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NECESSARY AND SUFFICIENT CONDITIONS IN THE MINIMAL CONTROL FIELD PROBLEM FOR LINEAR SYSTEMS

J. Casti

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I. Introduction

In the article [1], the following problem was posed: given the dynamical system

$$\dot{x}(t) = f(x,u,t) \quad , \qquad x(o) = c \quad , \qquad (*)$$

where x is an n-dimensional state vector, u is an m-dimensional control vector, and f is an n-dimensional vector function smooth enough to insure a unique solution to (*) for all piecewise continuous u, $t > 0$ determine a feedback control u, i.e. $u = u(x,t)$ such that i) using the feedback law $u(x,t)$, (*) is asymptotically stable and ii) the argument u contains the minimal number of components of x consistent with i). Obviously, this is a very complex problem and a clear-out solution for general f seems out of reach at present. Even for linear f and constant coefficients, i.e. $f(x,u,t) = Fx + Gu$, the problem is complicated by the fact that the solution is not invariant under coordinate transformations. Nonetheless, for the linear case some headway has been made. In [2] upper bounds for the necessary number of components of x that need appear (the "dimension" of the control field) are derived making use of special control laws. These bounds are easily

*International Institute for Applied Systems Analysis, Laxenburg, Austria and University of Arizona, Tucson, Arizona, U.S.A.

computable in terms of the characteristic vectors of F. If the condition i) were replaced by i') given a symmetric set of complex numbers Λ (i.e. if $\lambda \epsilon \Lambda$, then $\bar{\lambda} \epsilon \Lambda$), determine a linear feedback control law K of "minimal dimension," such that F + GK has Λ as its characteristic values, then a complete solution is given in [3]. This version of the problem substitutes a rigid placement of the controlled system's characteristic roots for the much weaker requirement of asymptotic stability.

In the current note, we return to the original problem (linear version) and give necessary and sufficient conditions for the eliminations of measurements of certain state variables in a stabilizing linear feedback law. Unfortunately, it does not seem possible to give any <u>single</u> set of operationally useful conditions which are both necessary and sufficient. However, we do present one set of necessary conditions and another set of sufficient conditions which are readily checked using the original problem data (the matrices F and G). Examples are also presented showing that, in some cases, these conditions do enable us to precisely determine the "minimal control field."

The standard method for coping with the foregoing "incomplete" measurement problem is to construct a so-called "observer" [4]. As is well known, the observer compensates for inabilities to measure certain parts of the state and gives the same asymptotic results as for the case of complete state information. However, from a practical engineering point of view, the introduction of an observer may be objectionable on economic as well as technical grounds. The added hardware and circuitry needed for the observer increases the cost, weight, size, and complexity of the sys'ems under design. Consequertly, it seems preferable to first analyze the system from the viewpoint presented above, and afterwards introduce an observer for those components of x which cannot be completely eliminated by the results of this paper and which are not physically measurable.

II. Necessary Conditions

a) Single-Input Systems

To begin with, consider the single-input linear system

$$\dot{x} = Fx + gu \quad , \qquad x(0) = c \quad , \qquad (\Sigma)$$

where x is an n-dimensional vector, g is an n-dimensional constant vector, u is a scalar control function, and F is an nxn constant matrix. We seek conditions such that a linear feedback control law k, i.e. $u = k'x$, may stabilize (Σ) and have some zero components (to avoid trivialities, we assume that F has at least one root with non-negative real pa t).

