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CATASTROPHE THEORY AND THE PROBLEM OF STELLAR COLLAPSE

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1. Introduction

Recently, a new mathematical tool called "catastrophe theory" has been developed by the topologists Thom, Zeeman, Mather, and others in an attempt to mathematically explain the discontinuities of observed behavior due to smooth changes in the basic parameters of physical, social, and biological processes. It has been shown that the number of mathematically distinct ways in which such discontinuities may arise is small when compared with the dimension of the process, and a complete classification of all distinct types has been made for processes depending upon five or less parameters.

The purpose of this note is two-fold: first, to serve as a very brief introduction to the subject of catastrophe theory and secondly, to illustrate the theory by applying it to the determination of equilibrium configurations for stellar matter which has reached the endpoint of thermonuclear evolution, the problem of "stellar collapse". It will be seen that catastrophe theory enables us to give a very satisfactory explanation for the observed phenomenon of unstable equilibrium configurations and the appearance of the so-called Chandrasekhar and Oppenheimer-Landau-Volkoff crushing points.

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2. Catastrophe Theory

In this section, we present a very brief discussion of the basic assumptions and results of catastrophe theory in a form most useful for applications. For details and proofs, we refer to the works [1-5].

Let $f = R^k \times R^n \to R$ be a smooth function representing a dynamical system Σ in the sense that R^k is the space of input variables (controls, parameters) while R^n represents the space of internal variables (states, behavior). We assume that $k \leq 5$, while n is unrestricted. The fundamental assumption is that Σ attempts to locally minimize f. We hasten to point out that in applications of catastrophe theory, it is not necessary to know the function f. In fact, in most cases f will be a very complicated function whose structure could never be determined. All we assume is that there exist such a function which Σ seeks to locally minimize.

Given any such function f, if we fix the point $c \in R^k$, we obtain a local potential function $f_c : R^n \to R$ and we may postulate a differential equation

$$\dot{x} = - \operatorname{grad}_x f,$$

where $x \in R^n$, $\operatorname{grad}_x f = \operatorname{grad} f_c = (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})$.

Thus, the phase trajectory of Σ will flow toward a minimum of f_c, call it x_c. The stable equilibria are given by the minima of f_c and, since there are usually several minima, x_c will be a multivalued function of c, i.e. $x_c : R^k \to R^n$ is <u>not</u> one-to-one.

The point of catastrophe theory is to analyze this multivaluedness by means of the theory of singularities of smooth mappings.

For completeness, and to round out the mathematical theory, we consider not only the minima, but also the maxima and other stationary values of f_c. Define the manifold $M_f \subset R^{k+n}$ as

$$M_f = \{(x,c): \mathrm{grad}_x\, f_c = 0\} .$$

let $\chi_f : M_f \to R^k$ be the map induced by the projection of $R^{k+n} \to R^k$. χ_f is called the catastrophe map of f. Further, let J be the space of C^∞-funtions on R^{k+n} with the usual Whitney C^∞-topology. Then the basic theorem of catastrophe theory (due to Thom) is

Theorem: There exists an open dense set $J_o \subset J$, called generic functions, such that if $f \in J_o$

i) M_f is a k-manifold;

ii) any singularity of χ_f is equivalent to one of a finite number of elementary catastrophes;

iii) χ_f is stable under small perturbations of f.

Remarks: 1) Here *equivalence* is understood in the following sense: maps $\chi: M \to N$ and $\bar{\chi}: \bar{M} \to \bar{N}$ are equivalent if there exist diffeomorphisms h,g such that the diagram

$$\begin{tikzcd} M \arrow[r, "\chi"] \arrow[d, "h"'] & N \arrow[d, "g"] \\ \bar{M} \arrow[r, "\bar{\chi}"'] & \bar{N} \end{tikzcd}$$

is commutative. If the maps $\chi, \bar{\chi}$ have singularities at $x \in M$, $\bar{x} \in \bar{M}$, respectively, then the singularities are equivalent if the above definition holds locally with $hx = \bar{x}$.

2) Stable means that χ_f is equivalent to χ_g for all g in a neighborhood of f in J (in the Whitney topology).

3) The number of elementary catastrophes depends only upon k and is given in the following table:

k	1	2	3	4	5	6
# elementary catastrophes	1	2	5	7	11	∞

A finite classification for k > 6 may be obtained under topological, rather than diffeomorphic, equivalence but the smooth classification is more important for applications.

