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A b s t r a c t

The problem addressed is given a group composed of N individuals and given a von Neumann-Morgenstern utility function for each individual in the group, how can these be aggregated to obtain a group von Neumann-Morgenstern utility function. The implications of a set of axioms, analogous to Arrow's, using individual cardinal utilities--as opposed to Arrow's ordinal rankings--are investigated. The result is a group cardinal utility function which explicitly requires interpersonal comparison of preference. Suggestions for who should make these comparisons and how they might be done are given.

1. Introduction

How should a group of individuals choose among a set of alternatives? Certainly there are a host of possible answers here ranging from formal aggregation schemes to informal discussion until a concensus emerges. The general problem--sometimes referred to as the social welfare problem--has drawn much attention from economists, sociologists, political scientists, etc.

The problem is often formalized along the following lines. A set of N individuals I_i, i = 1, 2, ..., N, must collectively select an alternative a_j from the set $A = \{a_j, j = 1, 2, ..., M\}$. It is assumed that each individual I_i can articulate his preferences, denoted by P_i. For instance, P_i could be a ranking of the M alternatives, or it could be a preference structure such as a von Neumann-Morgenstern utility function over the set of possible consequences of the alternatives, or it could be expected utilities associated with the alternatives. The problem is to obtain the group preferences P_G given the individual preferences P_i, i = 1, 2, ..., N. Thus, a function f is needed such that

$$P_G = f(P_1, P_2, \ldots, P_N). \qquad (1)$$

The usual approach has been to put reasonable restrictions on the manner in which the P_i are combined, and then derive the implications this places on f. For instance, one such restriction might be if $P_i = P$ for all i, then $P_G = P$, the common individual preference.

There are two versions of the problem formalized by (1) which are of interest in this paper. These will be referred to as the benevolent dictator problem and the participatory group problem. In the former case, the aggregation rule, that is the f in (1), is externally imposed by some individual--the benevolent dictator. In the participatory group, the group itself must internally generate the aggregation rule for selecting a best group alternative. The theoretical development is the same for both of these versions of the "social welfare problem," however the necessary input assessments needed to implement the results must be obtained in different manners.

In section 2, we briefly summarize aspects of Arrow's [1951, 1963] work on the social welfare problem. His formulation used the P_i as rankings of the alternatives. His result is that, in general, there is no f which satisfies five "reasonable" assumptions; and hence, the assumptions are incompatible. Arrow's formulation, since it used rankings, did not incorporate any concepts of strength of preference nor did it attempt to interpersonally compare preferences. Harsanyi [1955] was among the first to investigate assumptions leading to a group von Neumann-Morgenstern utility function. Recently, Sen [1970] has shown that formulations with the structure of (1) require interpersonal comparison of utility in order to achieve a group preference for all possible sets of individual preferences.

This paper tries to formulate the group decision problem in a manner analogous to Arrow except that the individual preferences P_i are utilities of the alternatives in the von Neumann-Morgenstern sense. In our model, we first assume the alternatives have certain known consequences. Uncertainties are then explicitly introduced in section 4. It is argued in section 3 that given five assumptions analogous to Arrow's, using cardinal utilities rather than rankings, it is always possible to define consistent aggregation rules for group cardinal utility function. Specific classes of such rules, which explicitly require interpersonal comparison of preference, are investigated in section 4. Suggestions for obtaining the necessary assessments to utilize these aggregation rules are given in section 5.

2. Arrow's Impossibility Theorem

Perhaps the best known work on group preferences is Arrow's [1951]. He proves that, in general, there is no procedure for obtaining a group ordering (i.e. ranking) of the various alternatives, call this P_G, based on individual group members orderings P_i that is consistent with five seemingly reasonable assumptions. Thus, there is no f satisfying (1) when the P_i's are rankings that is consistent with these five conditions:

Assumption A1. There are at least two individual members in the group, at least three alternatives, and a group ordering is specified for all possible individual member's orderings.

Assumption A2. If the group ordering indicates alternative a is preferred to alternative b for a certain set of individual orderings, then the group ordering must imply a is preferred to b if:

(i) the individual's orderings of alternatives other than a are not changed, and

(ii) each individual's ordering between a and any other alternative either remains unchanged or is modified in favor of a.