To express the basic necessary condition, the following notation will prove useful. Let A be an nxn matrix and b an nx1 vector. Then the quantity $[A|b]^{(i)}$ will denote the matrix A with its i^{th} column replaced by b, i.e.

$$[A|b]^{(i)} = \begin{bmatrix} a_{11} & \cdots & a_{1,i-1} & b_1 & a_{1,i+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2,i-1} & b_2 & a_{2,i+1} & \cdots & a_{2n} \\ \vdots & & & \vdots & & & \vdots \\ a_{n1} & \cdots & a_{n,i-1} & b_n & a_{n,i+1} & \cdots & a_{nn} \end{bmatrix} .$$

Furthermore, let $S_j(A)$ denote the sum of the principal minors of order j of the matrix A and let $\sigma_j([A|b]^{(i)})$ denote the sum of the principal minors of order j, <u>which contain components of b</u>, of the matrix $[A|b]^{(i)}$, $j = 1,2,\ldots,n$. For example, $S_1(A)$ = trace A, but $\sigma_1([A|b]^{(i)}) = b_i$. It is well known that the characteristic polynomial of F may be expressed as

$$\begin{aligned} \chi_F(z) &= z^n + \sum_{j=1}^{n} (-1)^j \; S_j(F) \; z^{n-j} \\ &= z^n + \sum_{j=1}^{n} a_j \; z^{n-j} \quad . \end{aligned}$$

Our necessary condition for minimality of the control law stabilizing Σ will be based upon the well known necessary condition for χ to be a stability polynomial--that all coefficients be positive. Since we are interested in the smallest number of components of x which can generate a stabilizing law, let us first consider the necessary condition for the dimensions of the minimal field to be one. This condition is given by

Theorem 1. *Let $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ be the non-positive coefficients of $\chi_F(z)$. Then a necessary condition for Σ to be stabilizable by a linear law* $k' = (0\ 0 \cdots 0\ k_j\ 0 \cdots 0)$ *is*

$$\sigma_i([F|g]^{(j)}) \neq 0 \quad , \qquad i = i_1, i_2, \ldots, i_k \quad .$$

Proof. Using a control law of the above form, the $i^{\underline{th}}$ coefficient in $\chi_{F+gk'}(z)$ is

$$a_i' = a_i + (-1)^i\ k_j \sigma_i([F|g]^{(j)}) \quad . \tag{1}$$

Hence, if $a_i \leq 0$, $\sigma_i([F|g]^{(j)})$ must be non-zero in order for any choice of k_j to influence the magnitude of a_i'.

Remarks. (i) The condition of Theorem 1 only enables us to eliminate certain components of x as potential "single-measurement" stabilizers. However, in view of (1), the string of inequalities implied by Theorem 1 is actually somewhat stronger since we must have

$$(-1)^i\ [S_i(F) + k_j \sigma_i([F|g]^{(j)}) > 0 \quad , \tag{2}$$

$$i = i_1, i_2, \ldots, i_k \quad .$$

As the example below illustrates, these stronger inequalities impose additional restrictions on k_j that may result in the further elimination of candidate components of x that satisfy the necessary condition of Theorem 1. For example, if no number k_j satisfies (2), then component j cannot stabilize Σ even if $\sigma_i([F|g]^{(j)}) \neq 0$ for $i = i_1, i_2, \ldots, i_k$.

ii) To test whether a "two-measurement" control field is possible, that is, a law of the form $k' = (0\ 0 \cdots 0\ k_j\ 0 \cdots 0\ k_\ell\ 0 \cdots 0)$, it is easy to see that the $i^{\underline{th}}$ coefficient of $\chi_{F+gk'}(z)$ is

$$a_i' = a_i + (-1)^i\, k_j\, \sigma_i([F|g]^{(j)}) + (-1)^i\, k_\ell\, \sigma_i([F|g]^{(\ell)}) \quad .$$

Thus, the appropriate necessary condition is $|\sigma_i([F|g]^{(j)})| + |\sigma_i([F|g]^{(\ell)})| > 0$, $i = i_1, i_2, \ldots, i_k$. Now instead of the single constraining inequality (2), we have

$$(-1)^i\ [S_i(F) + k_j\,\sigma_i([F|g]^{(j)}) + k_\ell\,\sigma_i([F|g]^{(\ell)})] > 0 \quad ,$$

$$i = i_1, i_2, \ldots, i_k \quad . \qquad (3)$$

Again, (3) may impose enough additional constraints that some possibly feasible components of x are eliminated.

