[\pi_2/\pi_1]$ is a counterpart of the standard arithmetic requirement that the divisor never vanishes. The next theorem shows that relations between integration and disintegration, are analogous to those between standard multiplication and division.

Theorem 5 *The following statements hold true.*

1) Let $\pi_1, \pi_2, \pi_3 \in \Pi$ and $Z^+(\pi_3) \subset Z^+(\pi_2) \subset Z^+(\pi_1)$. Then

$$[[\pi_3/\pi_2]/\pi_1] = [\pi_3/(\pi_2 \cdot \pi_1)] \tag{21}$$

where

$$[[\pi_3/\pi_2]/\pi_1] = \cup_{\pi'\in[\pi_3/\pi_2]}[\pi'/\pi_1].$$

2) Let $\pi_1, \pi_2, \pi_3 \in \Pi$, $Z^+(\pi_1) = Z^+(\pi_2) = Z^+(\pi_3)$ and $\pi_1 \cdot \pi_2 = \pi_1 \cdot \pi_3$. Then $\pi_2 = \pi_3$.

3) Let $\pi_1, \pi_2, \pi_3 \in \Pi$ and $Z^+(\pi_1) = Z^+(\pi_2) = Z^+(\pi_3)$. Then

$$(\pi_3 \cdot \pi_2)/\pi_1 = (\pi_3/\pi_1) \cdot \pi_2. \tag{22}$$

Proof. Prove statement 1. Let $\pi_1, \pi_2, \pi_3 \in \Pi$ and $Z^+(\pi_3) \subset Z^+(\pi_2) \subset Z^+(\pi_1)$. Take a $\pi \in [[\pi_3/\pi_2]/\pi_1]$. Then $\pi \in [\pi'/\pi_1]$ for some $\pi' \in [\pi_3/\pi_2]$. Hence, $\pi \cdot \pi_1 = \pi'$ and $\pi' \cdot \pi_2 = p_3$, implying $\pi \cdot \pi_1 \cdot \pi_2 = \pi_3$ or $\pi \cdot (\pi_1 \cdot \pi_2) = \pi_3$. Therefore, $\pi \in [\pi_3/(\pi_2 \cdot \pi_1)]$. We have stated that the set given in the left hand side of (21) is included in the set given in its right hand side. Prove the opposite inclusion. Take a $\pi \in [\pi_3/(\pi_2 \cdot \pi_1)]$. We have $\pi \cdot (\pi_2 \cdot \pi_1) = \pi_3$ or $(\pi \cdot \pi_1) \cdot \pi_2 = \pi_3$. Consequently, $\pi' = (\pi \cdot \pi_1) \in [\pi_3/\pi_2]$ and $\pi \in [\pi'/\pi_1]$. The desired opposite inclusion is proved. Thus, (21) is true.

Prove statement 2. Let $\pi_1, \pi_2, \pi_3 \in \Pi$, $Z^+(\pi_3) = Z^+(\pi_2) = Z^+(\pi_1)$ and $\pi_1 \cdot \pi_2 = \pi_1 \cdot \pi_3$. Obviously, $\pi_2 = (\pi_1 \cdot \pi_2)/\pi_1$ and $\pi_3 = (\pi_1 \cdot \pi_3)/\pi_1$. Hence, $\pi_2 = \pi_3$.

Prove statement 3. Let $\pi_1, \pi_2, \pi_3 \in \Pi$ and $Z^+(\pi_1) = Z^+(\pi_2) = Z^+(\pi_3)$ Denote

$$\pi = (\pi_3 \cdot \pi_2)/\pi_1.$$

By definition and by assumption

$$Z^+(\pi_3 \cdot \pi_2) = Z^+(\pi_1) = Z^+(\pi) \tag{23}$$

and $\pi \cdot \pi_1 = \pi_3 \cdot \pi_2$. Let $\pi' = \pi/\pi_2$. By definition and by (23)

$$Z^+(\pi') = Z^+(\pi) = Z^+(\pi_1) \tag{24}$$

and

$$\pi = \pi' \cdot \pi_2. \tag{25}$$

Hence, $\pi' \cdot \pi_2 \cdot \pi_1 = \pi_3 \cdot \pi_2$ or

$$(\pi' \cdot \pi_1) \cdot \pi_2 = \pi_3 \cdot \pi_2. \tag{26}$$

Using (24) and the assumption, we get

$$Z^+(\pi' \cdot \pi_1) = Z^+(\pi_1) = Z^+(\pi_2) = Z^+(\pi_3).$$

Then by statement 2 (26) yields $\pi' \cdot \pi_1 = \pi_3$. Hence, $\pi' = \pi_3/\pi_1$. Therefore, in view of (25), $\pi = \pi' \cdot \pi_2 = (\pi_3/\pi_1) \cdot \pi_2$. which proves (22). The proof is complete.

3 Integration and evaluation of probabilities

3.1 Impacts of most and least probable elements

Here we consider a transition from a probability $\pi_1 \in \Pi$ viewed as a prior stochastic estimate (see Section 1) to a posterior one, $\pi_1 \cdot \pi_2$, obtained through integration of π_1 with a $\pi_2 \in \Pi$. We show that, generically, integration raises the prior probability of every element z^* most probable in the probability space (Z, π_2) (Lemma 1), and reduces that of every element z_* least probable in (Z, π_2) (Lemma 2).

Lemma 1 *Let* $(\pi_1, \pi_2) \in \Pi^{(2)}$, $z^* \in Z$ *be such that* $\pi_1(z^*) > 0$, $\pi_2(z^*) = \max_{z' \in Z} \pi_2(z')$ *and there exist a* $z \in Z$ *satisfying* $\pi_2(z) < \pi_2(z^*)$ *and* $\pi_1(z)\pi_2(z) > 0$. *Then* $(\pi_1 \cdot \pi_2)(z^*) > \pi_1(z^*)$.

Proof. Let $Z^* = \{z' \in Z : \pi_2(z') = \pi_2(z^*)\}$. Note that $z \in Z \setminus Z^*$. By definition

$$\begin{aligned}(\pi_1 \cdot \pi_2)(z^*) &= \frac{\pi_1(z^*)\pi_2(z^*)}{\sum_{z' \in Z} \pi_1(z')\pi_2(z')} \\ &= \frac{\pi_1(z^*)}{\sum_{z' \in Z^*} \pi_1(z') + \pi_1(z)q(z) + \sum_{z' \in Z \setminus (Z^* \cup \{z\})} \pi_1(z')q(z')} \end{aligned} \tag{27}$$

where

$$q(z') = \frac{\pi_2(z')}{\pi_2(z^*)} \quad (z' \in Z).$$

Since $\pi_2(z^*) = \max_{z \in Z} \pi_2(z)$, we have $q(z') < 1$ for all $z' \in Z \setminus Z^*$, which, together with $\pi_1(z)\pi_2(z) > 0$ implies that $\pi_1(z)q(z) < \pi_1(z)$. Consequently, the denominator in (27) is smaller than $\sum_{z' \in Z} \pi_1(z') = 1$. Now (27) and assumption $\pi_1(z^*) > 0$ yield $(\pi_1 \cdot \pi_2)(z^*) < \pi_1(z^*)$. The lemma is proved.

Similarly, we prove the following symmetric lemma.

Lemma 2 *Let* $(\pi_1, \pi_2) \in \Pi^{(2)}$, $z_* \in Z$ *be such that* $\pi_1(z_*) > 0$, $\pi_2(z_*) = \min_{z' \in Z,\ \pi_2(z')>0} \pi_2(z')$ *and there exist a* $z \in Z$ *satisfying* $\pi_2(z) > \pi_2(z_*)$ *and* $\pi_1(z)\pi_2(z) > 0$. *Then* $(\pi_1 \cdot \pi_2)(z_*) < \pi_1(z_*)$.

