Chapter XX

Measuring Systemic Risk – Structural Approaches

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Abstract

The financial crisis of 2007/08 has demonstrated that factors for financial distress of large parts of the economy depend to a large extent on the interrelations between the financial institutions. Risks threatening the financial sector can be decomposed into risks based in the individual factors for single institutions and risks which can be attributed to the financial system as a whole. This part of the risks is called systemic risk. We review several approaches for quantifying systemic risk, most of them based on structural credit modeling. In particular we present an approach which is inspired by the fact that the joint probability distributions can be represented by their individual marginals and the copula function, which represents the interrelations.

Keywords: *copula function, marginal distributions, interrelation matrix, conditional value-at-risk*

> "Systemic risks are developments that threaten the stability of the financial system as a whole and consequently the broader economy, not just that of one or two institutions."
>
> —Ben Bernanke, ex-chairman of the US Federal Reserve Bank.

The Global financial crisis of 2007/08, often considered as the worst financial crisis since the Great Depression of the 1930s, resulted in a change of paradigms in the financial and banking sector. These crisis years saw collapses of large financial institutions, bailouts of banks by governments, and declines of stock markets. Triggered by the US housing bubble, which itself was caused by giving easy access to loans for subprime borrowers, financial distress spread over the banking sector and led to failure of key businesses and to the 2008–2012 global recession. Finally this also contributed to the European sovereign-debt crisis, with lots of aftereffects in our present times.

Uncertainties about bank solvency, declines in credit availability and reduced investor confidence had an impact on global stock markets. Governments responded with fiscal measures and institutional bailouts, which in the long term resulted in extreme public debts and necessary tax increases.

This negative experience demonstrates that the economy as a whole, but especially the financial sector is subject to risks, which are grounded in the interdependencies between the different economic actors and not in the performance of individual actors. This risk is generally phrased as “systemic risk”. While aspects of systemic risk (e.g. bank run and contagion) were always an issue in discussions about the financial system, the recent crises have increased the interest in the topic, not only in academic circles, but also among regulators and central banks.

Systemic Risk: Definitions

If one aims at measuring – and in a further step managing and mitigating – systemic risk, it is important to start with a definition. However, despite the consent that systemic risk is an important topic, which is reflected by an increasing number of related papers and technical reports, there is still not a single generally accepted definition.

As a first step, one should distinguish between systemic and systematic risk. Systematic risks are aggregate (macroeconomic) risks that cannot be reduced by hedging and diversification. Systemic risk on the other hand is a different notion. It refers to the risk of breakdown or at least major dysfunction of financial markets. The Group of Ten gave the following, often cited definition, see (Group of Ten, 2001).

> *Systemic financial risk is the risk that an event will trigger a loss of economic value or confidence in, and attendant increases in uncertainly about, a substantial portion of the financial system that is serious enough to quite probably have significant adverse effects on the real economy. Systemic risk events can be sudden and unexpected, or the likelihood of their occurrence can build up through time in the absence of appropriate policy responses. The adverse real economic effects from systemic problems are generally seen as arising from disruptions to the payment system, to credit flows, and from the destruction of asset values.*

This formulation describes many aspects related to systemic risk but can hardly be called a definition in the technical sense, as it is very broad and hard to quantify. In addition, it seems to confuse cause (confidence) and consequence (breakdown).

As an alternative, Kaufmann and Scott (Kaufmann & Scott, 2003) introduced the following definition:

> *Systemic risk refers to the risk or probability of breakdowns in an entire system, as opposed to breakdowns in individual parts or components, and is evidenced by co-movements among most or all the parts.*

In similar manner, but naming the cause and again considering larger consequences, the European Central Bank (European Central Bank, 2004) defines

> *Systemic risk: the risk that the inability of one institution to meet its obligations when due will cause other institutions to be unable to meet their obligations when due. Such a failure may cause significant liquidity or credit problems and, as a result, could threaten the stability of or confidence in markets.*

All discussed definitions focus on the banking or financial system as a whole, and relate systemic risk to the interconnectedness within the system. Often they stress the risk of spillovers from the financial sector to the real economy and the associated related costs. This effect is emphasized even more after the financial crisis, where exactly this happened: Adrian and Brunnermeier (Adrian & Brunnermeier, 2009) define systemic risk as

> *The risk that institutional distress spreads widely and distorts the supply of credit and capital to the real economy.*

A similar definition can be found in (Acharya, et al., 2009).

Given the described diversity of definitions, which are similar but also different with respect to their focus, it is hard to develop universally accepted measures for systemic risk. Different definitions refer to different important nuances of systemic risk, which means that on the operational level a robust framework for monitoring and managing systemic risk should involve a variety of risk measures related to these different aspects. See Hansen (Hansen, 2012) for a deeper discussion of the basic difficulties in defining and identifying systemic risk.

