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A DUAL-BASED PROCEDURE FOR
DYNAMIC FACILITY LOCATION

Tony J. Van Roy* and Donald Erlenkotter**

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International Institute for Applied Systems Analysis
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FOREWORD

The public provision of urban facilities and services often takes the form of a few central supply points serving a large number of spatially dispersed demand points: for example, hospitals, schools, libraries, and emergency services such as fire and police. A fundamental characteristic of such systems is the spatial separation between suppliers and consumers. No market signals exist to identify efficient and inefficient geographical arrangements, thus the location problem is one that arises in both East and West, in planned and in market economies.

This problem is being studied at IIASA by the Normative Location Modeling Task, which started in 1979. The expected results of this Task are a comprehensive state-of-the-art survey of current theories and applications, an established network of international contacts among scholars and institutions in different countries, a framework for comparison, unification, and generalization of existing approaches, as well as the formulation of new problems and approaches in the field of optimal location theory.

This paper is a result of collaboration between the Human Settlements and Services Area and the Resources and Environment Area which is hosting Professor Erlenkotter at IIASA. It focuses on the dynamic uncapacitated facility location problem, presenting three equivalent formulations of the problem and setting out a computationally efficient branch-and-bound solution procedure based on the dual ascent method. The authors also discuss extensions of the fundamental problem to allow for price-sensitive demands, linearized concave costs, interdependent projects, and multiple commodities.

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ABSTRACT

In dynamic facility location problems, one desires to specify the time-staged establishment of facilities at different locations so as to minimize the total discounted costs for meeting demands specified over time at various customer locations. We formulate a particular dynamic facility location problem as a combinatorial optimization problem. The formulation permits both the opening of new facilities and the closing of existing ones. A branch-and-bound procedure incorporating a dual ascent method is presented and shown, in computational tests, to be superior to previously developed methods. The procedure is comparable to the most efficient methods for solving static (single-period) location problems. Problems with as many as 25 potential facility locations, 50 customer locations, and 10 time periods have been solved within one second of CPU time on an IBM 3033 computer. Extensions of the dynamic facility location problem that allow price-sensitive demands, linearized concave costs, interdependent projects, and multiple commodities can also be solved by the dual ascent method. The method can serve as a component of a solution process for more difficult capacitated problems.

CONTENTS

A DUAL-BASED PROCEDURE FOR DYNAMIC FACILITY LOCATION

1. INTRODUCTION

Dynamic facility location problems deal with size, location, and time-phasing decisions for establishment of productive capacity. With economies-of-scale in the costs for facilities of different sizes, these decisions involve two interacting trade-offs. In the spatial dimension, making facilities smaller and more dispersed decreases distribution costs but raises facility costs. In the time dimension, making facilities smaller also increases facility costs, but allows deferral of some investment and possibly a reduction in total discounted costs.

Often dynamic facility problems are simplified into more tractable problems. One approach considers only the time-phasing and capacity-sizing problem, but neglects the location problem (see, e.g., Manne, 1967). This procedure is acceptable for problems with relatively insignificant transportation costs. A second approach eliminates the dynamics of the location problem and solves a static (single period) location problem for a specific time period (see, e.g., Geoffrion and Graves, 1974). This procedure can be used for problems in which dynamic characteristics are negligible (e.g. constant demand), or for problems where location choices can be modified at low cost.

Only in recent years have solution methods for general dynamic location problems been developed. Roodman and Schwarz (1975, 1977) and Eschenbach and Carlson (1975) generalized to a dynamic context the methods proposed by Efroymson and Ray (1966) and Khumawala (1972) for the static location problem. Erlenkotter and Rogers (1977) devised a dynamic programming algorithm with transportation subproblems. Approximate methods for dynamic location problems have also been proposed; a comparison of several such methods is given in Erlenkotter (1979).

This paper addresses the *dynamic uncapacitated facility location problem*, which was introduced by Roodman and Schwarz (1975), and in a slightly different form by Wesolowsky and Truscott (1975). We give three formulations for the problem and demonstrate their equivalence. Although one formulation is identical in structure to a static uncapacitated problem, the others reduce substantially the requirements for computer storage. For these other formulations, we present a branch-and-bound solution procedure incorporating a dual ascent method that extends approaches developed by Bilde and Krarup (1977) and Erlenkotter (1978) for static uncapacitated problems. Comparative testing shows that this method is computationally more efficient than the methods proposed by Roodman and Schwarz, with problems having as many as 25 facility locations, 50 customer locations, and 10 time periods solved within one second on an IBM 3033 computer. A new primal-dual adjustment procedure improves the results given in Erlenkotter (1978) for "difficult" problems and appears to be quite effective in the dynamic setting. We discuss extensions of the basic problem to allow for price-sensitive demands, linearized concave costs, interdependent projects, and multiple commodities. Finally we indicate how the method may be used in solving more difficult capacitated dynamic facility location problems.

2. THE DYNAMIC UNCAPACITATED FACILITY LOCATION PROBLEM

The dynamic uncapacitated facility location problem (DUFLP) has the objective of minimizing total discounted costs for meeting demands specified in different time periods at various customer locations, where costs include those for operation of facilities over time at several possible sites and for production and distribution of goods from facilities to customers. This problem may be formulated as the following extension of the (single period or static) uncapacitated facility location problem (UFLP):

$$\text{(PA)} \quad \min_{x,y} \sum_t \sum_i \sum_j c_{ij}^t \, x_{ij}^t + \sum_t \sum_i f_i^t \, y_i^t \tag{1}$$

$$\sum_i x_{ij}^t = 1 \qquad \text{all } j,t \tag{2}$$

$$x_{ij}^t \le y_i^t \qquad \text{all } i,j,t \tag{3}$$

$$y_i^t \le y_i^{t+1} \qquad \text{all } i \in I_0,\ 1 \le t \le T-1 \tag{4}$$

$$y_i^t \ge y_i^{t+1} \qquad \text{all } i \in I_c,\ 1 \le t \le T-1 \tag{5}$$

$$x_{ij}^t \ge 0;\ y_i^t \in \{0,1\} \qquad \text{all } i,j,t \tag{6}$$

where

i indexes the facility;

j indexes the customer;

t indexes the time period;

I is the set of facilities : $I = I_0 \cup I_c$;

I_0 is the set of facilities that may be opened;

I_c is the set of facilities that may be closed;

J is the set of "pseudo" customers : $J = \{(jt)\}$;

T is the time horizon;

x_{ij}^t is the fraction of customer j's demand in time period t delivered from facility i;

y_i^t = 1(0) when facility i is open (closed) in time period t;

c_{ij}^t is the cost of producing and shipping customer j's total demand from facility i in time period t;

f_i^t is the fixed cost for operating facility i in time period t.