3.2 Measures of concentration

For probabilities $\pi \in \Pi$, we now introduce measures of concentration – indicators that reach their highest values at the concentrated probabilities. A measure of concentration of a probability $\pi \in \Pi$ serves also for a measure of informativeness of π; the concentrated probabilities, π, carry maximum information on the occurrence of events in the probability space (Z, π); the higher is the measure of concentration of π, the more informative π is. As argued in Section 1, integration of probabilities is a tool for synthesizing pieces of knowledge provided by alternative, statistically inaccurate observation methods. If the result of integration of initially given non-conflicting probabilities (acting as prior stochastic estimates provided by alternative observation methods) has a higher measure of concentration than each of those probabilities does, we claim that the prior stochastic estimates enhance each other and provide, together, more information than each of them does, being taken separately.

An opposite situation assumes that the result of integration of initially given probabilities (prior stochastic stochastic estimates) has a lower measure of concentration than each of those probabilities – a synthesis of the prior stochastic estimates reduces information. This tells us that the underlying observation methods are in conflict; one of them is misleading. This puts us in a position to accept one of the methods and reject the other one; we need additional information to decide which of the two choices is correct.

An intermediate situation may occur, in which the result of integration of initially given probabilities has a higher measure of concentration than one of those probabilities and a lower measure of concentration than the other one. In this situation we may consider the latter method to be a tool for raising the accuracy of the former one.

Our primary interest is to characterize the former situation, in which the result of integration of probabilities π_1 and π_2 has a higher measure of concentration than each of those probabilities does; in this situation we will say that the pair (π_1, π_2) is consistent (with respect to the given measure of concentration).

Formally, we can define a measure of concentration to take value one on the concentrated probabilities and value zero on all the other ones. Such a discrete definition would not be informative because it would not capture the degree of closedness of any non-concentrated probability to a concentrated one. An informative measure of concentration would range the probabilities within a solid interval of reals, whose right end point would be occupied by the concentrated probability measures. We arrive at an informative definition if we require the measures of concentration to be continuous functions on Π.

Let us give an exact definition. We define a *measure of concentration* to be an arbitrary continuous function $\mu : \Pi \mapsto (-\infty, 1]$ such that $\mu(\pi) = 1$ if and only if π is concentrated.

We shall say that a $(\pi_1, \pi_2) \in \Pi^{(2)}$ is *consistent* with respect to a measure of concentration μ if $\mu(\pi_1 \cdot \pi_2) > \max\{\mu(\pi_1),\ \mu(\pi_2)\}$ and *inconsistent* with respect to μ if $\mu(\pi_1 \cdot \pi_2) < \min\{\mu(\pi_1),\ \mu(\pi_2)\}$.

Remark 10 Statement 3 of Theorem 1 implies that if a $(\pi_1, \pi_2) \in \Pi^{(2)}$ is consistent or inconsistent with respect to any measure of concentration, then neither π_1, nor π_2 is uniform.

The simplest measure of concentration is $\pi \mapsto \max_{z \in Z} \pi(z)$; we call it the *max measure of concentration*. In this pilot study, we deal with the max measure of concentration only.

Remark 11 Let us give a few additional examples of measures of concentration:

(i) $\pi \mapsto \max_{z \in Z} \pi(z) - \min_{z \in Z} \pi(z)$;

(ii) $\pi \mapsto \sum_{z \in Z} \pi^k(z)$ where $k > 1$;

(iii) $\pi \mapsto 1 - \sum_{z \in Z} [\xi(z) - \sum_{z' \in Z} \xi(z')\pi(z')]^2 \pi(z)$ where ξ is an arbitrary real-valued one-to one function on Z; the value of the latter measure of concentration on a $\pi \in \Pi$ is the dispertion of the random variable ξ on (Z, π);

(iv) $\pi \mapsto 1 + \sum_{z \in Z} \pi(z) \log \pi(z)$ (here it is set $\pi(z) \log \pi(z) = 0$ if $\pi(z) = 0$); the latter sum taken with the opposite sign is known as the entropy of π.

Remark 12 It is clear that the lowest possible value for the max measure of concentration equals $1/|Z|$ and is achieved on the uniform probability, $\bar{\pi}$.

Remark 13 It is easy to show that the max measure of concentration is a convex function on Π treated as a convex subset of the linear space of all real-valued functions defined on Z.

For brevity, a $(\pi_1, \pi_2) \in \Pi^{(2)}$ consistent (respectively, inconsistent) with respect to the max measure of concentration will be said to be *max consistent* (respectively, *max inconsistent*).

Using Lemma 1, we now characterize a typical situation, in which a $(\pi_1, \pi_2) \in \Pi^{(2)}$ is max consistent. Generically, this is the case if the sets of the most probable elements in the probability spaces (Z, π_1) and (Z, π_2) intersect.

Theorem 6 *Let $(\pi_1, \pi_2) \in \Pi^{(2)}$ and there exist a $z^* \in Z$ maximizing both π_1 and π_2 on Z. The following statements are true.*

1) z^* maximizes $\pi_1 \cdot \pi_2$ on Z.

2) If there exists a $z \in Z$ such that

$$0 < \pi_1(z) < \max_{z' \in Z} \pi_1(z'), \quad 0 < \pi_2(z) < \max_{z' \in Z} \pi_2(z'),$$

then (π_1, π_2) is max consistent.

Proof. Statement 1 follows straightforwardly from the definition of $\pi_1 \cdot \pi_2$. Let us prove statement 2. Obviously, $\pi_1(z^*) > 0$. By assumption $\pi_2(z) < \pi_2(z^*)$ and $\pi_1(z)\pi_2(z) > 0$. Thus, all the assumptions of Lemma 1 are satisfied. Applying Lemma 1, we get $(\pi_1 \cdot \pi_2)(z^*) > \pi_1(z^*) = \max_{z' \in Z} \pi_1(z')$. Changing the roles of π_1 and π_2 in the above argument, we come to a symmetric statement, $(\pi_1 \cdot \pi_2)(z^*) > \pi_2(z^*) = \max_{z' \in Z} \pi_2(z')$. The proof is complete.

A straightforward corollary from Theorem 6 is the following.

Corollary 4 *For every non-concentrated non-uniform* $\pi \in \Pi$, (π, π) *is max consistent.*

Let us extend the definition of max consistency of pairs $(\pi_1, \pi_2) \in \Pi^{(2)}$, to n-tuples $(\pi_1, \dots, \pi_n) \in \Pi^{(n)}$ for $n \geq 2$. Namely, for every natural $n \geq 2$, let us call a $(\pi_1, \dots, \pi_n) \in \Pi^{(n)}$ to be *max consistent* if

$$\max_{z \in Z} (\pi_1 \cdot \ldots \cdot \pi_n)(z) > \max\{\max_{z \in Z} \pi_1(z), \ldots \max_{z \in Z} \pi_1(z)\}.$$

Theorem 6 yields the following.

Corollary 5 *Let a* $n \geq 2$ *be natural,* $(\pi_1, \dots, \pi_n) \in \Pi^{(n)}$ *and there exist a* $z^* \in Z$ *maximizing* $\pi_1, \dots, \pi_n$ *on* Z *simultaneously. The following statements are true.*

1) z^* *maximizes* $\pi_1 \cdot \ldots \cdot \pi_n$ *on* Z.