In the present chapter we will focus on the first part of the definition by (Kaufmann & Scott, 2003), which summarizes the most important aspect of systematic risk in financial systems, without addressing more general economic aspects. Such an approach could be seen as "systemic risk in the narrow sense" and we state it (slightly modified) as follows:

Systemic risk is the risk of breakdowns in an entire system, as opposed to breakdowns in individual parts or components.

Three issues have to be substantiated, if one wants to apply such a definition in concrete situations.

The system: In financial applications the focus lies on parts of the financial system (like the banking system, insurance, hedge funds) or the financial system as a whole. Any analysis has to start with describing the agents (e.g. banks in the banking system) within the analyzed system. This involves their assets and liabilities and the main risk factors related to profit and loss.

For a systemic view it is important that the agents are not isolated entities at all. Systematic risk can be modeled by joint risk factors, influencing all profit and losses. Systemic risk in financial systems usually comes by mutual debt between the entities and the related leverage.

Breakdowns: In single period models breakdown is related to bankruptcy in a technical sense, i.e. that the asset value of an agent at the end of the period does not reach a certain level, e.g. is not sufficient to pay back the agents debt. A lower boundary than debt can be used to reflect the fact that confidence into a bank might fade away even before bankruptcy, which severely reduces confidence between banks. In a systemic view it is not sufficient to look at breakdowns of individual agents: relevant are events that lead to the breakdown of more than one agent.

Risk: Risk is the danger that unwanted events (here: breakdowns) may happen or that developments go in an unintended direction. Quantifiable risk is described by distributions arising from risk. For financial systems this may involve the probability of breakdowns or the distribution of payments necessary to bring back asset values to an acceptable level. Risk measures summarize favorable or unfavorable properties of such distributions.

It should be mentioned that such an approach assumes that a good distributional model for the relevant risk factors can be formulated and estimated. During this chapter we will stick to exactly this assumption. However it is clear that in practice it is often difficult to come up with good models and data availability might be severely restricted: additional risk (model risk) is related to the quality of the used models and estimations, see (Hansen, 2012) for a deeper discussion of this point.

From Structural Models to Systemic Risk

Structural models for default go back to (Merton, 2009) and build on the idea that default of a firm happens if the firm's assets are insufficient to cover contractual obligations (liabilities). Simple models (e.g. (Merton, 2009)) start by modeling a single firm in the framework of the Black-Scholes option pricing model, whereas more complex models extend the framework to multivariate formulations, usually based on correlations between the individual asset values. A famous example is Vasiceks asymptotic single factor model (see (Vasicek, 1987), (Vasicek, 1991) and (Vasicek, 2002)), which is very stylized but leads to a closed form solution.

In most structural default models it is not possible to calculate the portfolio loss explicitly; hence Monte-Carlo simulation is an important tool for default calculations. Even then the models usually make simplifying assumptions.

Consider a system consisting of k economic entities, e.g. banks, and let $A_1(t), A_2(t), \ldots, A_k(t)$ denote the asset processes, i.e. the asset values at time t for the individual entities. Furthermore, for each entity i a limit D_i, the distress barrier, defines default in the following sense: default occurs if the asset value of entity i falls below the distress barrier, i.e.

$$A_i(t) < D_i . \tag{2.1}$$

The relation between asset value and distress barrier is usually closely related to leverage, i.e. the ratio between debt and equity.

Finally, let $X_1(t), X_2(t), \ldots, X_k(t)$ with

$$X_i(t) = A_i(t) - D_i \tag{2.2}$$

denote *the distance to default* of the individual entities. Note that alternatively the distance to default can also be defined in terms of $X_i(t)$ as a percentage of asset value, divided by the asset volatility, see e.g. (Crosbie & Bohn, 2003).

In a one period setup – as used throughout this chapter - one is interested at values $A_i(T), X_i(T)$ at time T, the end of the planning horizon. Analyzing systemic risk then means analyzing the joint distribution of the distances to default $X_i(t)$, in particular their negative parts $X_i(T)^- = \max\{-X_i(T), 0\}$, and the underlying random risk factors are described by the joint distribution of asset values $A_i(T)$.

Many approaches for modeling the asset values exist in literature. In a classical finance setup one would use correlated geometric Brownian motions resulting in correlated log-normal distributions for the asset values at the end of the planning horizon. Segoviano Basurto proposes a Bayesian approach (Segoviano Basurto, 2006), for applications see also (Jin & Nadal de Simone, 2013). In the present paper we will use copula based models, as discussed later.

The second component of the approach, the distress barrier, is in the simplest case (Merton, 2009) modeled just by the face value of overall debt for each entity. Other approaches distinguish between short term and long term debt (longer than the planning horizon). Usually this is done by adding some reasonable fraction of long term debt to the full amount of short term debt, see e.g. (Servigny & Renault, 2007).

Still such classical credit default models (see e.g. (Guerra, et al., 2013)), although classified as systemic risk models, neglect an important aspect: economic entities like banks are mutually indebted and each amount of debt is shown as liability for one entity, but also as asset for another entity. Default of one entity (a reduction in liabilities) may trigger subsequent defaults of other entities by reducing their asset values. We call such models systemic models in the strict sense.