3. Discontinuity, Divergence, and the Cusp Catastrophe

Our critical assumption is that Σ, the system under study, seeks to minimize the function f, i.e. Σ is dissapative. Thus, the system behaves in a manner quite different than the Hamiltonian systems of classical physics. In this section we shall mention two striking features displayed by catastrophe theory which are not present in Hamiltonian systems but which are observed in many physical phenomena.

The first basic feature is discontinuity. If β is the image in R^k of the set of singularities of χ_f, then β is called the bifurcation set and consists of surfaces bounding regions of qualitatively different behavior similar to surfaces of phase transition. Slowly crossing such a boundary may result in a sudden change of behavior of Σ, giving rise to the term "catastrophe". Since the dimension of Σ does not enter into the classification theorem, all information about when and where such catastrophic changes in output will occur is carried in the bifurcation set β which, by conclusion i) of the Theorem, is a k-manifold. Hence, even though Σ may have a state space of inconceivably high dimension, the "action" is on a manifold of low dimension which may be analyzed by geometrical and analytical tools.

The second basic feature exhibited by catastrophe theory is the phenomenon of divergence. In systems of classical physics a small change in the initial conditions results in only a small change in the future trajectory of the process, one of the classical concepts of stability. However, in catastrophe theory the notion of stability is with respect to perturbations of the system itself (the function f), rather than just the initial conditions and so the Hamiltonian result may not apply. For example, in an homogeneous embryo adjacent tissues will differentiate.

Let us now illustrate the above ideas by consideration of the cusp catastrophe. It will turn out that a minor modification of this catastrophe is also the appropriate catastrophe for the main example of this paper, the problem of stellar collapse.

Let $k = 2$, $n = 1$, and let the control and behavior space have cordinates a, b, x, respectively.

Let $f\colon\ R^2 \times R^1 \to R$ be given by

$$f(a,b,x) = \frac{x^4}{4} + \frac{ax^2}{2} + bx \quad .$$

The manifold M_f is given by the set of points $(a,b,x) \subset R^3$ where

$$\text{grad}_x\, f(a,b,x) = 0 \ ,$$

i.e.

$$\frac{\partial f}{\partial x} = x^3 + a\,x + b = 0 \quad . \tag{1}$$

The map χ_f: $M_f \to R^2$ has singularities when two stationary values of f coalesce, i.e.

$$\frac{\partial^2 f}{\partial x^2} = 3x^2 + a = 0 \quad . \tag{2}$$

Thus, Eqs. (1) and (2) describe the singularity set S of χ. It is not hard to see that S consists of two fold-curves given parametrically by

$$(a,b,x) = (-3\lambda^2, 2\lambda^3, \lambda) \ , \qquad \lambda \neq 0 \ ,$$

and one <u>cusp</u> singularity at the origin. The bifurcation set β is given by

$$(a,b) = (-3\lambda^2, 2\lambda^3)$$

which is the cusp $4a^3 + 27b^2 = 0$. Since M_f and S are smooth at the origin, the cusp occurs in β and not in S. Figure 1 graphically depicts the situation.