2) If there exists a $z \in Z$ *such that*

$$0 < \pi_i(z) < \max_{z' \in Z} \pi_i(z')$$

for all $i \in \{1, \dots, n\}$, *then* $(\pi_1, \dots, \pi_n)$ *is max consistent.*

The following immediate implication from Corollary 4 concerns the topological structure of sets invariant with respect to integration. A $G \subset \Pi$ is said to be *invariant with respect to integration* if every $\pi_1, \pi_2 \in G$ are not conflicting and $\pi_1 \cdot \pi_2 \in G$ for every $\pi_1, \pi_2 \in G$.

Theorem 7 *Let a nonempty* $G \subset \Pi$ *do not contain concentrated probabilities and be invariant with respect to integration. Then one and only one statement holds true:*

(i) G *is one-element and its single element is the uniform probability,* $\bar{\pi}$;

(ii) G *is not closed in* Π.

Proof. Let G be one-element and π be its element. Then $\pi \cdot \pi = \pi$. Suppose $\pi \neq \bar{\pi}$. Then by Corollary 4 (π, π) is max consistent, i.e., $\max_{z \in Z} (\pi \cdot \pi)(z) > \max_{z \in Z} \pi(z)$, which is impossible in view of $\pi \cdot \pi = \pi$.

Let G be not one-element. Then G contains a non-uniform probability. Suppose G is closed. Let $p = \sup_{\pi \in G} \max_{z \in Z} \pi(z)$. As soon as G contains a non-uniform probability, p is greater than the value taken by the uniform probability, $p > 1/|Z|$. Since G is closed, there is a $\pi_* \in G$ such that $\max_{z \in Z} \pi_*(z) = p$. In view of $p > 1/|Z|$, π_* is non-iniform. Moreover, π_* is not concentrated, since G does not contain concentrated probabilities by assumption. Therefore, by Corollary 4 (π_*, π_*) is max consistent, i.e., $\max_{z \in Z} (\pi_* \cdot \pi_*)(z) > \max_{z \in Z} \pi_*(z) = p$. On the other hand, $(\pi_*, \pi_*) \in G$; hence, $\max_{z \in Z} (\pi_* \cdot \pi_*)(z) \leq p$. The contradiction completes the proof.

A straightforward corollary from Theorem is the following.

Corollary 6 *Let a nonempty $G \subset \Pi$ be invariant with respect to integration G do not contain concentrated probabilities and $G \neq \{\bar{z}\}$. Then G is not finite.*

As a complement to Corollary 6, let note that there are countable sets of non-concentrated probabilities in Π, wich are invariant with respect to integration – see the following example.

Example 1 Let a $\pi \in \Pi$ be non-uniform and not concentrated. Then $G = \{\pi^n : n \in \{1, 2, \ldots\}\}$ consists of non-concentrated probabilities, is countable and is invariant with respect to integration.

Remark 14 If $G \subset \Pi$ is invariant with respect to integration and G contains a concentrated probability, then there is no other concentrated probability contained in G, since every pair of concentrated probabilities having different concentration points is in conflict.

The situation, in which a $(\pi_1, \pi_2) \in \Pi^{(2)}$ is max inconsistent implies, generically, that elements having high probabilities in space (Z, π_1) have small probabilities in space (Z, π_1) and vice versa, which is an indication of a contradiction between π_1 and π_2 (see an example below).

Example 2 Let $Z = \{z_1, z_2\}$, $\pi_1(z_1) = 3/4$, $\pi_1(z_2) = 1/4$, $\pi_2(z_1) = 1/4$, $\pi_2(z_2) = 3/4$. Then $(\pi_1 \cdot \pi_2)(z_1) = (\pi_1 \cdot \pi_2)(z_2) = 1/2 < 3/4 = \max_{z \in Z} \pi_1(z) = \max_{z \in Z} \pi_2(z)$.

3.3 Marginal measure

For every non-concentrated $\pi \in \Pi$ let us introduce the set of all elements of Z whose probabilities in (Z, π) are less than one and bigger than zero, $Z_1^+(\pi) = \{z \in Z : 0 < \pi(z) < 1\}$, and define the *marginal measure* of π to be $\min_{z \in Z_1^+(\pi)} \pi(z)$.

Clearly, $\min_{z \in Z_1^+(\pi)} \pi(z) \leq 1/|Z|$ for every non-concentrated $\pi \in \Pi$, and $\min_{z \in Z_1^+(\pi)} \pi(z) = 1/|Z|$ if and only if π is the uniform probability, $\pi = \bar{\pi}$. In this context, the smaller is the marginal measure of a non-concentrated π, the 'less uniform' is π. Since total uniformity is interpreted as total non-informativeness, the probabilities whose marginal measures are small can be viewed as more informative than those whose marginal measures are high.

We shall say that a $(\pi_1, \pi_2) \in \Pi^{(2)}$ is *marginally consistent* if

$$\min_{z' \in Z_1^+(\pi_1 \cdot \pi_2)} (\pi_1 \cdot \pi_2)(z') < \min \left\{ \min_{z' \in Z_1^+(\pi_1)} \pi_1(z'), \min_{z' \in Z_1^+(\pi_2)} \pi_2(z') \right\}$$

and is *marginally inconsistent* if

$$\min_{z' \in Z_1^+(\pi_1 \cdot \pi_2)} (\pi_1 \cdot \pi_2)(z') > \max \left\{ \min_{z' \in Z_1^+(\pi_1)} \pi_1(z'), \min_{z' \in Z_1^+(\pi_2)} \pi_2(z') \right\}.$$

Remark 15 Statement 3 of Theorem 1 implies that if a $(\pi_1, \pi_2) \in \Pi^{(2)}$ is marginally consistent or marginally inconsistent, then neither π_1, nor π_2 is uniform.

Remark 16 It is easy to show that the marginal measure is a concave function on Π.

Lemma 2 allows us to characterize a typical situation, in which a $(\pi_1, \pi_2) \in \Pi^{(2)}$ is marginally consistent; generically, this takes place if the sets of the least probable elements in the probability spaces (Z, π_1) and (Z, π_2) intersect.

Theorem 8 *Let $(\pi_1, \pi_2) \in \Pi^{(2)}$, $\pi_1, \pi_2 \in \Pi$ be non-concentrated and there exist a $z_* \in Z$ minimizing π_1 on $Z_1^+(\pi_1)$ and π_2 on $Z_1^+(\pi_2)$ simultaneously. The following statements are true.*

1) z_* minimizes $\pi_1 \cdot \pi_2$ on $Z_1^+(\pi_1 \cdot \pi_2)$.

2) If there exists a $z \in Z$ such that

$$\pi_1(z) > \min_{z' \in Z_1^+(\pi_1)} \pi_1(z'), \quad \pi_2(z) > \min_{z' \in Z_1^+(\pi_2)} \pi_2(z'),$$

then (π_1, π_2) is marginally consistent.

A proof is similar to that of Theorem 6. A straightforward corollary from Theorem 8 is the following.

Corollary 7 *For every non-concentrated and non-uniform $\pi \in \Pi$, (π, π) is marginally consistent.*

Extending the definition of marginal consistency of pairs $(\pi_1, \pi_2) \in \Pi^{(2)}$ to n-tuples $(\pi_1, \dots, \pi_n) \in \Pi^{(n)}$ for $n \geq 2$, we will say that a $(\pi_1, \dots, \pi_n) \in \Pi^{(n)}$ is *marginally consistent* if

$$\min_{z \in Z} (\pi_1 \cdot \ldots \cdot \pi_n)(z) < \min\{\min_{z \in Z} \pi_1(z), \dots \min_{z \in Z} \pi_1(z)\}.$$

Theorem 8 yields the following.