Assumption A3. If an alternative is eliminated from consideration, the new group ordering for the remaining alternatives should be equivalent to the original group ordering for these same alternatives.

Assumption A4. For each pair of alternatives a and b, there is some set of individual orderings such that the group prefers a to b.

Assumption A5. There is no individual with the property that whenever he prefers alternative a to b, the group will also prefer a to b regardless of the other individual's orderings.

Luce and Raiffa [1957] examine the reasonableness of these assumptions and suggest that Assumption A3, referred to as Independence of Irrelevant Alternatives Assumption is the weakest of the five. The problem arises from interpreting (or misinterpreting) an individual's strength of preference of one alternative over another based on that individual's ranking of alternatives. In the next two sections, our formulation explicitly utilizes individual's strength of preferences and avoids this particular difficulty.

3. A Cardinal Utility Axiomatization

The specific problem addressed is as follows. For each individual I_i, i = 1, 2, ..., N, we are given the set of cardinal utilities $u_i(a_j)$ of the alternatives a_j, j = 1, 2, ..., M. We wish to obtain a set of group cardinal utilities $u_G(a_j)$ from the $u_i(a_j)$ consistent with five assumptions analogous to Arrow's. For decision purposes, the best group alternative is the one associated with the highest group utility. In terms of (1), the problem is to find a u such that

$$u_G = u(u_1, u_2, \ldots, u_N) \qquad (2)$$

that is consistent with five axioms:

Assumption B1. There are at least two individual members in the group, at least two alternatives, and group utilities are specified for all possible individual member's utilities.

Assumption B2. If the group utilities indicate alternative a is preferred to alternative b for a certain set of individual utilities, then the group utilities must imply a is preferred to b if:

(i) the individual's utilities of alternatives other than a are not changed, and

(ii) each individual's utilities for a either remains unchanged or is increased.

Assumption B3. If an alternative is eliminated from consideration, the new group utilities for the remaining alternatives should be equivalent to the original group utilities for these same alternatives.

Assumption B4. For each pair of alternatives a and b, there is some set of individual utilities such that the group prefers a to b.

Assumption B5. There is no individual with the property that whenever he prefers alternative a to b, the group will also prefer a to b regardless of the other individual's utilities.

As can be seen, the main distinction--and the only relevant one--between these assumptions and Arrow's is the substitution of group and individual utilities for his group and individual orderings. The interesting result is that there are many possible forms of u in (2) which satisfy Assumptions B1 - B5, whereas there were no f's in (1) consistent with Arrow's Assumptions A1 - A5. In the next section we will investigate some specific functional forms for combining the individual utilities which satisfy Assumptions B1 - B5. Here we will informally talk through properties of such forms to indicate that in fact some do exist.

If we let $u_G = u(u_1, u_2, \ldots, u_n)$, the critical property which must be satisfied by the group cardinal utility function u in order to be consistent with Assumptions B1 - B5 is

$$\frac{\partial u}{\partial u_i} > 0, \quad i = 1, 2, \ldots, N . \tag{3}$$

To illustrate, consider the special case where

$$u(u_1, u_2, \ldots, u_n) = \sum_{i=1}^{N} k_i u_i, \quad k_i > 0, \; i = 1, 2, \ldots, N . \tag{4}$$

We could first scale all the individual cardinal utilities for the alternatives from zero to one*. To assess the scaling constants--the k_i's in (4)--requires interpersonal comparison of utility. In the benevolent dictator problem, this comparison is done by the benevolent dictator himself, whereas in the participatory group problem, it is done by the group as a whole. We will return to this problem in section 5. For now, let us assume that u in (4) has been assessed. It is then simple to verify that in fact (3) is satisfied.