Such approaches with mutual debt have been proposed e.g. in (Chan-Lau, et al., 2009 B) or (Chan-Lau, et al., 2009 A). Models neglecting this aspect are systemic models in a broad sense; in fact they are restricted to the effects of systematic risk related to asset values.

The basic setup of systemic models in the strict sense can be described as follows: Let H_{ij}^0 denote the amount of debt between entities i and j, i.e. the amount of money borrowed by entity i from entity j. We also include debt to the non-bank sector, denoted by H_i for each entity i and credit C_i to the non-banking sector, both repayable (including interest) at the end of the planning horizon, time T. Furthermore, $S_i(T)$ is the value at time T of other financial assets hold by entity i. Then the asset value of entity i at the end of the planning horizon is given by

$$A_i^0(T) = S_i(T) + \sum_{j:j\neq i} H_{ji}^0 + C_i, \tag{2.3}$$

the distress barrier (in the simplest case) is

$$D_i^0 = \sum_{j:j\neq i} H_{ij}^0 + H_i \tag{2.4}$$

and the distance to default can be written as

$$X_i^0(T) = A_i(T) - D_i^0 = S_i(T) + C_i + \sum_{j:j\neq i} H_{ji}^0 - \sum_{j:j\neq i} H_{ij}^0 - H_i. \tag{2.5}$$

The random factors are the values $S_i(T)$ of financial assets, and (in an extended model) the credits C_i from outside the system, payable back at time

Again one could stop at this point and analyze the distances to default X_i^0, respectively the sum of all individual distances to default in the framework of classical default models. Systemic models in the strict sense however go farther.

Consider now all entities in distress (defaulted banks), i.e. $I_D = \{i : X_i^0(\mathrm{T}) < 0\}$. Each of these entities is closed down and the related debt has to be adjusted, because entity i cannot fully pay back its debts. In a simple setup this can be done by reducing all debts to other entities to

$$H_{ij}^1 = \left(1 - \frac{\left(X_i^0\right)^-}{D_i^0}\right) H_{ij}^0 \ , \tag{2.6}$$

$$H_i^1 = \left(1 - \frac{\left(X_i^0\right)^-}{D_i^0}\right) H_i^0 \ . \tag{2.7}$$

Here, the factor

$$LGD_i = \left(X_i^0\right)^- \Big/ D_i^0 \tag{2.8}$$

is an estimate for the loss given default of entity i.

It is now possible to calculate new asset values, new distress barriers and new distances to default, after the default of all entities in I_D^0. For this purpose we replace in (2.3)-(2.5) all occurrences of H_{ij}^0 by H_{ij}^1 and all occurrences of H_i^0 by H_i^1. This first default triggers further ones and starts a *loss cascade*: It may happen that after the first adjustment step new defaults can be observed, which results in a new set of bankrupt entities I_D^1 after the second round. In addition, bankruptcy of additional entities may reduce even further the insolvent assets of entities that already defaulted in the first round.

This process can be continued, leading to new values $X_i^k(T), H_{ij}^k, A_i^k(T), D_i^k(T)$ and a augmented set I_D^k of defaulted entities after each iteration k. The loss cascade terminates, when no additional entity is sent to bankruptcy in step k, i.e. $I_D^k = I_D^{k-1}$.

The sequences $\left[H_{ij}^k\right]$ and $\left[H_i^k\right]$ of debt are nonincreasing in k and furthermore are bounded from below by zero values for all components, which implies convergence of debt. At this point we have

$$S_i(T) + C_i + \sum_{j \neq i} \left(1 - \frac{X_j(T)^-}{\sum_{k \neq j} H_{jk}^0 + H_j} \right) H_{ji}^0 - \sum_{j \neq i} \left(1 - \frac{X_i(T)^-}{\sum_{j \neq i} H_{ij}^0 + H_i} \right) \left(H_{ij}^0 + H_i \right) = X_i(T)^+ - X_i(T)^- \tag{2.9}$$
$$X_i(T)^+ \geq 0,\ X_i(T)^- \geq 0.$$

This system describes the relation between the positive and negative parts of the distances to default $X_i(T) = X_i(T)^+ - X_i(T)^-$ for all entities i. It holds with probability 1 for all entities. Note that previous literature, e.g. (Chan-Lau, et al., 2009 B) or (Chan-Lau, et al., 2009 A), uses fixed numbers instead of the estimated loss given defaults in (2.9).

In fact, the system (2.9) is ambiguous and we search for the smallest solution, i.e. the optimization problem

$$\begin{aligned} &\min_{X(T)} \ \mathbf{E}\left[\sum_i X_i(T)^- \right] \\ &\textit{subject}\ \text{to} \quad (2.9) \end{aligned} \tag{2.10}$$

has to be solved in order to obtain the correct estimates for $X_i(T)^+$ and $X_i(T)^-$.