Corollary 8 *Let a $n \geq 2$ be natural, a $(\pi_1, \dots, \pi_n) \in \Pi^{(n)}$ be non-concentrated and there exist a $z_* \in Z$ minimizing $\pi_1, \dots, \pi_n$ on Z simultaneously. The following statements are true.*

1) z_* minimizes $\pi_1 \cdot \ldots \cdot \pi_n$ on Z.

2) If there exists a $z \in Z$ such that

$$\pi_i(z) > \max_{z' \in Z} \pi_i(z')$$

for all $i \in \{1, \dots, n\}$ then $(\pi_1, \dots, \pi_n)$ is marginally consistent.

The situation, in which a $(\pi_1, \pi_2) \in \Pi^{(2)}$ is marginally inconsistent, is, essentially, similar to the situation, in which (π_1, π_2) is max inconsistent; both situations imply, generically, that elements having high likelihoods in space (Z, π_1) have small likelihoods in space (Z, π_1) and vice versa. Example 2 given in the previous section is an elementary illustration.

3.4 Max raisers

Given a natural $n \geq 2$, a $\pi \in \Pi$ will be said to be a *max raiser* for a $(\pi_1, \dots, \pi_n) \in \Pi^n$ if $(\pi, \pi_i) \in \Pi^{(2)}$ and

$$\max_{z \in Z} (\pi \cdot \pi_i)(z) > \max_{z \in Z} \pi_i(z)$$

for all $i \in \{1, \dots, n\}$. The observation method lying behind a max raiser for $(\pi_1, \dots, \pi_n)$ improves, through integration, each of the observation methods lying behind $\pi_1, \dots, \pi_n$.

At first, we give a note on max raisers for pairwise max consistent n-tuples of probabilities.

Theorem 9 *Let an $n \geq 2$ be natural, a $(\pi_1, \dots, \pi_n) \in \Pi^n$ be such that (π_i, π_j) belong to $\Pi^{(2)}$ and be max consistent for every different $i, j \in \{1, \dots, n\}$. Then for every $i \in \{1, \dots, n\}$, π_i is a max raiser for $(\pi_1, \dots, \pi_n)$.*

Proof. Let $i, j \in \{1, \dots, n\}$ and $j \neq i$. We have $(\pi_i, \pi_i) \in \Pi^{(2)}$ by Remark 1, (v), and we have $(\pi_i, \pi_j) \in \Pi^{(2)}$ by assumption. The max consistency of (π_i, π_j) implies that π_i is non-uniform (see Remark 10); hence, (π_i, π_i) is max consistent by Corollary 4; therefore,

$$\max_{z \in Z} (\pi_i \cdot \pi_i)(z) > \max_{z \in Z} \pi_i(z).$$

Furthermore, (π_i, π_j) is max consistent by assumption, implying

$$\max_{z \in Z} (\pi_i \cdot \pi_j)(z) > \max_{z \in Z} \pi_j(z).$$

In view of the arbitrary choice of $j \neq i$, π_i is a max raiser for $(\pi_1, \dots, \pi_n)$. The proof is complete.

The next theorem shows that, generically, a probability giving a sufficiently high priority to an element having non-zero likelihoods in spaces $(Z, \pi_1), \dots, (Z, \pi_n)$ is a max raiser for $(\pi_1, \dots, \pi_n)$.

Theorem 10 *Let an $n \geq 2$ be natural, $(\pi_1, \dots, \pi_n) \in \Pi^n$, $\pi_1, \dots, \pi_n$ be non-concentrated and a $z_* \in Z$ be such that $\pi_i(z_*) > 0$ for all $i \in \{1, \dots, n\}$. Then every $\pi \in \Pi$ such that $\pi(z_*)$ is sufficiently close to one is a max raiser for $(\pi_1, \dots, \pi_n)$.*

Proof. Let a $\pi_* \in \Pi$ be concentrated at z_*. Clearly, $(\pi_*, \pi_i) \in \Pi^{(2)}$ for every $i \in \{1, \dots, n\}$. As follows from statement 4 of Theorem 1, for every $i \in \{1, \dots, n\}$ it holds that $\pi_* \cdot \pi_i = \pi_*$, implying

$$\max_{z \in Z} (\pi_* \cdot \pi_i)(z) = 1 > \max_{z \in Z} \pi_i(z);$$

the latter inequality is ensured by the fact that π_i is non-concentrated. Thanks to the continuity of integration (see Remark 4) and the continuity of the max concentration measure, the above inequality is preserved for all $i \in \{1, \dots, n\}$ if one replaces π_* with an arbitrary $\pi \in \Pi$ such that $\pi(z_*)$ is sufficiently close to one. The proof is complete.

The next theorem deals with probabilities sufficiently close to the uniform one.

Theorem 11 *Let an $n \geq 2$ be natural and a $\pi \in \Pi$ be non-uniform. Then π is a max raiser for any $(\pi_1, \dots, \pi_n) \in \Pi^n$ such that $\pi_1, \dots, \pi_n$ are sufficiently close to the uniform probability, $\bar{\pi}$.*

Proof. By Theorem 1 (see statement 3) we have $\pi \cdot \bar{\pi} = \pi$. Hence,

$$\max_{z \in Z} (\pi \cdot \bar{\pi}) = \max_{z \in Z} \pi(z) > \max_{z \in Z} \bar{\pi}(z) = 1/|Z|;$$

the latter inequality is ensured by the fact that π is not uniform. Thanks to the continuity of integration and continuity of the max concentration measure, the above inequality is preserved if one replaces $\bar{\pi}$ with arbitrary $\pi_1, \dots, \pi_2 \in \Pi$ sufficiently close to $\bar{\pi}$. The proof is complete.

Let us show that if $n < |Z|$, then, generically, for a given $(\pi_1, \dots, \pi_n) \in \Pi^n$ one can find a max raiser sufficiently close to the uniform probability; an interpretation is that n given observation methods can simultaneously be improved, being integrated with some observation method whose information value is poor.

Theorem 12 *Let $N = |Z|$, a natural n satisfy $2 \leq n < N$, $\pi_1, \ldots, \pi_n \in \Pi^n$, $Z = \{z_1, \ldots, z_N\}$, for every $i \in \{1, \ldots, n\}$ the maximizer of π_i in Z, z_{k_i}, is unique,*

$$v_{ik} = \begin{cases} 0 & \textit{if} \quad k \neq k_i \\ 1 & \textit{if} \quad k = k_i \end{cases} \qquad (i \in \{1, \ldots, n\},\ k \in \{1, \ldots, N\}),$$

and the rank of the matrix

$$A = \begin{pmatrix} \pi_1(z_1) - v_{11} & \pi_1(z_2) - v_{12} & \ldots & \pi_1(z_N) - v_{1N} \\ & & \ldots & \\ \pi_n(z_1) - v_{n1} & \pi_n(z_2) - v_{n2} & \ldots & \pi_n(z_N) - v_{nN} \\ 1 & 1 & \ldots & 1 \end{pmatrix} \tag{28}$$

is not smaller than $n+1$. Then for every $\varepsilon > 0$ there exists a max rasier π for $\pi_1, \ldots, \pi_n$ such that the distance in Π between π and the uniform probability, $\bar{\pi}$, is smaller than ε.