iii) Unfortunately, the foregoing conditions become rather unwieldly for high-dimensional systems when we want to test feasibility of a control field of dimension greater than one or two, since the number of combinations of components grows at an alarming rate, i.e. factorially. However, since all the operations and inequalities are linear, it should be possible to computationally check all possibilities for systems of moderate size--e.g. if $n = 20$ and we want to check the possibility of stabilizing by 10 components of x, we have $\binom{20}{10} = 184{,}756$ combinations to check (the worst case) and if it takes 10^{-3} seconds to check one combination, then the total time required is about 3 minutes.

Example. To illustrate application of Theorem 1 and its consequences, consider the system Σ described by the matrices

$$F = \begin{bmatrix} -10 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad g = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} .$$

By inspection, it is clear that this system can be stabilized by measurements of x_3 alone but it is instructive to use Theorem 1 to obtain this result. We have

$$\chi_F(z) = z^3 + 10z^2 - 5z - 38 \quad .$$

Since $a_2 < 0$, $a_3 < 0$, we have (in the notation of Theorem 1), $k = 2$, $i_1 = 2$, $i_2 = 3$. Our conjecture is that a control of the form $k' = (0\ 0\ k_3)$ will stabilize. Calculating the relevant quantities, we have

$$S_2(F) = -5 \quad ,$$

$$S_3(F) = 38 \quad ,$$

$$\sigma_2([F|g]^{(3)}) = \det \begin{bmatrix} -10 & 1 \\ 0 & 1 \end{bmatrix} + \det \begin{bmatrix} -2 & 1 \\ 0 & 1 \end{bmatrix} = -12 \neq 0$$

$$\sigma_3([F|g]^{(3)}) = \det \begin{bmatrix} -10 & 1 & 1 \\ 1 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix} = 19 \neq 0 \quad .$$

Thus, the necessary conditions of Theorem 1 are satisfied.

The strengthened condition (2) yields

$$-38 - 19\, k_3 > 0 \ ,$$

$$-5 - 12\, k_3 > 0 \ ,$$

$$10 - \quad k_3 > 0 \ ,$$

which together imply $k_3 < -5/12$.

It is easily calculated that the necessary conditions of Theorem 1 are <u>also</u> satisfied for measurements of x_1 alone or x_2 alone. However, the inequalities (2) yield

<u>x_1 alone</u>: $10 - k_1 > 0$, $-5 - k_1 > 0$, $6k_1 - 38 > 0$ which are inconsistent;

<u>x_2 alone</u>: $10 - k_2 > 0$, $-9k_2 - 5 > 0$, $22k_2 - 38 > 0$, which are also inconsistent. Thus, the only possibility to stabilize Σ by measuring a single coordinate is to measure x_3.

To check that indeed Σ can be stabilized by measurements of x_3, we use the necessary and sufficient conditions

$$a_1' a_2' > a_3' \ , \qquad a_1' > 0 \ , \qquad a_2' > 0 \ , \qquad a_3' > 0 \ .$$

We need only check the first of these inequalities as the others have already been seen to be fulfilled when $k_3 < -5/12$. The relevant inequality is

$$(10 - k_3)\,(-5 - 12k_3) > 2 - 19k_3$$

or

$$k_3^2 - 8k_3 - \frac{13}{3} > 0 \ .$$

This inequality is satisfied for

$$4 - \frac{1}{2}\sqrt{244/3} < k_3 < 4 + \frac{1}{2}\sqrt{244/3} \quad .$$

Since $\sqrt{244/3} \cong 9.018$, we see that any value of k_3 in the range

$$-1/2 - \varepsilon < k_3 < -5/12 \quad ,$$

will stabilize Σ, where $\varepsilon = \sqrt{244/3} - 9 \cong 0.018$.