This basic setup can be easily extended to deal with different definitions of the distress barrier, involving early warning barriers, or accounting for different types of debt, e.g. short term and long term as above.

Measuring systemic risk

The distances to default, derived from structural models, in particular from systemic models in the strict sense, can be used to measure systemic risk. In principle the joint distribution of distances to default for all involved entities contains (together with the definition of distress barriers) all the relevant information. We assume that the joint distribution is continuous and let $p(x) = p(x_1, x_2, \ldots, x_k)$ denote the joint density of the distances to default $X_1(T), X_2(T), \ldots, X_k(T)$ for all entities.

Note that the risk measures discussed in the following are often defined in terms of asset value, which is fully appropriate for systemic models in the broader sense. In view of the above discussion of systemic models in the strict sense, we instead prefer to use the distances to default or loss variables derived from the distance to default.

The first group of risk measures is based directly on unconditional and conditional default probabilities. See (Guerra, et al., 2013) for an overview of such measures. The simplest approach considers the *individual distress probabilities*

$$P_i^D = P(X_i(T) < 0) = \int_{x_i<0} \left[\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} p(x_1, x_2, \ldots, x_k) dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_k \right] dx_i \,. \qquad (3.1)$$

The term in squared brackets is the marginal density of $X_i(T)$, which means that it is not necessary to estimate the joint density for this measure. In similar manner one can consider joint distributions for any subset $I \subseteq \{1, \ldots, k\}$ of entities by using the related (joint) marginal density $p_I(x)$, which can be obtained by integrating the joint density $p(x)$ over all other entities, i.e. $j \notin I$.

Joint probabilities of distress for a subset I can be achieved by

$$P_I^D = P(\forall i \in I : X_i(T) < 0) = \int_{x_{i_1}<0} \cdots \int_{x_{i_{k_I}}<0} p_I(x_{i_1}, \ldots, x_{i_{k_I}}) dx_{i_1} \ldots dx_{i_{k_I}} \quad , \qquad (3.2),$$

where the set I contains the elements $i_1, i_2, \ldots, i_{k_I}$. Of special interest are the default probabilities of pairs of entities, see e.g. (Guerra, et al., 2013). Joint probabilities of distress describe tail risk within the chosen set I. If I represents the whole system, i.e. it contains all the entities, then the joint probability of distress can be considered as tail risk measure for systemic risk, see e.g. (Segoviano & Goodhart, 2009).

Closely related are *conditional probabilities of distress*, i.e. the probability that entity j is in distress, given that entity j is in distress, which can be written as

$$P_{j|i}^D = P(X_j(T) < 0 \mid X_i(T) < 0) = \frac{P_{\{i,j\}}^D}{P_i^D} \quad . \qquad (3.3)$$

These conditional probabilities can be presented by a matrix with $P_{j|i}^D$ as its ij-th matrix element, the *distress dependency matrix*.

While conditional distress probabilities contain important information, it should be noted, that they only reflect the two dimensional marginal distributions. Conditional probabilities are often used for analyzing the interlinkage of the system and the likelihood of contagion. However, such arguments should not be carried to extremes: finally, conditional probabilities do not contain any information about causality.

Another systemic measure related to probabilities is the probability of at least one distressed entity; see e.g. (Segoviano & Goodhart, 2009) for an application to a small system of four entities. It can be calculated as

$$P^1 = 1 - P(X_1 \geq 0, \ldots, X_k \geq 0) \tag{3.4}$$

(Guerra, et al., 2013) propose a asset-value-weighted average of individual probabilities of distress as an upper bound for the probability of at least one distressed entity. Probabilities of exact one, two or another number of distressed entities are hard to calculate for large systems because of the large number of combinatorial possibilities.

An important measure that bases on probabilities is the *banking stability index*, measuring the expected number of entities in distress, given that at least one entity is in distress. This measure can be written as

$$BSI = \frac{\sum_{i=1}^{k} P_i^D}{P^1} \; . \tag{3.5}$$

Other systemic risk measures base directly on the distribution of distances to default. (Adrian & Brunnermeier, 2009) propose a measure called *conditional-value-at risk*[1], $CoVaR^{=}$. It is closely related to value at risk, which is the main risk measure for banks under the Basel accord.

$CoVaR^{=}$ is based on conditional versions of the quantile at level β for an entity j given that entity i reaches the α-quantile. In terms of distances to default this reads

$$CoVaR^{=}_{\alpha,\beta}(j \mid i) = \inf\left\{\gamma \in \mathbb{R} : P\left(X_j(T) \leq \gamma \mid X_i(T) = VaR_\alpha(X_i(T))\right) \geq \beta\right\}, \tag{3.6}$$

where

$$VaR_\alpha(i) = \inf\left\{\gamma \in \mathbb{R} : P\left(X_i(T) \leq \gamma\right) \geq \alpha\right\} \; . \tag{3.7}$$

[1] Conditional-value-at risk should not be confused with general risk measure with the same name, which is also known as expected tail loss or average value at risk.