Proof. Let $\pi^* = (\pi(z_1), \ldots, \pi(z_N)) \in R^N$ for every $\pi \in \Pi$. For every $h = (h_1, \ldots, n_N) \in R^N$ such that

$$h_1 + \ldots + h_N = 0 \tag{29}$$

and every sufficiently small real λ we have

$$\bar{\pi}^* + \lambda h \in \Pi^* = \{\pi^* : \pi \in \Pi\}. \tag{30}$$

Let for every $p = (p_1, \ldots, p_N) \in R^N$

$$g_{ik}(p) = \frac{\pi_i(z_k) p_k}{\sum_{j=1}^N \pi_i(z_j) p_j} \quad (i \in \{1, \ldots, n\},\ k \in \{1, \ldots, N\}). \tag{31}$$

Obviously,

$$g_{ik}(\pi^*) = (\pi_i \cdot \pi)(z_k) \quad (i \in \{1, \ldots, n\},\ k \in \{1, \ldots, N\}) \tag{32}$$

for every $\pi \in \Pi$; in particular,

$$g_{ik}(\bar{\pi}^*) = (\bar{\pi}_i \cdot \pi)(z_k) = \pi(z_k) \quad (i \in \{1, \ldots, n\},\ k \in \{1, \ldots, N\}). \tag{33}$$

In view of (32) and (30) holding for all $h \in R^N$ satisfying (29) and for all sufficiently small real λ, we see that in order to finalize the proof it is sufficient to show that there exists an $h \in R^N$ such that for all sufficiently small $\lambda > 0$ we have

$$\max_{k=1,\ldots,N} g_{ik}(\bar{\pi}^* + \lambda h) > \max_{k=1,\ldots,N} \pi_i(z_k) \quad (i \in \{1, \ldots, n\})$$

or, equivalently,

$$g_{ik_i}(\bar{\pi}^* + \lambda h) > \pi_i(z_{ik_i}) \quad (i \in \{1, \ldots, n\}) \tag{34}$$

where $z_{ik_i} \in Z$ is the maximizer of π_i in Z (here we refer to the assumption that the latter maximizer is unique). Given an $h \in R^N$, for all sufficiently small $\lambda > 0$, (34) is equivalent to

$$g_{ik_i}(\bar{\pi}^*) + \langle \text{grad } g_{ik_i}(\bar{\pi}^*), h \rangle \lambda > \pi_i(z_{ik_i}) \quad (i \in \{1, \ldots, n\})$$

and, thanks to (33), to

$$\langle \text{grad } g_{ik_i}(\bar{\pi}^*), h \rangle > 0 \quad (i \in \{1, \ldots, n\}); \tag{35}$$

here grad $g_{ik}(\bar{\pi}^*)$ is the gradient of $p \mapsto g_{ik}(p)$ at $\bar{\pi}^*$ and $\langle \cdot, \cdot \rangle$ is the scalar product in R^N.

We will complete the proof by showing that there exists an $h \in R^N$ satisfying (29), for which (35) holds. Referring to (31), we find that

$$\text{grad } g_{ik_i}(\bar{p}) = (\gamma_i^{(1)}, \dots, \gamma_i^{(N)})$$

where

$$\gamma_i^{(k)} = -\frac{\pi_i(z_{k_i})\pi_i(z_k)(1/N)}{\left(\sum_{j=1}^N \pi_i(z_j)(1/N)\right)^2} = -N\pi_i(z_{k_i})\pi_i(z_k) \quad \text{for} \quad k \neq k_i,$$

$$\gamma_i^{(k_i)} = \frac{\pi_i(z_{k_i}) \sum_{j=1}^N \pi_i(z_j)(1/N) - \pi_i(z)\pi_i(z_k)(1/N)}{\left(\sum_{j=1}^N \pi_i(z_j)(1/N)\right)^2} = N[\pi_i(z_{k_i}) - \pi_i^2(z_{k_i})];$$

here we take into account that $\sum_{j=1}^N \pi_i(z_j) = 1$. Thus, for an arbitrary $h = (h_1, \dots, n_N) \in R^N$ we have

$$\langle \text{grad } g_{ik_i}(\bar{\pi}^*), h \rangle = -N\pi_i(z_{k_i}) \left(\sum_{k=1}^{k_i-1} \pi_i(z_k)h_k + (\pi_i(z_{k_i}) - 1)h_{k_i} + \sum_{k=k_i+1}^{N} \pi_i(z_k)h_k \right) \tag{36}$$

$$(i \in \{1, \dots, n\}).$$

Let $a_1, \dots, a_n < 0$. Consider the following system of algebraic equations with respect to $h_1, \dots, h_N$:

$$\sum_{k=1}^{k_i-1} \pi_i(z_k)h_k + (\pi_i(z_{k_i}) - 1)h_{k_i} + \sum_{k=k_i+1}^{N} \pi_i(z_k)h_k = a_k \tag{37}$$

$$(i \in \{1, \dots, n\}),$$

$$h_1 + \ldots + h_N = 0. \tag{38}$$

Rewrite it in a matrix form as

$$Ah^T = a^T \tag{39}$$

where A is given by (28), $a = (a_1, \dots, a_n, 0)$ and T marks transposed row vectors. By assumption the rank of A is not smaller than $n+1$ – the number of its rows (the number of equations in (37), (38)), the latter number is not smaller than N – the number of its columns (the number of the unknown values in (37), (38)), and $N \geq n+1$. Therefore, the system of equations (37), (38) (equation (39)) is solvable. Let $h = (h_1, \dots, h_N)$ be a solution to (37), (38). Since $a_1, \dots, a_n < 0$, the right hand sides in (36) are positive; consequently, inequalities (35) hold. The proof is complete.

4 Integrable sequences of probabilities

4.1 Integral limits

We call a sequence $(\pi_i)_{i=1}^\infty$ in Π *integrable* if $(\pi_1, \dots, \pi_n)$ lies in $\Pi^{(n)}$ (is not in conflict) for every natural n; note that for an integrable sequence $(\pi_i)_{i=1}^\infty$ the integration result $\pi_1 \cdot \ldots \cdot \pi_n$ is defined correctly for every natural n.

Given an integrable sequence $(\pi_i)_{i=1}^\infty$ in Π, we call every partial limit of sequence $(\pi_1 \cdot \ldots \cdot \pi_n)_{n=1}^\infty$ in Π a *partial integral limit* of $(\pi_i)_{i=1}^\infty$; in case the partial integral limit of $(\pi_i)_{i=1}^\infty$ is unique, we call the latter the *integral limit* of $(\pi_i)_{i=1}^\infty$.

Remark 17 As mentioned in Section 1, Π is a compactum. Therefore, every integrable sequence in Π has a partial limit.

An integrable sequence in Π will be said to be *integrally convergent* if its integral limit exists, and *integrally divergent* otherwise. An integrally convergent sequence in Π will be said to be *integrally concentrated* if its integral limit is concentrated.

Interpretations of the above definitions are straightforward. If a sequence $(\pi_i)_{i=1}^{\infty}$ is integrally concentrated, observation methods $1, 2\ldots$ lying, respectively, behind probabilities $\pi_1, \pi_2, \ldots$ improve each other, through integration, to a degree, at which we get complete information on the observed element in Z. If $(\pi_i)_{i=1}^{\infty}$ is integrally convergent and not integrally concentrated, methods $1, 2\ldots$, being integrated, 'find a compromise' and provide a definite though incomplete information on the observed element. If $(\pi_i)_{i=1}^{\infty}$ is integrally divergent, methods $1, 2\ldots$ disagree.

Let us give an example of an integrally divergent sequence in Π.