Let us check the basic assumptions. B1 is trivially satisfied by (4). Because (3) is satisfied, Assumption B2 is also satisfied. Increasing an individual's utility for alternative a can only increase the group's utility. Assumption B3 is a little more involved. As shown in section 5, selecting values for the k_i's in (4) is dependent only on the alternatives for which the u_i's are zero or one. However, we have already assessed the k_i's and we will not change these, even if the alternatives associated with a utility of either zero or one

*Note that this assumes that the utilities of each of the individuals are bounded.

for any individual are deleted. Hence, clearly if an alternative is dropped from consideration, the group utility of those remaining will not be affected. Thus, the new group utilities for the remaining alternatives are equivalent to--in this case they are identical to--the original group utilities for these same alternatives. It also is the case that if a new alternative is added to the list, it may have negative utility or a utility greater then one for some of the individuals given the scaling we have previously established. This is fine, and again, it will not affect the group utilities for the original alternatives.

If each of the individuals prefer alternative a to alternative b, the group must prefer a to b using (4) so Assumption B4 is satisfied. Assumption B5 is also satisfied by (4) because there is obviously some small amount ε such that if individual I_i prefers a to b by a utility margin of ε, and also if all the other individuals prefer b to a, then alternative b will be preferred to a by the group as a whole. Although the reasoning is a bit more involved, one can similarly check to see that any group utility function consistent with (3) will satisfy the Assumptions B1 - B5.

4. Some Specific Cardinal Utility Functions

We have established that group cardinal utility function consistent with Assumptions B1 - B5 do exist. The arguments of these functions are different individual's cardinal utilities of certain alternatives. Let us now expand our problem to include uncertain alternatives*, that is, an alternative can now indicate which of the a_j's may result and the associated probabilities, which will be denoted p_j.

In general, the different individuals associated with a particular problem may be in disagreement about the values of the p_j's for any particular alternative. In this paper, we will simply skip this important issue and address those problems where there is agreement on the probabilities. Thus, given $u_G(a_j)$, $a_j = 1, 2, \ldots, M$, and the probabilities p_j for any uncertain alternative, the expected group utility for this alternative is $\sum_{i=1}^{M} u_G(a_j)\cdot p_j$. We have not ruled out the possibility that M is infinite, and in this case the summation is simply replaced by an integral sign.

We want to look at axiomatizations which satisfy Assumptions B1 - B5. Consider the additional assumption which can be stated as

Assumption C1. In situations where the utilities of N-1 of the individual's are fixed for two alternatives, the utilities of the N^{th} individual shall guide the group decisions.

In a little more technical terms, we could say that Assumption C1 implies that if $u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_N$ are fixed at any level and alternatives are considered which differ only in u_i, then the alternative with the highest mathematical expectation

*For this interpretation, it may be more convenient to think of the certain alternatives as being tautological to the consequences which they imply.

for u_i should be chosen. Since u is to be a group cardinal utility function, Assumption C1 also implies that with u_1, ..., u_{i-1}, u_{i+1}, ..., u_N fixed, if alternative a leads to a probability distribution $p_a(u_i)$ and alternative b leads to a probability distribution $p_b(u_i)$, the group utility function should be such that the expected group utility of a is greater than that of b if and only if the expected utility of a to individual I_i is greater than his expected utility of b. Thus, since utility functions are unique up to positive linear transformation (von Neumann and Morgerstern [1953]), u must be linear in each of the u_i so Assumption C1 means

$$u(u_1, u_2, \ldots, u_N) = f_i(u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_N) + g_i(u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_N)u_i,$$
$$g_i > 0, \text{ for } i = 1, 2, \ldots, N. \tag{5}$$

Given $A \equiv \{a_j, j = 1, 2, \ldots, M\}$ is the set of certain alternatives under consideration, the following result, proven in Keeney [1972] in the context of multiattribute utility and in Keeney and Kirkwood [1973] in a group decision context holds.