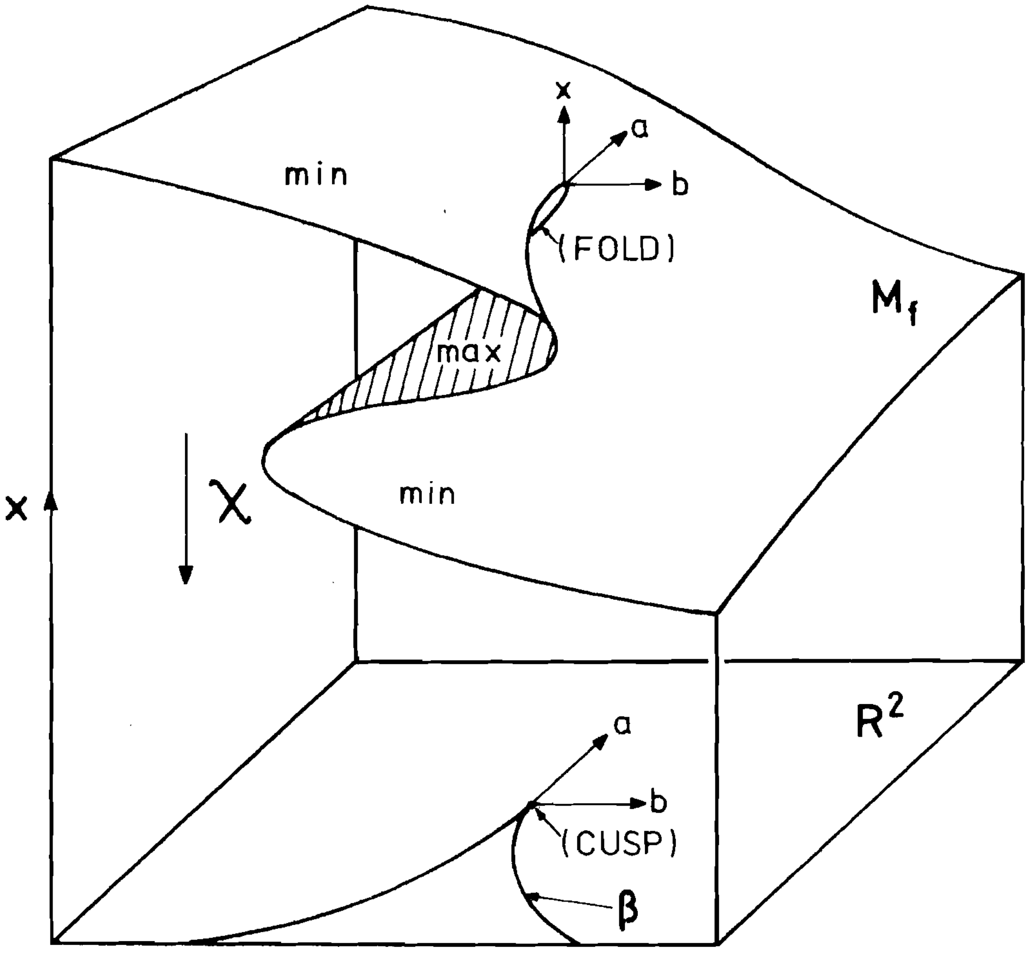

FIGURE 1. THE CUSP CATASTROPHE

It is clear from the figure that if the control point (a,b) is fixed <u>outside</u> the cusp, the function f has a unique minimum, while if (a,b) is inside the cusp, f has two minima separated by one maximum. Thus, over the inside of the cusp, M_f is triple-sheeted.

The phenomenon of smooth changes in (a,b) resulting in discontinuous behavior in x is easily seen from Figure 1 by fixing the control parameter a at some negative value, then varying b. At entrance to the inside of the cusp nothing unusual is observed in x, but upon further change in b, resulting in an exit from the cusp, the system will make a catastrophic jump from the lower sheet of M_f to the upper, or vice-versa, depending upon whether b is increasing or decreasing. The cause of the jump is the bifurcation of the differential equation $\dot{x} = -\text{grad}_x f$, since the basic assumption is that Σ always moves so as to minimize f. As a result, no position on the middle sheet of maxima can be maintained and Σ must move from one sheet of minima to the other.

An hysteresis effect is observed when moving b in the opposite direction from that which caused the original jump, i.e. the jump phenomenon will occur only when exiting the interior of the cusp from the side opposite to that where the cusp region was entered.

To see the previously mentioned divergence effect, consider two control points (a,b) with $a > 0$, $b \gtrless 0$. Maintaining the b values fixed, with decreasing a the point with positive b follows a trajectory on the lower sheet of M_f, while the other point moves on the upper sheet. Thus, two points which may have been arbitrarily close to begin with, end up at radically different positions depending upon which side of the cusp point they pass.

While the cusp is only one of several elementary catastrophes, it is perhaps the most important for applications. In Table 2, we list several other types for $k \leq 4$, but refer the reader to [6] for geometrical details and applications.

Name	potential function f	control space dimension	behavior space dimension
fold	$x^3 + ux$	1	1
cusp	$x^4 + ux^2 + vx$	2	1
swallowtail	$x^5 + ux^3 + vx^2 + wx$	3	1
butterfly	$x^6 + ux^4 + vx^3 + wx^2 + tx$	4	1
hyperbolic umbilic	$x^3 + y^3 + uxy + vx + wy$	3	2
elliptic umbilic	$x^3 - xy^2 + u(x^2+y^2) + vx + wy$	3	2
parabolic umbilic	$x^2y + y^4 + ux^2 + vy^2 + wx + ty$	4	2

Table 2: The Elementary Catastrophes for $k \leq 4$.

4. Stellar Collapse

We turn now to the main application of this paper, the determination of equilibria configurations for stars which have exhausted their nuclear fuel and have entered the collapsing stage of their existence. It is a well observed phenomenon that there exist several possible final configuations for such stars, depending upon their initial mass and internal pressures. According to present theory, the possible stable final states are white dwarfs, neutron stars, and black holes.