The contribution of entity i to the risk of entity j then is calculated as

$$\Delta CoVaR_{\alpha}^{=}(j\,|\,\mathrm{i}) = CoVaR_{\alpha,\alpha}^{=}(j\,|\,i) - CoVaR_{0.5,0.5}^{=}(j\,|\,i), \tag{3.8}$$

i.e. the conditional value at risk at level α is compared to the conditional value at risk at the median level. From all the $\Delta CoVaR_{\alpha}^{=}(j\,|\,i)$ values, it is possible to construct another kind of dependency matrix.

This idea can also be applied to the system as a whole: if $X_j(T)$ is replaced by $X(T) = \sum_{i=1}^{k} X_i(T)$, the distance to default of the whole system, (3.6)-(3.8) leads to a quantity $\Delta CoVaR_{\alpha}^{=}(j\,|\,i)$ that measures the impact of entity i on the system. In this way one is able to analyze notions like "too big to fail" or "too interconnected to fail".

In contrast to probability based measures, $CoVaR$ emphasizes the role of potential monetary losses. This approach can be carried forward, leading to the idea that systemic risk should be related to the losses arising from adverse events. Given a model for the distances to default $X_i(T)$, the overall loss of the system can be written as

$$L^{tot} = \sum_{i=1}^{k} X_i(T)^{-} \quad . \tag{3.9}$$

L^{tot} covers all credit losses in the whole system, both from interbank credits and from credits to the public.

From the viewpoint of a state this notion of total loss may be seen as too extensive. One may argue that only losses guaranteed by the state are really relevant. Definition (3.9) therefore depicts a situation in which a state guarantees all debt in the system, which can be considered as unrealistic. However in most developed countries the state guarantees saving deposits to a high extend, and anyhow society as a whole will have to bear the consequences of lost debt from outside the banking system. Therefore a further notion of loss is given by

$$L^{sav} = \sum_{i=1}^{k} LGD_i \cdot H_i \quad , \tag{3.10}$$

which describes the amount of lost non-banking debt. For model (2.9) loss given default is given by (2.8).

In general, the notion of loss depends on the exact viewpoint (loss to whom). We will therefore use the symbol L to represent any kind of loss variable in the following discussion of systemic risk measures.

An obvious measure is expected loss, i.e. the (discounted) expectation of the risk variable L. For simple structural models like (2.2) this measure can be calculated from the marginal distribution of asset values, respectively of distances to default. Modeling the joint distributions is not necessary. Note that this is different for the strict systemic model (2.9).

The expectation can be calculated with respect to an observed (estimated) model, or with respect to a risk neutral (martingale) model. Using observed probabilities may account insufficiently for risk, which contradicts the aim of systemic risk measurement. Using risk neutral valuation seems reasonable from a finance point of view and has been used e.g. in (Gray & Jobst, 2010) or (Gray, et al., 2010). However it should be kept in mind that the usual assumptions underlying contingent claims analysis, in particular that the acting investor is a price taker, are not valid if the investor has to hedge the whole financial system, which clearly would be the case when hedging the losses related to systemic risk.

Using expectation and the concept of loss cascades, (Cont, et al., 2010) define a *Contagion Index* as follows: they define first the total loss of a loss cascade triggered by a default of entity i and the contagion index of entity i as the expected total loss conditioned on all scenarios which trigger the default of entity i.

Clearly the expectation does not fully account for risk. An obvious idea is to augment expectation by some risk measure ρ, which, with weight a, leads to

$$\pi_\rho(L) = E\left[L\right] + a\rho\left(L\right). \tag{3.11}$$

Typical choices of ρ are dispersion measures like the variance or the standard deviation. Such measures are examples of classical premium calculation principles in insurance. Further, more general premium calculation principles are e.g. the distortion principle or the Esscher premium principle. For an overview on insurance pricing see e.g. (Furmann & Zitikis, 2008). In the context of systemic risk, the idea to use insurance premiums was proposed in (Huang, et al., 2009). In this paper empirical methods were used for extracting an insurance premium from high frequency credit default swap data. Even more general, it should be noted that any monetary risk measure, in particular coherent measures of risk, can be applied to the overall loss in a system. See (Kovacevic & Pflug, 2014) for an overview and references.

In this broad framework, an important class of risk measures is given by the *quantiles* of the loss variable L:

$$Q_\alpha(L) = \inf\left\{l : P\left\{L \le l\right\} \ge \alpha\right\}. \tag{3.12}$$

With probability α the loss will not higher than the related quantile.

Quantiles are closely related to the value-at-risk (VaR), which measures quantiles for the deviation of the loss from the expected loss. Note the slight difference between (3.12) and (3.7), because (3.7) is stated in terms of distance to default and (3.12) in terms of loss.