Example 3 (an integrally divergent sequence). Let $Z = \{z_1, z_2\}$, $\pi^{(1)}, \pi^{(2)} \in \Pi$ be such that

$$\pi^{(1)}(z_1) > \pi^{(1)}(z_2) > 0, \quad \pi^{(2)}(z_2) > \pi^{(2)}(z_1) > 0$$

and $(\pi_i)_{i=1}^{\infty}$ be defined by

$$\begin{aligned}
\pi_i &= \pi^{(1)} \quad (i \in \{1, \ldots, k_{2^j-1}\},\ j \in \{1, 2, \ldots\}),\\
\pi_i &= \pi^{(2)} \quad (i \in \{k_{2^j-1}+1, \ldots, k_{2^j}\},\ j \in \{1, 2, \ldots\})
\end{aligned}$$

where $1 < k_1 < k_2 < k_3 \ldots$. It is clear that $(\pi_i)_{i=1}^{\infty}$ is integrable. Let

$$\pi_j^* = \pi_1 \cdot \ldots \cdot \pi_{k_j}, \quad \pi_{j,j+1}^* = \pi_{k_j+1} \cdot \ldots \cdot \pi_{k_{j+1}} \quad (j = 1, 2, \ldots), \tag{40}$$

$$q^{(1)} = \frac{\pi^{(1)}(z_2)}{\pi^{(1)}(z_1)}, \quad q^{(2)} = \frac{\pi^{(2)}(z_2)}{\pi^{(2)}(z_1)}.$$

Obviously,

$$q^{(1)} < 1, \quad q^{(2)} > 1.$$

We have

$$\pi_1^*(z_1) = \frac{\pi^{(1)k_1}(z_1)}{\pi^{(1)k_1}(z_1) + \pi^{(1)k_1}(z_2)} = \frac{1}{1 + q^{(1)k_1}}.$$

Take a positive sequence $(\varepsilon_j)_{j=1}^{\infty}$ convergent to zero. Set k_1 to be so large that $q^{(1)k_1} < \varepsilon_1$. Then

$$\pi_1^*(z_1) > \frac{1}{1+\varepsilon_1}. \tag{41}$$

Furthermore,

$$\pi_{1,2}^*(z_1) = \frac{\pi^{(2)(k_2-k_1)}(z_1)}{\pi^{(2)(k_2-k_1)}(z_1) + \pi^{(2)(k_2-k_1)}(z_2)} = \frac{1}{1 + q^{(2)(k_2-k_1)}} \tag{42}$$

and

$$\begin{aligned}
\pi_2^*(z_1) &= (\pi_1^* \cdot \pi_{1,2}^*)(z_1)\\
&= \frac{\pi_1^*(z_1)\pi_{1,2}^*(z_1)}{\pi_1^*(z_1)\pi_{1,2}^*(z_1) + \pi_1^*(z_2)\pi_{1,2}^*(z_2)}
\end{aligned}$$

$$
\begin{aligned}
&< \frac{\pi_1^*(z_1)\pi_{1,2}^*(z_1)}{\pi_1^*(z_2)\pi_{1,2}^*(z_2)} \\
&= \frac{\pi_1^*(z_1)}{\pi_1^*(z_2)}\frac{\pi_{1,2}^*(z_1)}{1-\pi_{1,2}^*(z_1)}
\end{aligned} \tag{43}
$$

Based on (42) and on the fact that $q^{(2)} > 1$, we set $k_2 > k_1$ to be so large that the right hand side in (43) is smaller than ε_2. Then

$$
\pi_2^*(z_1) < \varepsilon_2. \tag{44}
$$

Arguments similar to those used to state (41) and (44) lead us to choices of $k_3, k_4, \ldots$ such that

$$
\pi_{2j-1}^*(z_1) > \frac{1}{1+\varepsilon_{2j-1}}, \quad \pi_{2j}^*(z_1) < \varepsilon_{2j} \quad (j \in \{1,2,\ldots\}).
$$

Therefore, $\lim_{j\to\infty}\pi_{2j-1}^*$ is concentrated at z_1 and $\lim_{j\to\infty}\pi_{2j}^*$ is concentrated at z_2. By (40) both limits are partial integral limits of $(\pi_i)_{i=1}^\infty$. Consequently, $(\pi_i)_{i=1}^\infty$ is integrally divergent.

Theorems 13, 14 and 15 given below provide simple conditions ensuring a sequence of probabilities to be integrally concentrated. Theorem 13 follows straightforwardly from the fact that concentrated probabilities are zeros with respect to integration understood as multiplication (see Theorem 1, statement 4).

Theorem 13 *Let $(\pi_i)_{i=1}^\infty$ be an integrable sequence in Π and there be a natural k such that π_k is concentrated. Then $(\pi_i)_{i=1}^\infty$ is integrally concentrated and its integral limit is π_k.*

The next theorem following from the definition of a measure of concentration provides a general criterion for an integrable sequence to be integrally concentrated.

Theorem 14 *Let μ be a measure of concentration. An integrable sequence $(\pi_i)_{i=1}^\infty$ in Π is integrally concentrated if and only if $\lim_{n\to\infty}\mu(\pi_1\cdot\ldots\cdot\pi_n) = 1$.*

The next theorem states that if all the probabilities in an integrable sequence give visibly highest values to a same element, z_*, the sequence is integrally concentrated at z_*. An interpretation is that observation methods that unambiguously agree on prioritizing a certain element enhance their common probabilistic priority through integration so that the common priority turns into certainty.

Theorem 15 *Let a sequence $(\pi_i)_{i=1}^\infty$ in Π be integrable and there be a $z_* \in Z$ and a positive $q < 1$ such that $\pi_i(z)/\pi_i(z_*) < q$ for all $z \in Z\setminus\{z_*\}$ and all natural i. Then $(\pi_i)_{i=1}^\infty$ is integrally concentrated and its integral limit is concentrated at z_*.*

Proof. For every $z \in Z$ and every natural n let

$$
v_n(z) = \frac{(\pi_1\cdot\ldots\cdot\pi_n)(z)}{(\pi_1\cdot\ldots\cdot\pi_n)(z_*)} = \frac{\pi_1(z)\ldots\pi_n(z)}{\pi_1(z_*)\ldots\pi_n(z_*)}. \tag{45}
$$

Obviously, $v_n(z) \le q^n$ for every $z \in Z\setminus\{z_*\}$ $(n \in \{1,2,\ldots\}$. Therefore, $(\pi_1\cdot\ldots\cdot\pi_n)(z) \to 0$ as $n\to\infty$. Consequently, $(\pi_1\cdot\ldots\cdot\pi_n)_{n=1}^\infty$ converges in Π to the probability whose concentration point is z_*. The proof is complete.

The next example shows that if in assumptions of Theorem 15 we set $q = 1$ (implying that the observation methods lying behind $\pi_1, \pi_2, \ldots$ may have no clear consensus in prioritizing z_*), the statement of the theorem is no longer valid.

Example 4 Let $Z = \{z, z_*\}$, $\pi_1, \pi_2, \ldots \in \Pi$ and $q_i = \pi_i(z)/\pi_i(z_*) \in (0,1)$ for all natural i. Furtheremore, let $q_{i+1} > q_i$ for all natural i, and the series $\sum_{i=1}^{\infty} |\log q_i|$ converge. Then sequence $(v_n)_{n=1}^{\infty}$ given by (45) takes values in $(0,1)$ and decreases; consequently, $v_n \to v \in [0,1)$ as $n \to \infty$. Therefore, $(\pi_i)_{i=1}^{\infty}$ is integrable and integrally convergent. Since $\sum_{i=1}^{\infty} |\log q_i|$ converges, $v = \lim_{n\to\infty} q_1 \ldots q_n > 0$. Hence, the integral limit of $(\pi_i)_{i=1}^{\infty}$ is not concentrated.

Let us give a few conditions sufficient for an integrable sequence of probabilities to be integrally convergent. The next statement follows straightforwardly from definitions and the continuity of integration (see Remark 4).

Theorem 16 *Let $(\pi_i)_{i=1}^{\infty}$ be an integrable sequence in Π, there be a natural $k \geq 2$ such that $(\pi_i)_{i=k}^{\infty}$ is integrally convergent and π be its integral limit. Then $(\pi_i)_{i=1}^{\infty}$ is integrally convergent and its integral limit is $\pi_1 \cdot \ldots \cdot \pi_{k-1}\pi$.*

The next theorem generalizes Theorem 15.