Before showing the relevance of catastrophe theory, let us briefly summarize the steps in the analysis of equilibrium configurations as outlined in [7]. We consider the equation of hydrostatic equilibrium

$$-\frac{dp}{dr} = \frac{\rho(r)GM(r)}{r^2} ,$$

where M(r) is the mass effective in producing gravitational pull at the distance r, i.e. the mass included within the sphere of radius r, $\rho(r)$ is the density of matter at radius r, G is the gravitational constant, and p(r) is the pressure at radius r.

One catalogs equilibrium configuations by the value of the central density in the following way: the central density ρ_0 is fixed and Eq. (3) is integrated from r = 0 to r = r*, where r* is that value such that p(r*) = 0. The value M(r*) is the total mass. Another value of the central density is then chosen and the process repeated. In Fig. 2, the curve of mass as a function of central density is displayed:

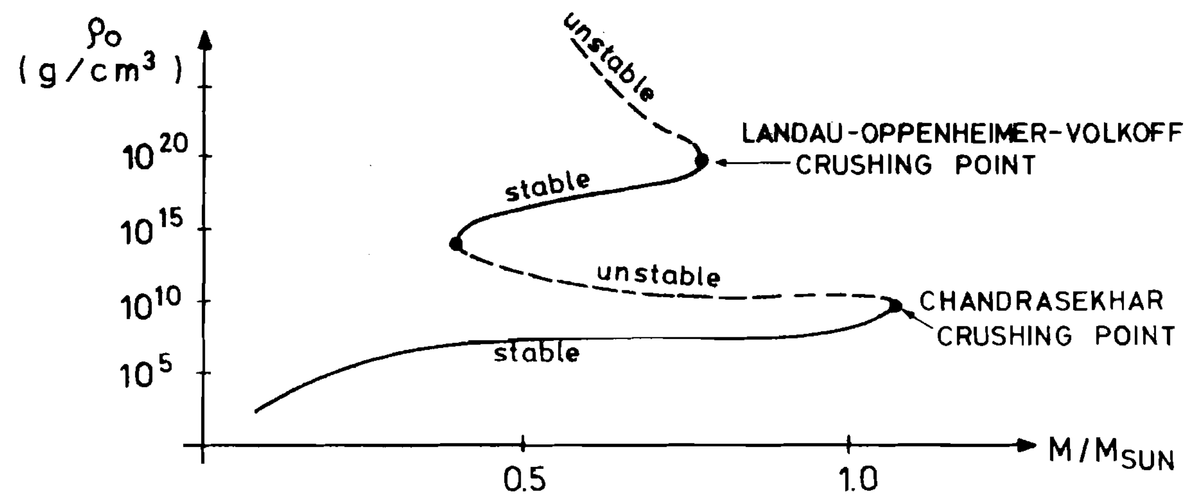

FIGURE 2. EQUILIBRIUM CONFIGURATIONS OF STELLAR MATTER [7]

From the standpoint of physics, as well as catastrophe theory, the most interesting features of Fig. 2 are the two crushing points separating the stable and unstable equilibrium configurations. In all cases, the situation in which there is a decrease in total mass with increasing central density signifies an unstable system.

Upon comparing Figs. 1 and 2, we observe that for a = constant < 0 in Figure 1, the "slice" of M_f for constant b gives a representation for the curve of Fig. 2, omitting the unstable region of densities beyond the Landau - Oppenheimer - Volkoff (L-O-V) point. Thus, taking the control space to be the mass-pressure plane and the behavior space as the central density, we postulate that all of the structural information in Fig. 2 (and more) may be accounted for by a minor modification of the cusp catastrophe. The modification is necessary to describe the unstable positions beyond the L-O-V point, as well as the stable configuations of even greater densities which are not depicted.

In order to account for all positions, in effect we use two copies of Fig. 1, suitably glued together, to form a single manifold having two cusps and four folds. The geometrical picture is shown in Fig. 3. From the standpoint of current theory, the most striking feature of Fig. 3 is the unstable region between neutron stars and black holes. It's not entirely clear what the proper physical interpretation of this region should be, but it's most likely a transition phase corresponding to a star on the borderline of becoming a black hole, a situation highly dependent, of course, on the pressure/mass relationship as is clearly indicated by the diagram. In addition, the cusp catastrophe shows that the Chandrasekhar and L-O-V critical points are actually bifurcation sets in the mass/pressure plane corresponding to the branches of the cusps.