$Q_\alpha(L)$ can also be interpreted in an economic way as follows. Assume that a fund is build up in order to cover systemic losses in the banking system. If we ask the question, how large the fund should be such that it is not exhausted with probability α over the planning period, then the answer will be $Q_L(\alpha)$. This idea can also be reversed. Assume now that a fund of size q has been accumulated to deal with systemic losses. Then the *probability that the fund is not exhausted* , i.e.

$$F_L(q) = P\left(L \leq q\right) \tag{3.13}$$

is a reasonable systemic risk measure. Clearly, $F_L(q)$ is the distribution function of the loss and q is the quantile at level $F_L(q)$.

Unfortunately, quantiles do not contain any information about those $(1-\alpha)100$ percent cases, in which the loss lies above the quantile. Two different distributions, which are equal in their negative tails, but very different in the positive tails, are treated equally.

The average value-at-risk (AVaR) avoids some drawbacks of quantiles. It is defined for a parameter α, which again is called level. The AVaR averages the bad scenario,

$$AVaR_\alpha\left(L\right) = E\left[L|L > Q_\alpha\left(L\right)\right] \ . \tag{3.14}$$

The latter formula justifies the alternative name conditional value-at-risk (CVaR), which is frequently used particularly in finance. In insurance, the AVaR is known as conditional tail expectation or expected tail loss.

The effect of individual banks can be analyzed in obvious manner by defining conditional versions of the quantile or $AVaR$ loss measures, i.e. by conditioning the overall loss on the distance to default of an individual bank in the style of $CoVaR$, see (3.6) above.

Systemic Risk and Copula Models

The distinction between risk factors which are related to individual performances and risk factors which are a consequence of the interrelations of the economic agents has its parallel in a similar distinction for probability distributions or stochastic processes:

Suppose that $X_1(t),\ \dots X_k(t)$ describe the performance processes of k economic agents. The individual (marginal) processes are assumed to follow certain stochastic models as discrete Markov processes, diffusion models or jump-diffusion models. The joint distribution however depends on the copula process, which links the marginal processes.

To simplify, suppose only a single-period model is considered and that the performance after one period is $X_1,\dots X_k$. If this vector has marginal cumulative distribution functions $F_1,\dots,F_k$ (meaning that $P(X_i \le u) = F_i(u)$), then the joint distribution of the whole vector can be represented by

$$P(X_1 \le u_1,\dots,X_k \le u_k) = C\left(F_1(u_1),\dots,F_k(u_k)\right) , \tag{3.15}$$

where C is called the copula function. Typical families of copula functions are the normal copula, the Clayton copula, the Gumbel copula or – more generally – the group of Archimedean copulas.

While the marginal distributions describe the individual performances, the copula function models the interrelations between them and can thus be seen as representing the systemic component. In particular, the relation between underperformance of agent i and agent j can be described on the basis of the copula. To this end, we use the notion of conditional value-at-risk (CoVaR), see above. Following (Mainink & Schaaning, 2014) we use the notations

$$CoVaR^{=}_{\alpha,\beta}(Y|X) = VaR_\beta(Y \mid X = VaR_\alpha(X))$$

for the notion of Conditional Value-at-Risk introduced by (Adrian & Brunnermeier, 2009) and

$$CoVaR_{\alpha,\beta}(Y|X) = VaR_\beta(Y \mid X \le VaR_\alpha(X))$$

for the variant introduced by (Girardi & Ergün, 2012). Keep in mind that we work here with profit&loss variables and not with pure loss variables. The latter variant can be expressed in terms of the conditional copula

$$C_{V|U\le\alpha}(v)=\frac{P(V\le u,U\le\alpha)}{P(U\le\alpha)}=\frac{C(\alpha,v)}{\alpha}.$$

Its inverse

$$C^{-1}_{V|U\le\alpha}(\beta)=\inf\{v:C_{V|U\le\alpha}(v)\ge\beta\}$$

and the marginal distribution of X can be used to write the $CoVaR$ in the following way:

$$CoVaR_{\alpha,\beta}(Y|X)=F_X^{-1}\left(C^{-1}_{V|U\le\alpha}(\beta)\right).$$

For the $CoVaR^{=}_{\alpha,\beta}(Y|X)$ the conditional copula

$$C_{V|U=\alpha}(v)=\frac{\partial}{\partial v}C(\alpha,v)$$

is needed. With

$$C^{-1}_{V|U=\alpha}(\beta)=\inf\{v:C_{V|U=\alpha}(v)\ge\beta\}$$

one gets

$$CoVaR^{=}_{\alpha,\beta}(Y|X)=F_X^{-1}\left(C^{-1}_{V|U=\alpha}(\beta)\right).$$

Notice that both notions of $CoVaR$ depend only on the copula and the marginal distribution of X.

If underperformance of an agent means that its performance falls below an α-quantile, then $C_{V|U\le\alpha}(\alpha)=\dfrac{C(\alpha,\alpha)}{\alpha}$

indicates the probability that also the other agent underperforms. $CoVaR_{\alpha,\alpha}\left(X_i|X_j\right)$ gives the necessary risk reserve for agent i to survive a possible default of agent j. For a system of k agents, the notion of $CoVaR_{\alpha,\beta}$ can be generalized in a straightforward manner to k components.