<u>Theorem 1</u>. If assumption C1 is satisfied, then the group cardinal utility function u_G defined by $u_G(a_j) = u[u_1(a_j), u_2(a_j), \ldots, u_N(a_j)]$ can be calculated from

$$u(u_1, u_2, \ldots, u_N) = \sum_{i=1}^{N} k_i u_i + \sum_{\substack{i=1 \\ \ell>i}}^{N} k_{i\ell} u_i u_\ell + \ldots$$

$$+ k_{12 \ldots N} u_1 \cdots u_N \qquad (6)$$

where

(1) u_i, $i = 1, 2, \ldots, N$, is scaled from zero to one for the set of alternatives,

(2) u is scaled by $u(0, 0, \ldots, 0) = 0$ and $u(1, 1, \ldots, 1) = 1$,

(3) $k_i > 0$, for $i = 1, 2, \ldots, N$, and

(4) $k_i + \sum_{\ell=1}^{N} k_{i\ell} + \sum_{\substack{i=1 \\ \ell>i \\ m>\ell}}^{N} k_{i\ell m} + \ldots + k_{12\ldots N} > 0$, for $i=1,2,\ldots,N$.

Condition (4) of the theorem follows from (5) since g_i must be positive.

The interesting fact about u in (6) is that it satisfies the conditions B1 - B5 which are analogous to Arrow's assumptions. Assumption B1 is clearly met, and since from (5), $\partial u/\partial u_i = g_i > 0$, for all i, Assumption B2 is also satisfied. B4 and B5 are consistent with (6) for the same reasoning as discussed in the last section. The only assumption which is not clearly satisfied is B3.

If an alternative is eliminated from consideration for which u_i is not equal to zero or one, then of course we have the same utility function as in (6) so the remaining alternatives have the same utility as previously and B3 is met.

The difficulty occurs if an alternative is eliminated for which u_i is zero or one for some i. We can consider all cases sequentially so assume two alternatives have been eliminated from consideration for which $u_1 = 0$ for one of these and $u_1 = 1$ for the other. Now we wish to find a new group utility function u' defined on the utilities of alternatives denoted by $u_1', u_2', \ldots, u_N'$ where the u_i' are scaled from zero to one. Thus, we have $u_i' = u_i$ for i = 2, 3, ..., N. Note that usually there will be amounts of u_1, let me call them u_1^o and u_1^*, where $0 < u_1^o < u_1^* < 1$, which now coincide with $u_1' = 0$ and 1 respectively*. By this, we mean u_1 for some alternative a was u_1^o and u_1' for a is 0. Assumption C1 must still hold so the group utility function u' must be of the form

$$u'(u_1', u_2', \ldots, u_N') = \sum_{i=1}^{N} k_i' u_i' + \sum_{\substack{i=1 \\ \ell > i}}^{N} k_{i\ell}' u_i' u_\ell' + \ldots + k_{12\ldots N}' u_1' \cdots u_N' \tag{7}$$

as in Theorem 1. In order for u and u' to give equivalent utilities to all possible alternatives, they must be equivalent utility functions. Hence, since cardinal utility functions are unique up to positive linear transformation, the question is where or not a λ_1 and $\lambda_2 > 0$ exist such that

$$u'(u_1', u_2', \ldots, u_N') = \lambda_1 + \lambda_2\, u(u_1, u_2, \ldots, u_N) \tag{8}$$

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*This case where individual I_1 ends up indifferent between all the remaining alternatives, that is where $u_1^o = u_1^*$, is uninteresting since I_1's preferences will have no effect on the groups preferences.

where both u' and u are scaled zero to one consistent with (7) and (6) respectively. But we know that

$$u'(0, 0, \ldots, 0) = 0 = \lambda_1 + \lambda_2\, u(u_1^o, 0, \ldots, 0) \tag{9}$$

and

$$u'(1, 1, \ldots, 1) = 1 = \lambda_1 + \lambda_2\, u(u_1^*, 1, \ldots, 1) \tag{10}$$

if such a λ_1 and λ_2 exist. Since we can just solve (6) for $u(u_1^o, 0, \ldots, 0)$ and $u(u_1^*, 1, \ldots, 1)$, these can be substituted into (9) and (10) to give us numerically values of λ_1 and λ_2. From this all the k_i', k_{ij}', and so on of (7) can be evaluated directly from (6) by using (8). Thus, if scaling is consistently handled, Assumption B3 is satisfied by (6).

Now let us look at a stronger assumption than C1, which is a natural extension of the premise.