Customer demands are required to be met by (2); constraints (3) indicate that a customer j may be served by facility i only if facility i is open; constraints (4) require a facility $i \in I_0$ opened at time t_0 to remain open; similarly, the constraints (5) keep a facility $i \in I_c$ closed once it has been closed. Constraints (4) and (5) are complicating constraints that prevent solution of the problem by decomposing it into T (single period) UFLP's, one for each period. As in the UFLP, an optimal solution will have integer-valued x_{ij}.

The formulation addressed by Roodman and Schwarz (1975, 1977) differs from (PA) only by aggregating (summing) the constraints (3) over the index j for each i and t. The integer solution for such an aggregated formulation is the same as for (PA), but the linear programming relaxation which deletes the integrality restrictions in (6) generally is weaker than that for (PA).

An alternative formulation is given by (P):

$$\text{(P)} \quad \min_{x,z} \sum_t \sum_i \sum_j c_{ij}^t \, x_{ij}^t + \sum_t \sum_i F_i^t \, z_i^t \tag{7}$$

$$\sum_i x_{ij}^t = 1 \qquad \text{all } j,t \tag{8}$$

$$x_{ij}^t \leq \sum_{t' \varepsilon \bar{T}_{it}} z_i^{t'} \qquad \text{all } i,j,t \tag{9}$$

$$x_{ij}^t \geq 0 \; ; \; z_i^t \varepsilon \{0,1\} \qquad \text{all } i,j,t \tag{10}$$

where

$z_i^t = 1$, $i \varepsilon I_0$ implies facility i is open in period t and in the *following* periods;

$z_i^t = 1$, $i \varepsilon I_c$ implies facility i is open in period t and in all the *preceding* periods;

F_i^t, $i \varepsilon I_0$ gives all future fixed costs for opening facility i at t;

F_i^t, $i \varepsilon I_c$ gives all past fixed costs for having facility i open through t;

$\bar{T}_{it} = \{1,2, \ldots, t\}$ for $i \varepsilon I_0$;

$\bar{T}_{it} = \{t, t+1, \ldots, T\}$ for $i \varepsilon I_c$.

We demonstrate the equivalence between (P) and (PA) by defining:

$$z_i^t = y_i^t - y_i^{t-1} \geq 0 \qquad \text{for } i \varepsilon I_0, \text{ all } t \text{ and } y_i^0 \equiv 0$$

$$z_i^t = y_i^t - y_i^{t+1} \geq 0 \qquad \text{for } i \in I_C, \text{ all } t \text{ and } y_i^{T+1} \equiv 0.$$

Then

$$y_i^t = \sum_{t' \in \bar{T}_{it}} z_i^{t'} , \tag{11}$$

$$\sum_t f_i^t y_i^t = \sum_t f_i^t \sum_{t' \in \bar{T}_{it}} z_i^{t'} = \sum_t \Big(\sum_{t' \in T_{it}} f_i^{t'} \Big) z_i^t = \sum_t F_i^t z_i^t$$

where

$$T_{it} = \{t, t+1, \ldots, T\} \text{ for } i \in I_0$$

and

$$T_{it} = \{1, 2, \ldots, t\} \quad \text{for } i \in I_C ,$$

and (PA) becomes (P). Similarly (P) can be written as (PA) by using (11) and substituting also:

$$f_i^t = F_i^t - F_i^{t+1} \qquad \text{for } i \in I_0, \text{ all } t \text{ and } F_i^{T+1} \equiv 0$$

$$f_i^t = F_i^t - F_i^{t-1} \qquad \text{for } i \in I_C, \text{ all } t \text{ and } F_i^0 \equiv 0.$$

Although formulation (P) includes the static UFLP as a special case where T = 1, it also can be derived from a static UFLP formulation. This formulation is

$$\text{(PD)} \qquad \underset{x,z}{\text{Min}} \sum_t \sum_i \sum_j \sum_\tau c_{ij}^{t\tau} x_{ij}^{t\tau} + \sum_\tau \sum_i F_i^\tau z_i^\tau \tag{12}$$

$$\sum_i \sum_\tau x_{ij}^{t\tau} = 1 \qquad \text{all } j,t \tag{13}$$

$$x_{ij}^{t\tau} \leq z_i^{\tau} \qquad \text{all } i,j,t,\tau \qquad (14)$$

$$x_{ij}^{t\tau} \geq 0 \qquad \text{all } i,j,t,\tau \qquad (15)$$

$$z_i^{\tau} \in \{0,1\} \qquad \text{all } i,\tau \qquad (16)$$

where

$x_{ij}^{t\tau}$ is the fraction of customer j's demand in time period t delivered from the facility at i established in period τ;

z_i^{τ} is defined as in (P);

$c_{ij}^{t\tau}$ is the cost of producing and shipping customer j's total demand in period t from facility i established in time period τ;

F_i^{τ} is defined as in (P).

Formulation (PD) is clearly a "static" UFLP with pseudo facility locations ($i\tau$) and pseudo customers (jt). To create the "dynamic" structure, we specify the data as follows:

$$c_{ij}^{t\tau} = +\infty \qquad \text{for } t < \tau,\ i \in I_0$$

$$\text{and } t > \tau \quad i \in I_c \quad .$$

This ensures that a facility will not supply its customers before it is opened, or if initially opened, after it is closed.

To show that (P) is a special case of (PD), we note that (P) requires further that

$$c_{ij}^{t\tau} \equiv c_{ij}^{t} \qquad \text{for } \tau \in \bar{T}_{it} \; .$$

If we now define

$$x_{ij}^{t} = \sum_{\tau \in \bar{T}_{it}} x_{ij}^{t\tau}$$

clearly the optimal integer solutions for (P) and (PD) are identical. Furthermore, even though constraints (9) represent an aggregation of the constraints (14) in formulation (PD), the solutions to the linear programming relaxations of (P) and (PD), with $z_i^t \in \{0,1\}$ replaced by $0 \leq z_i^t \leq 1$, are also identical. This is seen by showing that the solution for the aggregated formulation (P) corresponds to one optimal for (PD) given the structure of the costs $c_{ij}^{t\tau}$. For $i \in I_0$, such a solution may be constructed recursively by defining $x_{ij}^{t\tau} = \min \{z_i^{\tau}, x_{ij}^{t} - \sum_{\tau' < \tau} x_{ij}^{t\tau'}\}$ for $\tau \leq t$. Clearly such a solution is feasible for (PD) and gives the same objective value since the $c_{ij}^{t\tau}$ are the same for all relevant τ. A similar procedure may be applied to $i \in I_c$.