Example 1. Consider a financial institution A, which faces a gamma-distributed loss with mean 10 and variance 20. Then A's unconditional 99% VaR is 23.8.

If A's performance related to B's performance with a normal copula with correlation ρ, then A's conditional VaR (the CoVaR) increases with increasing ρ, see the table below.

ρ	0.0	0.2	0.5	0.8
$\mathrm{CoVaR}_{0.99,0.99}$	23.8	26.6	30.0	31.4

Table 1.

Example 2. Consider a system of 7 banks, where the performances $X_i, i = 1,..,7$ are related by a normal copula stemming from a correlation matrix with all off-diagonal elements ρ (the diagonal elements are 1). Suppose that the first bank defaults if its performance drops below the 5% quantile. Given the copula one may determine the number of other banks which also fall below the 5% quantile, i.e. default as a consequence of the first banks's default. The following pictures show the distribution of these numbers for the choice of $\rho = 0, \rho = 0.2, \rho = 0.5$ and $\rho = 0.8$. One may observe that in the independent case ($\rho = 0$) the other banks are practically not affected by the default of one bank, while for higher correlated cases a contagion effect to other banks can be easily seen.

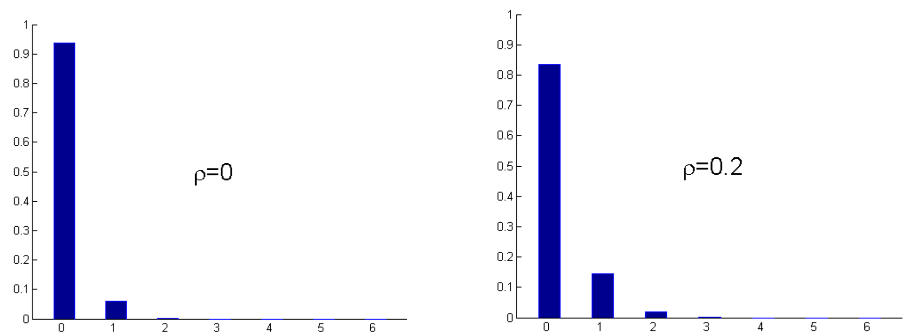

Figure 1.

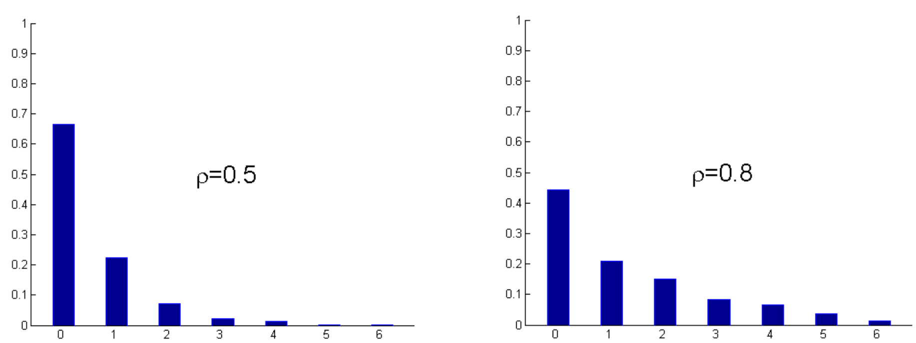

Figure 2.

A very interdependent banking system carries a high systemic risk. It has therefore been proposed to limit the dependencies by creating quite independent subsystems. The following Example gives an evidence for this argument.

Example 3. Here we consider 7 banks each of which has a performance given by a negative gamma distribution with mean 100 and variance 200, but shifted such that with probability 5% a negative performance happens, which means bankruptcy. The total losses of the system are calculated on the basis of a normal copula linking the individual losses. By assuming that the government (or the tax payer) takes responsibility for covering total losses up to the 99% quantile, this quantile (the 99% VaR) can be seen as a quantization of the systemic risk.

In the following pictures, we show in the upper half a visualization of the correlations (which determine the normal copula) by the thickness of the arcs connecting the 7 nodes representing the banks. The lower half shows the distribution of the total systemic losses, where also the 99%VaR is indicated. As one can see, the higher correlation increases the systemic risk. If the system is divided into independent subsystems, the systemic risk decreases.