Assumption C2. In situations where the utilities of N-2 of the individuals are fixed for two alternatives, the utilities of the two remaining individuals shall guide the group decisions.

In a logical manner similar to the way we arrived at (5) as a mathematical statement of Assumption C1, a mathematical statement of Assumption C2 is

$$u(u_1, u_2, \ldots, u_N) = f_{ij}(u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_{j-1}, u_{j+1}, \ldots, u_N) + g_{ij}(u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_{j-1}, u_{j+1}, \ldots, u_N)\, h_{ij}(u_i, u_j), \text{ for } i = 1,2,\ldots,N, j>i, \tag{11}$$

where $g_{ij} > 0$. As proven in Keeney [1974] in the context of multiattribute utility theory, we have

<u>Theorem 2</u>. If Assumption C2 is satisfied, then the group cardinal utility function u_G defined by $u_G(a_j) \equiv u[u_1(a_j), u_2(a_j), \ldots, u_N(a_j)]$ can be evaluated from

$$u(u_1, u_2, \ldots, u_N) = \sum_{i=1}^{N} k_i u_i + k \sum_{\substack{i=1 \\ \ell > i}}^{N} k_i k_\ell u_i u_\ell + \ldots + k^{N-1} k_1 k_2 \ldots k_N u_1 u_2 \ldots u_N \qquad (12)$$

where

(1) u_i is scaled from zero to one for the set of alternatives,

(2) u is scaled by $u(0, 0, \ldots, 0) = 0$ and $u(1, 1, \ldots, 1) = 1$,

(3) $k_i > 0$, $i = 1, 2, \ldots, N$, and

(4) $k = 0$, if $\sum_{i=1}^{N} k_i = 1$, and k is the solution to

$$1 + k = \prod_{i=1}^{N} (1 + kk_i), \text{ if } \sum_{i=1}^{N} k_i \neq 1 .$$

Note that (12) is a special case of (6) proving that Assumption C2 is in fact stronger than C1. Also of interest is that if $k = 0$, then (12) is the additive form

$$u(u_1, u_2, \ldots, u_N) = \sum_{i=1}^{N} k_i u_i , \qquad (13)$$

and if $k \neq 0$, we can multiply (12) by k, add 1, and factor to obtain the multiplicative form of the group utility function

$$1 + ku(u_1, u_2, \ldots, u_N) = \prod_{i=1}^{N} (1 + kk_i u_i) . \qquad (14)$$

Since (12) is a special case of (6) and (6) satisfies Assumptions B1 - B5, then clearly (12) does also.

Let us investigate the implications of one more assumption.

<u>Assumption C3</u>. In situations where the utilities of N-2 of the individuals are fixed for two alternatives, the group should be indifferent if individually, each of other two individuals is indifferent.

A mathematical statement equivalent to Assumption C3 is that the two lotteries

$$\begin{cases} 1/2 & (u_i, u_j, \bar{u}_{ij}) \\ 1/2 & (u_i', u_j', \bar{u}_{ij}) \end{cases} \quad \text{and} \quad \begin{cases} 1/2 & (u_i, u_j', \bar{u}_{ij}) \\ 1/2 & (u_i', u_j, \bar{u}_{ij}) \end{cases}$$

where $\bar{u}_{ij} \equiv (u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_{j-1}, u_{j+1}, \ldots, u_N)$ must be indifferent for all $\bar{u}_{ij}$, u_i, u_j, u_i', u_j'. Notice that with each lottery, individual I_i has the same expected utility and individual I_j has the same expected utility.

<u>Theorem 3</u>. If Assumption C3 is satisfied, the group cardinal utility function u_G defined by

$u_G(a_j) = u[u_1(a_j), u_2(a_j), \ldots, u_N(a_j)]$ can be evaluated from

$$u(u_1, u_2, \ldots, u_N) = \sum_{i=1}^{N} k_i u_i \qquad (15)$$

where

(1) u_i is scaled from zero to one for the set of alternatives,

(2) u is scaled by $u(0, 0, \ldots., 0) = 0$ and $u(1, 1, \ldots, 1) = 1$, and

(3) $k_i > 0$, $i = 1, 2, \ldots, N$, with $\sum_{i=1}^{N} k_i = 1$.