We have shown that three seemingly different formulations of the DUFLP are equivalent, and one of the equivalent forms is a " static" UFLP. Each seems to offer somewhat different insight into the nature of the problem. However, before turning to solution methods, we point out that one significant shortcoming restricts the applicability of these and related formulations (Wesolowsky and Truscott, 1975). In the static UFLP, facility size decisions are determined simultaneously with location decisions and the capacities established are fully utilized. In the dynamic problem, capacities established in earlier periods become constraints on production in subsequent periods: full capacity utilization in every period is unlikely. By ignoring these capacity decisions, the DUFLP assumes that capacity adjustment in each period is perfectly flexible.

Effectively the DUFLP is a *project sequencing problem* (Erlenkotter and Rogers, 1977) that determines the order in which fixed costs should be incurred to open or close facility sites rather than the more general dynamic location problem addressed in Erlenkotter (1979). Although this degree of capacity flexibility may be unrealistic, the DUFLP can be used as a component of an approach that does incorporate capacity amounts explicitly as discussed in Section 5.

3. A DUAL ASCENT METHOD

Since the formulation (PD) provides an equivalent static UFLP, a solution of the DUFLP could be attempted by applying directly the dual-based method of Erlenkotter (1978) for the UFLP. However, there are two reasons for developing a modified approach. First, computer storage requirements for data are determined mainly by the number of cost coefficients: as many as $|I|\cdot|J|$ c_{ij}^{t}'s for formulation (P) and $|I|\cdot|J|\cdot(T+1)/2$ $c_{ij}^{t\tau}$'s for formulation (PD). Addressing formulation (P) economizes by avoiding the separate storage and processing of identical cost elements. In-core solution of the larger problems examined in Section 4 would not have been possible with formulation (PD). Second, in the construction of solutions and in the branching procedure, it is desirable to enforce explicitly the condition that a facility should be opened (or closed) no more than once. This condition is not exploited if the DUALOC code of Erlenkotter (1978) is used to solve (PD).

We solve (P) by a branch and bound method with lower bounds obtained via the linear programming relaxation of (P), with the integrality restrictions in (10) deleted. As in Bilde and Krarup (1977) and Erlenkotter (1978), instead of solving the LP relaxation optimally we use a heuristic dual ascent method, applied to a "condensed dual" of (P) or (PA). The condensed dual problem may be obtained by taking the dual of (P):

$$\max_{v,w} \sum_t \sum_j v_j^t$$

$$v_j^t - w_{ij}^t \leq c_{ij}^t \qquad \text{all } i,j,t$$

$$\sum_j \sum_{t' \varepsilon T_{it}} w_{ij}^{t'} \leq F_i^t \qquad \text{all } i,t$$

$$w_{ij}^t \geq 0 \qquad \text{all } i,j,t$$

where v and w are the vectors of dual variables corresponding to (8) and (9) respectively.

We may set

$$w_{ij}^t = \max \{0, v_j^t - c_{ij}^t\} \qquad \text{all } i,j,t \tag{17}$$

to obtain the *condensed dual problem*:

$$\text{(D)} \quad v(D) = \underset{v}{\text{Max}} \sum_t \sum_j v_j^t \tag{18}$$

$$\sum_j \sum_{t' \varepsilon T_{it}} \max \{0, v_j^{t'} - c_{ij}^{t'}\} \leq F_i^t \qquad \text{all } i,t\ . \tag{19}$$

Alternatively, (D) may be derived from (PD) as in Erlenkotter (1978) with terms having infinite $c_{ij}^{t\tau}$ deleted and the redundant index τ dropped for those remaining.

Define from (19) the slack variable

$$s_i^t \equiv F_i^t - \sum_j \sum_{t' \varepsilon T_{it}} \max \{0, v_j^{t'} - c_{ij}^{t'}\} \geq 0 \qquad \text{all } i,t\ ; \tag{20}$$

then the complementary slackness (CS) conditions for the LP relaxation of (P) are:

$$s_i^t z_i^t = 0 \qquad \text{all } i,t \tag{21}$$

$$w_{ij}^t\left(\sum_{t'\varepsilon\bar{T}_{it}} z_i^{t'} - x_{ij}^t\right) = 0 \qquad \text{all } i,j,t \tag{22}$$

$$v_j^t\left(\sum_i x_{ij}^t - 1\right) = 0 \qquad \text{all } j,t \tag{23}$$

$$x_{ij}^t\left(v_j^t - c_{ij}^t - w_{ij}^t\right) = 0 \qquad \text{all } i,j,t \quad . \tag{24}$$

The dual ascent procedure uses the condensed dual formulation (D) for finding a set of dual feasible $\{v_j^{t+}\}$ [and corresponding $\{w_{ij}^{t+}\}$ by (17)]. Then a primal feasible solution $\{z_i^{t+}\}$, $\{x_{i+j}^t\}$ can be constructed corresponding to $\{v_j^{t+}\}$ such that the CS conditions (21), (23), and (24) are satisfied and the number of violations of (22) is kept small. We first show how to construct a primal feasible solution $\{z_i^{t+}\}$, $\{x_{i+j}^t\}$. Define

$$I^* = \{(it) : s_i^{t+} = 0\}$$

and

$$I_t^* = \{i : (it') \varepsilon I^* \text{ and } t' \varepsilon \bar{T}_{it}\}$$

where s_i^{t+} is given by (20) and $v_j^t = v_j^{t+}$. Thus I_t^* denotes the candidate set of potential open facilities at time t. Also,

$$I_t^+ = \{i : \text{facility } i \text{ is open at time } t\}$$

$$I^+ = \{(it) : i \varepsilon I_{t'}^+ \text{ for some } t' \text{ and}$$

$$t = \min\ (\max)\ \{t' : i \varepsilon I_{t'}^+\} \text{ for } i \varepsilon I_0(I_C)\} \quad .$$

We then set $z_i^t = 1$ for each $(it)\ \varepsilon\ I^+$.