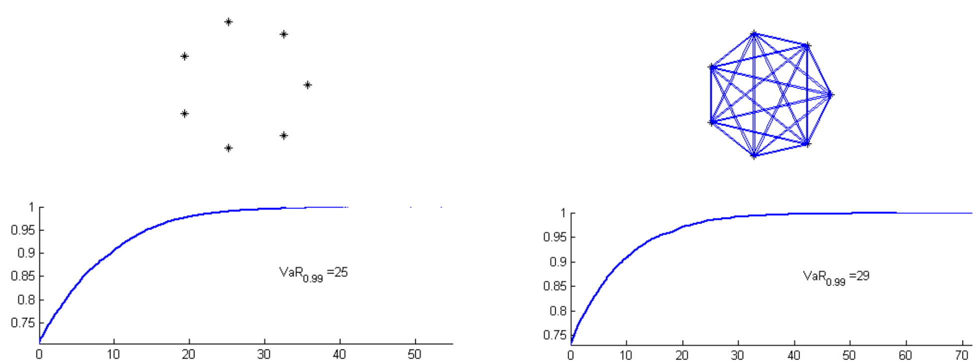

Figure 3. Left: all banks are independent, $VaR_{0.99}$=25
Right: all correlations are $\rho = 0.2$, $VaR_{0.99}$=29

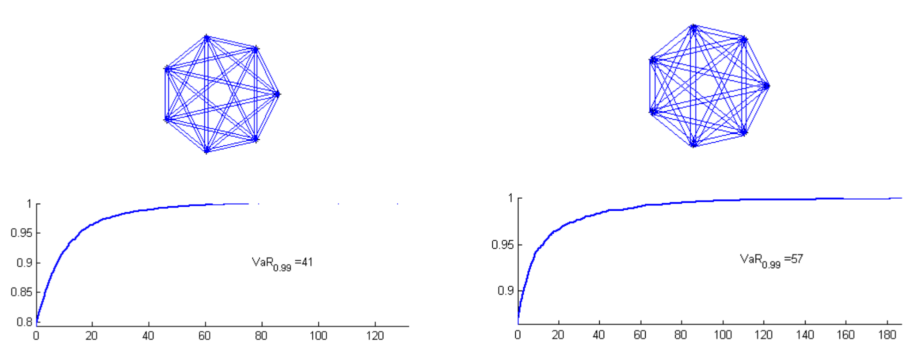

Figure 4. Left: all correlations are $\rho = 0.5$, $VaR_{0.99}$=41
Right: all correlations are $\rho = 0.8$, $VaR_{0.99}$=57

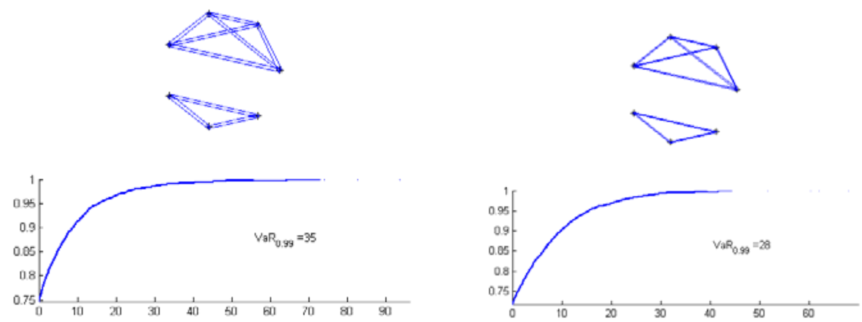

Figure 5. The system consists of two independent subsystems with internal correlations ρ. Left: $\rho = 0.2$, $VaR_{0.99}$=28; Right: $\rho = 0.5$, $VaR_{0.99}$=35

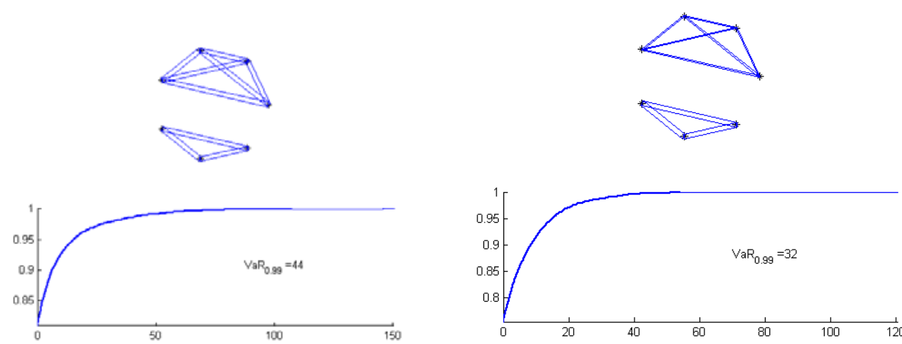

Figure 6. The system consists of two independent subsystems with internal correlations ρ. Left: $\rho = 0.8$, $VaR_{0.99}$=44; Right: one subsystem has $\rho = 0.2$, the other $\rho = 0.8$, $VaR_{0.99}$=32

Conclusions

Systemic financial risk is an important issue in view of the distress the banking systems all over the world has experienced in the recent years of crises. Even if breakdowns are prevented by the government, the related societal costs are extremely high.

We described the measurement of systemic risk, based on the structural approach originating from structural credit risk models. In particular the cascading effects which are caused by mutual debt between the individual banks in the system were analyzed in detail. Furthermore we related the notion of systemic risk to the copula structure, modeling dependency between the performances of the individual banks. The effects of different levels of dependency on the total systemic risk in terms of the value at risk of total losses were demonstrated by examples.

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