Theorem 17 *Let a sequence $(\pi_i)_{i=1}^{\infty}$ in Π be integrable, all π_i ($i \in \{1, 2, \ldots\}$) have the common set of maximizers, Z_*, and there be a positive $q < 1$ such that $\pi_i(z)/\pi_i(z_*) < q$ for all $z \in Z \setminus Z_*$, all $z_* \in Z_*$ and all natural i. Then $(\pi_i)_{i=1}^{\infty}$ is integrally convergent and its integral limit, π, is uniform on Z_*, i.e., $\pi(z) = 1/|Z_*|$ for all $z_* \in Z_*$ and $\pi(z) = 0$ for all $z \in Z \setminus Z_*$.*

We omit the proof (similar to that of Theorem 15). The next theorem states that a sequence of probabilities is integrally convergent if all the probabilities in the sequence prioritize the elements in the same order.

Theorem 18 *Let a sequence $(\pi_i)_{i=1}^{\infty}$ in Π be integrable and there be a sequence $(z_k)_{k=1}^{N}$ in Z such that $\{z_1, \ldots, z_N\} = Z$ and $\pi_i(z_1) \leq \ldots \leq \pi_i(z_N)$ for all natural i. The following statements hold true.*

1) $(\pi_i)_{i=1}^{\infty}$ is integrally convergent and for its integral limit, π, it holds that $\pi(z_1) \leq \ldots \leq \pi(z_N)$.

2) If there exist a positive $q < 1$, an $l \in \{1, \ldots, N-1\}$ and a subsequence $(\pi_{i_m})_{m=1}^{\infty}$ of $(\pi_i)_{i=1}^{\infty}$ such that $\pi_{i_m}(z_l)/\pi_{i_m}(z_N) < q$ for all natural m, then $\pi(z_1) = \ldots = \pi(z_l) = 0$.

Proof. Prove statement 1. Let $j \in \{1, \ldots, N\}$ be the minimum of all $k \in \{1, \ldots, N\}$ such that $\pi_i(z_k) > 0$ for all natural i. Obviously, for every natural $k < j$ we have $(\pi_1 \cdot \ldots \cdot \pi_n)(z_k) = 0$ for all sufficiently large n. If $j = N$, we get $(\pi_1 \cdot \ldots \cdot \pi_n)(z_k) = 1$ for all sufficiently large n, which completes the proof. Let $j \leq N-1$. For every $k \in \{j, \ldots, N-1\}$ and every natural n let

$$v_{kn} = \frac{(\pi_1 \cdot \ldots \cdot \pi_n)(z_k)}{(\pi_1 \cdot \ldots \cdot \pi_n)(z_{k+1})} = \frac{\pi_1(z_k) \ldots \pi_n(z_k)}{\pi_1(z_{k+1}) \ldots \pi_n(z_{k+1})}.$$

By assumption for every $k \in \{j, \ldots, N-1\}$ $v_{k1} \leq 1$ and sequence $(v_{kn})_{n=1}^{\infty}$ is non-increasing; hence, $v_{kn} \leq 1$ for all natural n. Consequently,

$$(\pi_1 \cdot \ldots \cdot \pi_n)(z_k) \leq (\pi_1 \cdot \ldots \cdot \pi_n)(z_{k+1})$$

for all $k \in \{j, \ldots, N-1\}$ and all natural n. To complete the proof, it is sufficient to show that sequence $((\pi_1 \cdot \ldots \cdot \pi_n)(z_k))_{n=1}^{\infty}$ converges for every $k \in \{j, \ldots, N-1\}$. Let us note here that for every $k \in \{j, \ldots, N-1\}$ the non-increasing sequence $(v_{kn})_{n=1}^{\infty}$ converges.

Let us prove that sequence $((\pi_1 \cdot \ldots \cdot \pi_n)(z_N))_{n=1}^{\infty}$ is non-decreasing; here we slightly modify the argument used in the proof of Lemma 1. By definition

$$(\pi_1 \cdot \pi_2)(z_N) = \frac{\pi_1(z_N)\pi_2(z_N)}{\sum_{k=1}^{N} \pi_1(z_k)\pi_2(z_k)} = \frac{\pi_1(z_N)}{\sum_{k=1}^{N-1} \pi_1(z_k)q(z_k) + \pi_1(z_N)}$$

where $q(z_k) = \pi_2(z_k)/\pi_2(z_N)$ for $k \in \{1, \ldots, N-1\}$. Since $q(z_k) \leq 1$ for $k \in \{1, \ldots, N-1\}$, we get $(\pi_1 \cdot \pi_2)(z_N) \geq \pi_1(z_N)$. Similarly, using induction, we state that $(\pi_1 \cdot \ldots \cdot \pi_{n+1})(z_N) \geq (\pi_1 \cdot \ldots \cdot \pi_n)(z_N)$ for an arbitrary natural n. Thus, sequence $((\pi_1 \cdot \ldots \cdot \pi_n)(z_N))_{n=1}^{\infty}$ is non-decreasing; therefore, it converges. Furthermore,

$$(\pi_1 \cdot \ldots \cdot \pi_n)(z_{N-1}) = v_{N-1\ n}(\pi_1 \cdot \ldots \cdot \pi_n)(z_N)$$

for all natural n. Since both $(v_{N-1\ n})_{n=1}^{\infty}$ and $((\pi_1 \cdot \ldots \cdot \pi_n)(z_N))_{n=1}^{\infty}$ converge, $((\pi_1 \cdot \ldots \cdot \pi_n)(z_{N-1}))_{n=1}^{\infty}$ does, too. Similarly, using induction, we state that $((\pi_1 \cdot \ldots \cdot \pi_n)(z_k))_{n=1}^{\infty}$ converges for every $k \in \{j, \ldots, N-1\}$. Statement 1 is proved.

Prove statement 2. Let the assumptions of statement 2 be satisfied. Then for all sufficiently large natural n it holds that

$$w_n = \frac{(\pi_1 \cdot \ldots \cdot \pi_n)(z_l)}{(\pi_1 \cdot \ldots \cdot \pi_n)(z_N)} = \frac{\pi_1(z_l) \ldots \pi_n(z_l)}{\pi_1(z_N) \ldots \pi_n(z_N)} \leq q^{s(n)}$$

where $s(n)$ is the maximum of all i_m ($m \in \{1, 2, \ldots\}$) such that $i_m \leq n$. Since $s(n) \to \infty$ as $n \to \infty$, we have $w_n \to 0$ as $n \to \infty$. Consequently, $\pi(z_l) = \lim_{n\to\infty}(\pi_1 \cdot \ldots \cdot \pi_n)(z_l) = 0$. Since $\pi(z_1) \leq \ldots \leq \pi(z_l)$, we get $\pi(z_1) = \ldots = \pi(z_l) = 0$. Statement 2 is proved.

Taking into account Theorem 16, we immediately deduce from Theorem 18 that for the integral convergence of a sequence of probabilities it is sufficient that the absolute majority of those prioritize the elements in the same order.

Corollary 9 *Let a sequence $(\pi_i)_{i=1}^{\infty}$ in Π be integrable and there be a sequence $(z_k)_{k=1}^{N}$ in Z such that $\{z_1, \ldots, z_N\} = Z$ and $\pi_i(z_1) \leq \ldots \leq \pi_i(z_N)$ for all sufficiently large natural i. Then $(\pi_i)_{i=1}^{\infty}$ is integrally convergent and statement 2 of Theorem 18 holds true.*

4.2 Permutations in infinite integration

Here we show that finite permutations in integrable sequences of probabilities do not change the results of integration (Corollary 10), whereas infinite ones can possess the opposite property (Example 5).