Notice that requiring $I^+ \subseteq I^*$ or $I_t^+ \subseteq I_t^*$ for all t implies that CS condition (21) is satisfied, and also that the definition for I^+ implies that a particular facility i is opened (closed) only once or it is not opened (closed) at all.

Now, we can present the *primal procedure* for constructing a primal feasible solution corresponding to $\{v_j^{t+}\}$. For a given, but arbitrary, sequence $(jt)_q$, $q = 1,\ldots|J|$, we perform:

step 1

a. set $I^+ = \emptyset$, $I_t^+ = \emptyset$; $(jt) \leftarrow (jt)_1$; $q \leftarrow 1$.

b. find I^*, I_t^* .

step 2

a. for each t, include in I_t^+ all $i \in I_t^*$ for which a single $v_j^{t'+} \geq c_{ij}^{t'}$ for some j and $t' \in \bar{T}_{it}$.

b. update I^+ .

step 3

a. for (jt), if there is no facility $i^+ \in I_t^+$ with $v_j^t \geq c_{i+j}^t$, augment $I_{t'}^+$, all $t' \in T_{it}$, with $i \in I_t^*$ having minimum c_{ij}^t for (jt) ; update I^+ .

b. $(jt) \leftarrow (jt)_{q+1}$; $q \leftarrow q + 1$; go to step 3a if $q \leq |J|$, otherwise go to step 4 with $(jt) \leftarrow (jt)_1$.

step 4

a. assign (jt) to facility $i \in I_t^+$ with lowest c_{ij}^t and record this assignment as i(jt) .

b. $(jt) \leftarrow (jt)_{q+1}$; $q \leftarrow q + 1$; go to step 4a if $q \leq |J|$, otherwise terminate .

As a result, the primal procedure gives a feasible integer solution with cost $v^+(P)$. The objective function value of the corresponding dual solution is denoted by $v^+(D)$. Note that the role of step 3 is, if possible, to keep the open set of facilities I^+ smaller than the eligible set I^*. As in Erlenkotter (1978), this will reduce the gap between $v^+(P)$ and $v^+(D)$.

The condensed dual problem (D) is the basis for a dual ascent procedure similar to those in Bilde and Krarup (1977) and Erlenkotter (1978) for solving the (static) uncapacitated facility location problem. As in (PD), one may interpret the condensed dual (D) as having pseudo customers (jt) and pseudo facilities (it), with fixed costs F_i^t. The objective is to increase the dual variables v_j^t associated with the pseudo customers until their sum is maximal, thereby absorbing the fixed costs F_i^t. To do so, we sort the demand costs c_{ij}^t for each pseudo customer (jt) into nondecreasing order, and we increase the dual variables v_j^t consecutively from one demand cost level to the next higher c_{ij}^t. Each time v_j^t is increased, the corresponding slacks $s_i^{t'}$ given by (20) are decreased until v_j^t is "blocked" from increasing by one or more zero slacks. The procedure is general in that it may start with any feasible solution $\{v_j^t\}$ to (D) and restrict changes to a subset of pseudo customers J^+. We reindex the c_{ij}^t for each pseudo customer (jt) in nondecreasing order as c_j^{tk}, $k = 1,2,\ldots k_{jt}$, where k_{jt} denotes the number of facility-to-customer links for (jt). The mapping i(kjt) indicates the original facility index. For convenience, we also include a dummy source with c_{ij}^t arbitrarily large for all (jt).

We now give the *dual ascent procedure*:

step 1

initialize with any feasible solution $\{v_j^t\}$ to (D), such that $v_j^t \geq c_j^{t1}$ for each (jt) and $s_i^t \geq 0$ for all (it). For each (jt), define $k(jt) = \min\{k : v_j^t \leq c_j^{tk}\}$. If $v_j^t = c_j^{tk(jt)}$, increase k(jt) by 1 .

step 2

initialize $(jt) \leftarrow (jt)_1$ and $q \leftarrow 1$; $\delta = 0$.

step 3

if $(jt) \not\in J^+$, go to step 7 .

step 4

set $\Delta_j^t = \min\limits_{it'} \{s_i^{t'} : v_j^t - c_{ij}^t \geq 0 \ , \ t' \in \bar{T}_{it}\}$.

step 5

if $\Delta_j^t > c_j^{tk(jt)} - v_j^t$, set $\Delta_j^t = c_j^{tk(jt)} - v_j^t$ and $\delta = 1$, and increase k(jt) by 1 .

step 6

decrease $s_i^{t'}$ by Δ_j^t for all i and $t' \varepsilon \bar{T}_{it}$
with $v_j^t - c_{it}^t \geq 0$; then increase v_j^t by Δ_j^t .

step 7

if $q \neq |J|$, $q \leftarrow q+1$, $(jt) \leftarrow (jt)_q$, return to step 3 .

step 8

if $\delta = 1$, return to step 2. Otherwise stop.

Different alternative sequences may be used for (jt) , q = 1,..., |J|.

When the dual ascent procedure terminates, a dual-feasible solution $\{v_j^{t+}\}$ to (D) is produced and a primal solution corresponding to $\{v_j^{t+}\}$ may be constructed with the primal procedure described above. If $v^+(P) = v^+(D)$ the integer primal solution has been verified as optimal and the algorithm is terminated. If $v^+(P) > v^+(D)$ we try to increase $v^+(D)$ by an adjustment procedure. Since $v^+(P) \neq v^+(D)$ there exists a "pseudo" customer (jt) with a CS violation of (22); that is, if we reduce v_j^{t+} by one unit the slack variables $s_i^{t'}$ of at least two "blocking" facilities (i.e. having $s_i^{t'} = 0$ and $z_i^{t'} = 1$) will be increased. As a result it may be possible that we can increase by one unit the dual variables of more than one pseudo customer and improve the dual objective value. To specify the procedure formally, we define the following additional notation:

$$I_j^{t*} = \{i : \exists t' \varepsilon \bar{T}_{it} \mid (it') \varepsilon I^* \text{ and } v_j^t \geq c_{ij}^t\}$$

$$I_j^{t+} = \{i : \exists t' \varepsilon \bar{T}_{it} \mid (it') \varepsilon I^+ \text{ and } v_j^t > c_{ij}^t\}$$

$$J_i^{t+} = \{(jt') : I_j^{t'*} = \{i\} \text{ for } t' \varepsilon T_{it}\} \quad .$$