This result was first proven by Harsanyi [1955] and is very similar to a proof by Fishburn [1965] in a multiattribute utility context. Obviously (15) is a special case of (6) and as such, it satisfies the Assumptions B1 - B5. The power of the apparently innocuous Assumption C3 stems from the fact that it assumes the "balance" of utilities among the individual is unimportant. This was not implied by either Assumption C1 or Assumption C2.

5. Interpretation and Assessment of the Group Utility Functions

The assessments necessary to implement the formulations of the last section come from different sources for the two versions--the benevolent dictator and the participatory group--of group decision problems defined at the beginning of the paper. In both cases the cardinal utilities of the certain alternatives come from the individuals who make up the groups; each invidivual articulates his own utilities. The more difficult assessments concern obtaining the scaling constants, that is the k's in (6), (12), and (15). In the benevolent dictator model, the benevolent dictator himself must make these judgments, whereas the group as a whole must assess the k's in the participatory group model.

Assessing the k's requires interpersonal comparison of preferences. This is the heart of the issue. To be as simple as possible and yet address the point, consider the benevolent dictator who must assess the k's in (15). Since the individual's utilities are scaled from zero to one, we can arbitrarily set $u(0, 0, \ldots, 0) = 0$ and $u(1, 1, \ldots, 1) = 1$, where u is actually the benevolent dictator's utility function. Thus, the benevolent dictator must consider questions like which of $(1, 0, \ldots, 0)$ or $(0, 1, 0, \ldots, 0)$ he prefers. It is easily to show from (15) that $u(1, 0, \ldots, 0) = k_1$ and $u(0, 1, 0, \ldots, 0) = k_2$ so if the former is preferred, then $k_1 > k_2$. With similar considerations, a ranking of the k_i can be developed. These considerations are not easy since the benevolent dictator must conjur up in his mind what a $u_1 = 0$ and a $u_1 = 1$ means to individual I_1 and what a $u_2 = 0$ and $u_2 = 1$ means to I_2, and then superimpose his own value structure about how important he thinks it is to change u_1

from 0 to 1 versus u_2 from 0 to 1, etc. Suppose k_1 is greater than k_2, then the benevolent dictator must ask himself, how much u_1, call it u_1^* is such that $(u_1^*, 0, \ldots, 0)$ is indifferent to $(0, 1, 0, \ldots, 0)$. By using (15) and equating utilities of these circumstances, we find $k_1 u_1^* = k_2$. A similar procedure is repeated for each of the u_i's which provides us with a set of N-1 equations and N unknowns, the k_i's. From Theorem 3, the Nth equation is $\sum_{i=1}^{N} k_i = 1$ from which the values of the k_i's can be found.

The same type of thinking must be followed in the participatory group decision model by each of the individuals in the group. However, in addition, they must somehow arrive at a concensus for the k's. Sometimes this may not be possible and thus the model could not be used as intended. When one uses the more general utility functions (6) and (12) rather than (15), it is necessary that the assessors consider impacts on two or more individuals at a time in order to arrive at the scaling factors. That is, in general, one must find pairs of circumstances $(u_1', u_2', \ldots, u_N')$ and $(u_1'', u_2'', \ldots, u_N'')$ for which the assessor is indifferent. Then naturally $u(u_1', u_2', \ldots, u_N') = u(u_1'', u_2'', \ldots, u_N'')$ gives us one equation with the number of unknowns equal to the number of scaling constants. The idea is to generate the number of independent equations equal to the number of scaling constants and then solve for them. Kirkwood [1972] discusses assessment of the scaling constants in more detail.

It is a difficult problem for the decision maker--the benevolent dictator in the benevolent dictator model or the group as a whole in the participatory group model--to make the requisite interpersonal comparisons of utility. An excellent discussion of this issue is found in Harsanyi [1974]. We made no pretence that interpersonal utility comparisons are easy, but they are often implicitly made in group decisions. When one can formalize this aspect of the process, the group utility functions discussed in this paper do provide a means for integrating these preferences which satisfies many reasonable conditions.

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