What interpretation to attach to the cusp points themselves is also unclear. In Fig. 3, the pressure/mass/density coordinate axis is drawn to indicate that a catastrophe will occur for all positive pressures and masses in accordance with current theory;

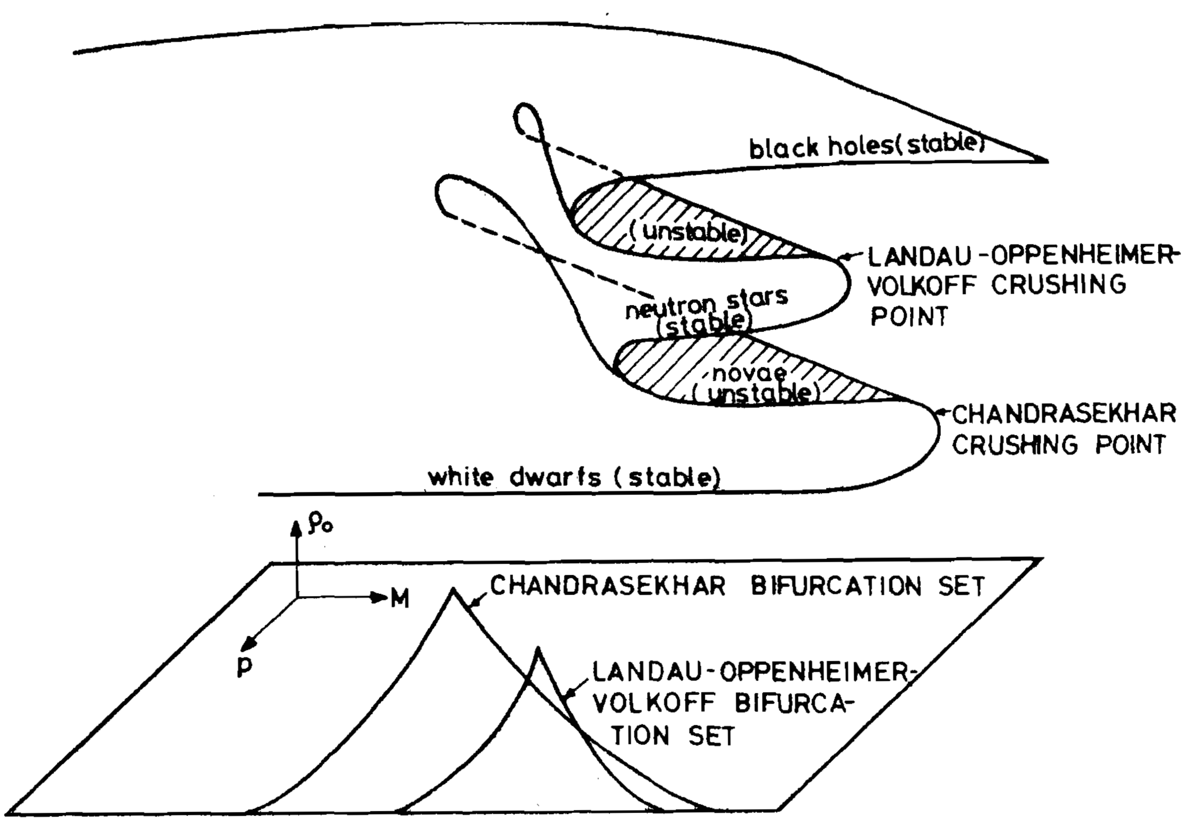

FIGURE 3.CATASTROPHE MANIFOLD FOR STELLAR COLLAPSE

however, the mathematics would admit of a smooth transition from the white dwarf region to the black holes should future observation "translate" the coordinate frame in the direction of negative p. It would be an interesting and worthwhile exercise to precisely locate the coordinate frame and to determine the precise equations for the Chandrasekhar and L-O-V bifurcation sets by means of the currently available data.

5. Discussion

In this note, we have shown that catastrophe theory may be useful in synthesizing the global picture of various physical phenomena. Obviously, there are many similar situations in which such a picture may prove to be useful in explaining observations and (hopefully) in predicting new ones. For some examples along these lines see [3,4]. In future reports, we shall investigate the applicability of some of the other catastrophes listed in Table 2 to physical phenomena, especially those areas where "phase transition" - type behavior plays an important role.

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