If $|I_j^{t+}| > 1$, we have a CS violation; if $|I_j^{t+}| \leq 1$ for all (jt) then the primal solution corresponding to I^+ is optimal; if $|I_j^{t*}| = 1$ then a single constraint (19) blocks v_j^t from further increase. A best source i(jt) and a second-best source i'(jt) are given by

$$c_{i(jt)j}^t = \min_{i \in I_t^+} c_{ij}^t \qquad \text{for all (jt)}$$

$$c_{i'(jt)j}^t = \min_{\substack{i \in I_t^+ \\ i \neq i(jt)}} c_{ij}^t \qquad \text{for all (jt) with } |I_j^{t+}| > 1$$

and we define

$$c_j^{t-} = \max_i \{c_{ij}^t : v_j^t > c_{ij}^t\} \quad .$$

We now give the *primal-dual adjustment procedure*:

step 1

initialize $(jt) \leftarrow (jt)_1$, $q \leftarrow 1$; set $v_D = v^+(D)$ and $v_P = v^+(P)$; set $\delta = 0$.

step 2

if $|I_j^{t+}| \leq 1$ go to step 9 .

step 3

if $J_{i(jt)}^{t+} = \emptyset$ and $J_{i'(jt)}^{t+} = \emptyset$ go to step 9 .

step 4

for each (it') with $t' \varepsilon \bar{T}_{it}$ and $v_j^t > c_{ij}^t$, increase $s_i^{t'}$ by $v_j^t - c_j^{t-}$; then decrease v_j^t to c_j^{t-} .

step 5a

set $J^+ = J_{i(jt)}^{t+} \cup J_{i'(jt)}^{t+}$ and execute the dual ascent procedure.

5b

augment J^+ by (jt) and repeat the dual ascent procedure.

5c

set $J^+ = J$ and repeat the dual ascent procedure.

step 6

if v_j^t has not resumed its original value, return to step 2.

step 7

execute the primal procedure.

step 8

if neither $v^+(D) > v_D$ nor $v^+(P) < v_P$, set $\delta \leftarrow \delta + 1$; otherwise set $\delta = 0$ and update v_D and v_P .

step 9

if $v_D \geq v_P$, $\delta = \delta_{max}$, or $q = |J|$, stop; otherwise $(jt) \leftarrow (jt)_{q+1}$, $q \leftarrow q+1$, return to step 2.

Further improvement may be possible by repeating the primal-dual adjustment procedure as long as the dual objective value $v^+(D)$ improves.

This primal-dual adjustment procedure tends to be more efficient than the dual adjustment procedure of Erlenkotter (1978), even for static UFLP's. The essential difference is that the procedure in Erlenkotter (1978) constructs a primal solution only at the termination of the adjustment procedure, whereas here a primal solution is obtained at step 7 each time the dual solution is modified. For more difficult problems where typically $I^+ \neq I^*$, the procedure of Erlenkotter (1978) sometimes adjusts the dual solution assuming more facilities in I^* open than are required, and corresponding violations of complementary slackness condition (22) that are unnecessary. For the static problems of Erlenkotter (1978), incorporating the primal-dual adjustment procedure reduced the solution time for each of the two difficult 100 location problems by more than two-thirds, to slightly less than one second on an IBM 360/91 computer. The number of dual solutions required to solve the most difficult 33 location problem was reduced from 37 to 16. Similar performance on dynamic problems is reported in the next section.

A second innovation in the primal-dual adjustment procedure is the use of a counter δ for the number of times in succession that adjustment has been attempted with no primal or dual improvement, and the abandonment of adjustment if no improvement is attained after δ_{max} trials. Setting $\delta_{max} = 2$ worked well in experiments, and all computational tests were conducted with this value.

If $v^+(D) < v^+(P)$ after termination of the primal-dual adjustment procedure, we initiate a *branch and bound procedure* to complete the discovery and verification of an optimal solution. Subproblem separation is based on the y_i^t variables in formulation (PA): this permits effective restriction of other integer variables to values of zero or one in the branching process. An obvious choice for y_i^t in branching is the one that contributes the largest magnitude of complementary slackness violation. Initially the branching facility is always fixed open, and a LIFO backtracking scheme is used. The primal-dual adjustment procedure is repeated at the inital node as long as the dual value increases, and at all subsequent nodes only if the primal solution improves.

4. COMPUTATIONAL RESULTS

The solution procedure for the DUFLP has been implemented in a FORTRAN IV computer code, called DYNALOC, and tested on IBM 360/91 and IBM 3033 computers. The only DUFLP's for which comparative results are available are the problems solved by Roodman and Schwarz (1975, 1977). We have solved those problems, and in addition provide results for a new set of dynamic location problems developed from the static problems of Kuehn and Hamburger (1963).

Core memory requirements for DYNALOC can be calculated as

$$30{,}000 + 6\overline{MN}\ (T + 1) + 2N\ (5T + 2) + 4M\ (2T + 5) \text{ bytes}$$

where T is the number of time periods, N is the number of customers, M is the number of potential facility locations and $\overline{MN}$ is the total number of facility-customer links, assumed the same for each period. We store only possible facility-customer link costs, i.e., $\overline{MN} < M \times N$; for each time period. A 50 facility, 100 customer problem for 10 time periods and with an average of 15 potential facilities per customer ($\overline{MN} = 1500$) typically requires 150K bytes. For comparison, solution of such a problem with DUALOC (Erlenkotter, 1978) using the formulation (PD) would require 900K bytes.

In the presentation of the detailed results for problem sets, we shall use the following terminology: "*level*" denotes the maximum depth in the branch and bound tree needed to solve the problem; "*nodes*" gives the total number of explored nodes in the branch and bound tree; "*duals*" is the number of times the dual ascent procedure completed a solution $\{v_j^{t+}\}$.

Computational performance for DYNALOC on the Roodman-Schwarz problems is given in Table 1, together with their results for these problems. Problems 101-108 involve only facility phase-out

Table 1. Computational results for Roodman - Schwarz problems.