As noted in Corollary 1, the result of integration of a finite number of probabilities is insensitive to the order, in which these probabilities are integrated. A straightforward implication is that any finite permutation in a sequence of probabilities does not change the set of its partial integral limits.

Corollary 10 *Let $(\pi_i)_{i=1}^{\infty}$ be an integrable sequence in Π, $(i_1, i_2, \ldots, i_k)$ be an arbitrary permutation in $(1, 2, \ldots, k)$ for some natural k and*

$$\pi_j^* = \begin{cases} \pi_{i_j} & \text{if } \ j \in \{1, \ldots, k\} \\ \pi_j & \text{if } \ j \in \{k+1, k+2, \ldots\} \end{cases}.$$

Then the sets of all integral partial limits of $(\pi_i)_{i=1}^{\infty}$ and $(\pi_i^*)_{i=1}^{\infty}$ coincide.

The next example shows that generally, the above statement does not hold for infinite permutations in sequences of probabilities.

Example 5 Let $Z = \{z_1, z_2\}$ and $\pi_i^{(1)}, \pi_i^{(2)} \in \Pi$ ($i \in \{1, 2, \ldots\}$) be such that

$$\pi_i^{(1)}(z_1) = 1 - \varepsilon_i, \quad \pi_i^{(1)}(z_2) = \varepsilon_i, \quad \pi_i^{(2)}(z_1) = \varepsilon_i, \quad \pi_i^{(2)}(z_2) = 1 - \varepsilon_i$$

where $\varepsilon_i \in (0, 1)$ and $\lim_{i\to\infty} \varepsilon_i = 0$. For every natural i and $j \geq i$ denote

$$\pi_{ij}^{(1)} = \pi_i^{(1)} \cdot \ldots \cdot \pi_j^{(1)}, \quad \pi_{ij}^{(2)} = \pi_i^{(2)} \cdot \ldots \cdot \pi_j^{(2)}.$$

By Theorem 15

$$\pi_{ij}^{(1)} \to \pi^{(1)} \text{ in } \Pi, \quad \pi_{ij}^{(2)} \to \pi^{(2)} \text{ in } \Pi \tag{46}$$

where $\pi^{(1)}$ and $\pi^{(2)}$ are concentrated at z_1 and z_2, respectively. Take natural $k_1, k_2 \ldots$ and let

$$\begin{aligned}
\pi'_j &= \pi_j^{(1)} \quad (j \in \{1, \ldots, k_1\}),\\
\pi'_{k_1+1} &= \pi_1^{(2)},\\
\pi'_j &= \pi_j^{(1)} \quad (j \in \{k_1 + 2, \ldots, k_1 + 2 + k_2\}),\\
\pi'_{k_1+k_2+3} &= \pi_2^{(2)},\\
&\ldots\\
\pi'_j &= \pi_j^{(1)} \quad (j \in \{m_s, \ldots, m_{s+1}\}),\\
\pi'_{m_{s+1}+1} &= \pi_{s+1}^{(2)},\\
&\ldots
\end{aligned}$$

where

$$m_s = \sum_{l=1}^{s} k_l + s + 1.$$

Obviously, $(\pi'_j)_{j=1}^{\infty}$ is integrable. Take positive $\delta_1, \delta_2, \ldots$ such that $\lim_{s\to\infty} \delta_s = 0$. Based on (46), the fact that $\pi^{(1)}$ is a zero with respect to integration (understood as multiplication) and the continuity of integration, set, sequentially, $k_1, k_2, \ldots$ to be so large that

$$\begin{aligned}
(\pi'_1 \cdot \ldots \cdot \pi'_{m_{s+1}+1})(z_1) &= (\pi'_1 \cdot \ldots \cdot \pi'_{m_s+1} \cdot \pi^{(1)}_{m_s+1\ m_{s+1}} \cdot \pi^{(2)}_{m_{s+1}+1})(z_1)\\
&= (\pi^{(1)}_{m_s+1\ m_{s+1}} \cdot (\pi'_1 \cdot \ldots \cdot \pi'_{m_s+1} \cdot \pi^{(2)}_{m_{s+1}+1}))(z_1)\\
&> 1 - \delta_s.
\end{aligned}$$

Then, using Theorem 6, for every $k \in \{m_{s+1} + 2, \ldots, m_{s+3}\} = \{m_{s+2}, \ldots, m_{s+3}\}$ we get

$$\begin{aligned}
(\pi'_1 \cdot \ldots \cdot \pi'_k)(z_1) &= ((\pi'_1 \cdot \ldots \cdot \pi'_{m_{s+1}+1}) \cdot \pi'_{m_{s+2}} \cdot \ldots \cdot \pi'_k)(z_1)\\
&= ((\pi'_1 \cdot \ldots \cdot \pi'_{m_{s+1}+1}) \cdot \pi^{(1)}_{m_{s+2}} \cdot \ldots \cdot \pi^{(1)}_k)(z_1)\\
&> (\pi'_1 \cdot \ldots \cdot \pi'_{m_{s+1}+1})(z_1)\\
&> 1 - \delta_s.
\end{aligned}$$

Thus, $(\pi'_j)_{j=1}^{\infty}$ integrally converges to $\pi^{(1)}$. Now define $\pi''_1, \pi''_2, \ldots \in \Pi$ by changing the places of $\pi_j^{(1)}$ and $\pi_j^{(2)}$ in the definition of $\pi'_1, \pi'_2, \ldots$. Using a similar argument, we state that $(\pi''_j)_{j=1}^{\infty}$ integrally converges to $\pi^{(2)}$. It is clear that $(\pi''_j)_{j=1}^{\infty}$ is the result of an infinite permutation in $(\pi'_j)_{j=1}^{\infty}$. Thus, an infinite permutation in $(\pi'_j)_{j=1}^{\infty}$ changes its integral limit.

Conclusion

The study presented here is motivated by problems of assessment of diverse inaccurate data characterizing uncertain complex systems. The paper suggests an approach to integration of pieces of information provided by alternative error-corrupted observation methods modeled by probability distributions. Our departure point is the posterior Bayesian probability in the product of the probability spaces associated with the observation methods.

We find that integration as a binary operation in the set of all probabilities on a given finite carrier possesses algebraic properties similar to those of multiplication; a particular implication is that the set of all probabilities providing a non-zero likelihood to all elementary events forms a topological abelian group, in which the uniform probability acts as the unit.

We pay special attention to cases, in which integration raises the information value of the integrated probabilities, imlying that the underlying observation methods enhance, via integration, each other in resolution; roughly speeking, such cases assume that the integrated probabilities give extreme likelihoods to same elements. In this analysis, the max measure of concentration plays an important role.

In the final section, we study infinite processes, in which new probabilities are sequentially integrated with the existing ones. We state, in particular, that if all sequentially integrated probabilities, $\pi_1, \pi_2, \ldots$, give an unambiguous priority to a same elementary event, z_*, the results of sequential integration raise the likelihood of z_* to one. Another observation is that if $\pi_1, \pi_2, \ldots$ prioritize the elementary events identically, the results of sequential integration sharpen prioritization and converge.

The paper presents a pilot study, dealing with the simplest objects. Questions for further exploration include: extensions to infinite sets of elementary events; listing the results of integration of standard probability distributions (an 'integration calculus'); assessment of the impact of integration on the information value of the integrated probabilities in terms of various measures of concentration; extensions to random processes; analysis of the impact of localized permutations in sequences of probabilities on their integral convergence properties; problems of optimal integration; and other.

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