Problem number	Problem parameters			Optimal solution value	Roodman & Schwarz		DYNALOC			
	M	N	T		CPU I[a]	CPU II[a]	CPU[b]	Levels	Nodes	Duals
101	8	30	5	62970	42.9	-	.05	0	1	6
102	8	40	5	73473	40.8	-	.25	2	5	28
103	8	50	5	109824	127.8	-	.74	6	19	62
104	12	40	5	79893	314.2	160.6	.59	4	15	57
105	12	50	5	110742	654.3	302.0	.51	3	11	37
106	8	30	7	86418	184.7	-	.57	5	15	57
107	8	40	7	108224	521.7	211.8	1.65	8	31	131
108	8	50	7	149672	175.4	61.9	.28	0	1	19
205	12	30	5	38030	-	141.4	.45	4	9	71
206	12	30	8	49362	-	173.3	.60	4	13	53
207	15	30	8	50054	-	169.2	.20	0	1	20
208	15	30	6	31346	-	85.5	.01	0	1	1
209	15	30	1	6380	-	8.9	.01	0	1	5
210	25	50	1	10226	-	249.8	.08	1	3	20
Total CPU time (seconds)					1960.6[c]		5.99			

[a] CPU time in seconds on Honeywell G 635 as reported in Roodman and Schwarz (1975) for I and in Roodman and Schwarz (1977) for II.
[b] CPU time in seconds on IBM 3033, excluding input-output time.
[c] The total time is obtained by taking the best for approaches I and II.

(closing) decisions; 205-208 involve only phase-in (opening) decisions, and 209-210 are static UFLP's. (As is evident from formulation (PD), pure phase-out problems, with no constraints (4), may be transformed into equivalent pure phase-in problems, with constraints (5) absent, by renumbering the periods in reverse order.) Problems 106-108, listed by Roodman and Schwarz as 8 period problems, are indicated here as 7 period problems since in each case the last period has no demands. This period must be deleted to replicate their solutions, which have no facilities open in the last period, since constraint (2), also present in their formulation, does not permit such solutions. Even allowing for the slower computer used by Roodman and Schwarz, the results in Table 1 indicate that DYNALOC is faster by more than an order of magnitude.

Comparisons for some of these dynamic problems also demonstrate the previously discussed superiority of the primal-dual adjustment procedure to the (pure) dual adjustment procedure of Erlenkotter (1978). Table 2 presents comparative results for three of the Roodman-Schwarz problems.

Table 2. Comparison of dual versus primal-dual adjustment.

Problem	Primal-dual adjustment		Dual adjustment[a]	
	CPU[b]	Duals	CPU[b]	Duals
RS 102	.25	28	.64	85
RS 104	.59	57	1.74	194
RS 106	.57	57	7.62	927

[a] Erlenkotter (1978).
[b] CPU time in seconds on IBM 3033, excluding input-output time.

The second set of dynamic facility location problems was derived from those of Kuehn-Hamburger (1963) by specifying mutually offsetting demand growth and discount rates. Details of these problems are as follows:

1. all costs c_{ij}^t and F_i^t are calculated in dollars and rounded to integers;
2. $c_{ij}^1 = c_{ij}^2 = \ldots = c_{ij}^T$ for all i,j; defined as identical in all periods to the c_{ij}'s of the (static) Kuehn-Hamburger problems;
3. only facility opening decisions are considered ($I_c = \emptyset$);
4. the fixed costs are computed to compensate for the terminal effects of the finite time horizon. The fixed costs f_i given by the (static) Kuehn-Hamburger problems can be interpreted as the equivalent annual cost for investment. Then we compute the F_i^t's as:

$$F_i^t = \sum_{t'=t}^{T} \frac{f_i}{(1+r)^{t'-1}} \quad .$$

Hence r = 0.2 specifies an annual discount rate of 20%. Since the c_{ij}^t are held constant in terms of present value, the corresponding effective annual percentage demand increase is also 20%.

Each of the K-H problems has 24 potential warehouses and 50 demand locations in each period, not counting those warehouses available for zero fixed cost at factories. The first four have a factory and warehouse at Indianapolis, the second four at Jacksonville, the third four at both Baltimore and Indianapolis, and the last four a factory, but no warehouse, at Indianapolis.

Table 3 summarizes DYNALOC's performance on all the K-H problems with ten time periods and values for r of 0.1, 0.2, and 0.3. These 25 x 50 x 10 problems have 250 pseudo facilities and 500 pseudo customers. The CPU time on the IBM 3033 for a single problem ranges from 0.05 to 1.08 seconds, the average being 0.43 seconds. Only three of the 48 problems required branching.

Table 3. Computational results for 10-period Kuehn-Hamburger problems.

	r = 0.1					r = 0.2					r = 0.3				
Problem number	Optimal solution value	Levels	Nodes		CPU[a]	Optimal solution value	Levels	Nodes	Duals	CPU[a]	Optimal solution value	Levels	Nodes	Duals	CPU[a]
1	7580852	0	1	1	.05	7332055	0	1	1	.05	7172941	0	1	1	.05
2	8073654	0	1	27	.96	7731857	0	1	18	.72	7502799	0	1	16	.66
3	8443420	0	1	24	.81	8049263	0	1	22	.86	7769630	2	5	22	.94
4	8850519	0	1	27	.87	8417930	0	1	29	1.02	8086360	0	1	21	.80
5	10533198	0	1	8	.34	10267399	0	1	4	.17	10100440	0	1	5	.23
6	11021068	0	1	8	.31	10671522	0	1	5	.22	10432797	0	1	3	.13
7	11378062	0	1	3	.12	10990806	0	1	7	.29	10703746	0	1	5	.21
8	11823657	0	1	11	.35	11388979	0	1	11	.41	11049800	0	1	7	.27
9	5800386	0	1	2	.10	5560542	0	1	2	.10	5407175	0	1	2	.10
10	6227991	0	1	30	1.08	5919710	0	1	19	.76	5704843	0	1	16	.68
11	6519180	0	1	24	.75	6189709	1	3	28	1.08	5936713	2	5	23	.94
12	6846832	0	1	21	.62	6499958	0	1	25	.86	6211390	0	1	17	.63
13	7651525	0	1	1	.05	7389860	0	1	1	.05	7224883	0	1	1	.05
14	8180750	0	1	1	.06	7816381	0	1	1	.05	7574425	0	1	1	.05
15	8600269	0	1	4	.16	8167396	0	1	4	.16	7868040	0	1	3	.13
16	9098073	0	1	16	.51	8597387	0	1	14	.47	8232065	0	1	12	.45
Mean					.45					.45					.39

[a]Seconds on IBM 3033, excluding input-output time: overall mean CPU time is 0.43 sec.; the total charged CPU time including input and output for all problems was 22.87 sec., which yields mean total time of 0.47 sec.

5. EXTENSIONS OF THE BASIC PROBLEM (P)

In Sections 3 and 4 we presented a dual ascent method embedded in a branch and bound scheme for solving the dynamic facility location problem (P). Because of its structure a computationally efficient method could be developed. In this section, we investigate extensions that still may be solved by a dual ascent/ primal-dual adjustment procedure. In particular, we discuss the cases of price-sensitive demands, concave facility costs, interdependent projects, and multiple commodities. We also suggest how such a procedure may be employed to solve capacitated dynamic location problems.

For static UFLP's, it has been established in Erlenkotter (1977) that problems with *price-senstive demand functions* at the various demand locations can be converted into equivalent problems with fixed demands, corresponding to (PA) with T = 1. Exactly the same procedure may be applied to dynamic problems, where demand functions now are specified for each customer j in each time period t. The c_{ij}^{t} now would be interpreted as the negative of the optimal discounted benefit contribution assuming that facility i supplies customer j in period t. The presence of the linking constraints (4) and (5) has no effect on this transformation. Thus DYNALOC is applicable directly to DUFLP's with price-sensitive demands after the transformation is made.

It is customary to handle *concave facility costs* by a piecewise linear approximation, as illustrated in Figure 1. In the single period location problem (Efroymson and Ray, 1966), i.e., problem (P) with T = 1, we simply use the segments i0, i1, i2, ... as alternative facilities with fixed costs $\bar{f}_{i0}$, $\bar{f}_{i1}$, $\bar{f}_{i2}$, ... with the variable costs a_{i0}, a_{i1}, a_{i2} included in the respective customer costs c_{i0j}, c_{i1j}, c_{i2j}. Since concave costs imply $\bar{f}_{i0} < \bar{f}_{i1} < \bar{f}_{i2} < \ldots$ and $c_{i0j} > c_{i1j} > c_{i2j} > \ldots$, the optimal solution always has facility i0 open if throughput is between 0 and v_i; facility i1 open if throughput is between v_1 and v_2 etc. However, in the dynamic problem only facility i0 of the set {i0, i1, i2,...} must satisfy constraints (4) and (5) of (P).

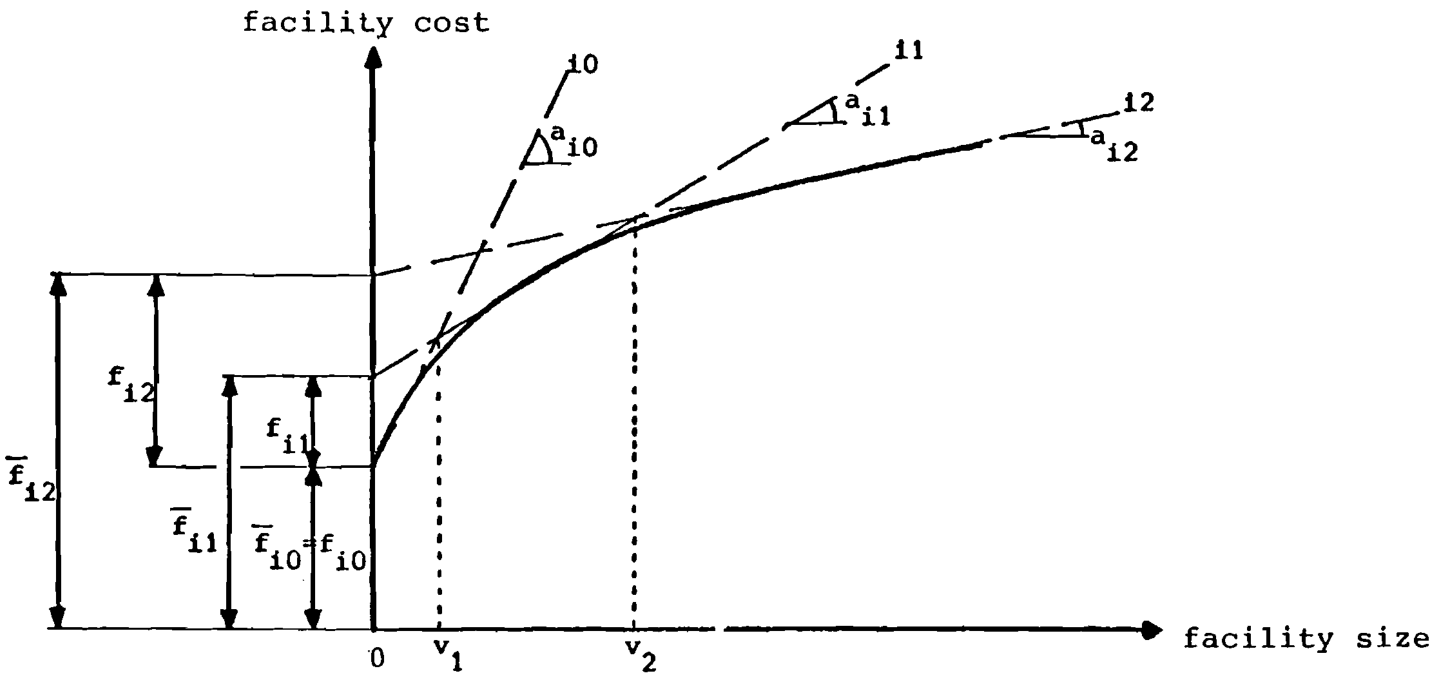

Figure 1. Concave facility costs.

Therefore, the following modifications to (P) are made:

1. use $f_{i0} = \bar{f}_{i0}$ as the fixed cost for the basic facility corresponding to the first segment of the linearization;
2. use $f_{ik} = \bar{f}_{ik} - f_{i0}$ as the fixed cost for facilities corresponding to subsequent segments;
3. require only the basic facilities i0 to satisfy constraints (4) and (5);
4. require the facilities ik ($k \neq 0$) to satisfy

$$y_{i0}^{t} \geq y_{ik}^{t} \qquad \text{all } t, k \neq 0, i,$$

or

$$\sum_{t' \in \bar{T}_{it}} z_{i0}^{t'} \geq y_{ik}^{t} \qquad \text{all } t, k \neq 0, i \quad . \tag{25}$$

Then the condensed dual becomes

$$v(D) = \underset{\substack{v \\ s \geq 0}}{\text{Max}} \sum_{j} \sum_{t} v_{j}^{t} \tag{26}$$

$$\text{s.t.} \quad \sum_{j} \sum_{t' \in T_{it}} \max\{0, v_{j}^{t'} - c_{i0j}^{t'}\} \leq F_{i0}^{t} - \sum_{k} \sum_{t' \in T_{it}} s_{ik}^{t'} \tag{27}$$

$$\text{all } i0, t$$

$$\sum_{j} \max\{0, v_{j}^{t} - c_{ikj}^{t}\} \leq f_{ik}^{t} + s_{ik}^{t} \qquad \text{all } t, k \neq 0, i \tag{28}$$

where s_{ik}^{t} is the dual variable associated with constraint (25). This is a combination of static and dynamic location problems. The dual ascent method of Section 3 can be implemented with minor modifications. Initially one sets all $s_{ik}^{t} = 0$. A plant ik, $k \neq 0$, cannot block unless the basic facility i0 blocks.

Interdependent projects frequently occur in dynamic location problems. Interdependencies arise, for example, in water resource planning problems where the output of a power plant depends on construction of a reservoir upstream. Some approaches, e.g., Erlenkotter and Rogers (1977), do not permit such interdependencies. As we will show later, interdependent facilities may also arise in multicommodity problems. To deal with such interdependencies we modify problem (P) by adding *logical constraints* of the type $y_i^t \leq y_j^t$. Again only minor changes to the dual ascent procedure are needed. For example, assume that facility A cannot be constructed unless facilities B and C are open, i.e., A only if B and C, or:

$$y_A^t \leq y_B^t \quad \text{and} \quad y_A^t \leq y_C^t \qquad \text{all } t \ ,$$

or

$$\sum_{t' \varepsilon \bar{T}_{At}} z_A^{t'} \leq \sum_{t' \varepsilon \bar{T}_{Bt}} z_B^{t'} \quad \text{and} \quad \sum_{t' \varepsilon \bar{T}_{At}} z_A^{t'} \leq \sum_{t' \varepsilon \bar{T}_{Ct}} z_C^{t'}$$

$$\text{all } t \ .$$

By introducing these additional constraints on the z variables, the condensed dual problem can easily be constructed. The modified constraints in (D) are:

$$\sum_j \sum_{t' \varepsilon T_{At}} \max(0, v_j^{t'} - c_{ij}^{t'}) \leq F_A^t + \sum_{t' \varepsilon T_{At}} (s_{AB}^{t'} + s_{AC}^{t'})$$

$$\sum_j \sum_{t' \varepsilon T_{Bt}} \max(0, v_j^{t'} - c_{ij}^{t'}) \leq F_B^t - \sum_{t' \varepsilon T_{Bt}} s_{AB}^{t'}$$

$$\sum_j \sum_{t' \varepsilon T_{Ct}} \max(0, v_j^{t'} - c_{ij}^{t'}) \leq F_C^t - \sum_{t' \varepsilon T_{Ct}} s_{AC}^{t'} \ .$$

To accommodate these constraints in the dual ascent method of Section 3, one initially sets all s_{AB}^t and s_{AC}^t equal to zero. Then blocking of customers by facility A is avoided by increasing s_{AB}^t or s_{AC}^t until B and C block.

We can also solve *multicommodity DUFLP's*, e.g., fire equipment location problems (Schilling, et al, 1979). Let i be the facility index and k the index for a type of equipment (commodity). Then the DUFLP can be formulated as:

$$\text{(MP)} \quad \min_{x\,y} \sum_{ti} f_i^t y_i^t + \sum_{tik} f_{ik}^t y_{ik}^t + \sum_{tijk} c_{ijk}^t x_{ijk}^t \qquad (29)$$

$$\sum_i x_{ijk}^t = 1 \qquad \text{all } t,j,k \qquad (30)$$

$$x_{ijk}^t \leq y_{ik}^t \qquad \text{all } t,i,j,k \qquad (31)$$

$$y_{ik}^t \leq y_i^t \qquad \text{all } t,i,k \qquad (32)$$

$$y_i^t \leq y_i^{t+1} \qquad \text{all } i \in I_0,\ 1 \leq t \leq T-1 \qquad (33)$$

$$y_i^t \geq y_i^{t+1} \qquad \text{all } i \in I_c,\ 1 \leq t \leq T-1 \qquad (34)$$

$$x_{ijk}^t \geq 0 \quad ; \quad y_i^t,\ y_{ik}^t \in \{0,1\} \quad . \qquad (35)$$

The identification of the symbols is as in Section 2. Formulation (MP) implies that facilities (buildings) cannot be opened and closed with perfect flexibility [see (33) and (34)]. On the other hand, equipment may be assigned with greater flexibility. Thus constraints (32), (33), and (34) can be handled as in the case with concave costs. If there exist types of equipment k which, once assigned to that facility, remain assigned to that facility, we add constraints of form (33) and (34) for all ik. Then the problem is one with interdependent "projects".

Finally, the dual ascent method can also be used for solving *capacitated dynamic facility location problems*, i.e.

$$\text{(CP)}\quad \min_{x,y} \sum_t \sum_i \sum_j c_{ij}^t x_{ij}^t + \sum_t \sum_i f_i^t y_i^t$$

s.t. (2), (3), (4), (5), (6), and

$$\sum_i d_j^t x_{ij}^t \le a_i^t y_i^t \qquad \text{all } i,t \tag{36}$$

where d_j^t is customer j's demand in period t, and a_i^t is facility i's delivery capability in period t. Guignard and Spielberg (1979) recently presented a dual ascent method for solving (CP) for a single period (T = 1). DYNALOC could be embedded in a similar approach for multiple periods. Since DYNALOC itself deals with the dynamic constraints (4) and (5), only minor algorithmic changes would be required. Similarly one can modify the procedure proposed by Van Roy and Gelders (1979) to deal with a (static) facility location problem with general side constraints of the form of (36). A procedure similar to that of Van Roy and Gelders (1979) would solve (CP) by a sequence of problems (P) derived from a Lagrangian relaxation of (36), each of which can be solved by DYNALOC. The same procedure could be followed for capacitated dynamic problems with price-sensitive demands (Erlenkotter and Trippi, 1976), where the Lagrangian problem would be transformed into an equivalent DUFLP as in Erlenkotter (1977).

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