b) Multiple-Input Systems

Now we assume that Σ has $1 < m \leq n$ inputs, i.e. G is an nxm matrix and K is an mxn matrix. Again let $a_{i_1}, \ldots, a_{i_k}$ be the non-positive coefficients of $\chi_F(z)$. Then the counterpart of Theorem 1 is

Theorem 1'. A necessary condition for Σ to be stabilizable by a linear law

$$K = \begin{bmatrix} 0 & \cdots & 0 & k_{1j} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & k_{2j} & 0 & \cdots & 0 \\ \vdots & & \vdots & & & & \vdots \\ 0 & \cdots & 0 & k_{mj} & 0 & \cdots & 0 \end{bmatrix}$$

is

$$\sum_{r=1}^{m} |\sigma_i([F|G^{(r)}]^{(j)})| > 0 \quad , \qquad i = i_1, i_2, \ldots, i_k \quad ,$$

where $G^{(r)}$ denotes the r^{th} column of G.

Proof. It is easily verified that the i^{th} coefficient in $\chi_{F+GK}(z)$ is

$$a_i' = a_i + \sum_{\ell=1}^{m} k_{\ell j}\, \sigma_i([F|G^{(\ell)}]^{(j)}) \quad .$$

Consequently, no measurements on x_j can influence a_i' if all the terms $\sigma_i([F|G^{(\ell)}]^{(j)})$ vanish, $\ell = 1,2,\ldots,m$.

In a completely analagous manner, we can obtain similar results for the case of more than one state measurement.

III. Sufficient Conditions

To derive sufficient conditions for the elimination of state variables from a stabilizing feedback law, we utilize the integral equation representation for the solution of Σ:

$$x(t) = e^{Ft}c + \int_0^t e^{F(t-s)}GK\, x(s)\, ds \quad .$$

We now assume that F is a normal matrix, i.e. F is diagonalizable as

$$F = T\, \mathrm{diag}(\lambda_1,\lambda_2,\ldots,\lambda_n)\, V' \quad ,$$

where $\mathrm{Re}\lambda_1 > \mathrm{Re}\lambda_2 > \cdots > \mathrm{Re}\lambda_p \geq 0 > \mathrm{Re}\lambda_{p+1} > \cdots > \mathrm{Re}\lambda_n$.

Standard results in the theory of integral equations allow us to express x(t) in the form

$$x(t) = e^{Ft}c + \int_0^t R(t,s)\, e^{Fs}c\, ds \quad , \tag{4}$$

where the resolvent kernel R(t,s) satisfies the integral equation

$$R(t,s) = e^{F(t-s)}GK + \int_0^t e^{F(t-s')}GK\, R(s',s)\, ds' \quad .$$

The representation (4) plus the normality assumption on F enable us to write the i^{th} component of x as

$$x_i(t) = \sum_{j=1}^{n} e^{\lambda_j t} (c, V_{(j)})\, t_{ij} + \int_0^t \sum_{j=1}^{n} r_{ij}(t,s) \left[\sum_{\ell=1}^{n} e^{\lambda_\ell s} (c, V_{(\ell)}) t_{\ell j} \right] ds \quad , \tag{5}$$

$$i = 1,2,\ldots,n \quad ,$$

where $T = [t_{ij}]$, $V_{(j)}$ is the j^{th} row of V, and (,) denotes the usual vector inner product. We are now in a position to assert

Theorem 2. Assume that $t_{ij} = 0$, $j = 1,2,\ldots,p$ and that Σ is controllable. Then component x_i may be omitted from a stabilizing linear feedback law.

Proof. If $t_{ij} = 0$, $j = 1,\ldots,p$, no increasing exponentials appear in the first term of (5). Also, the condition implies that no increasing exponentials will appear in any of the integrals which involve the i^{th} column of R. Thus, in selecting a stabilizing K, we may choose it so that the i^{th} column of R is zero. However, the integral equation for R shows that this implies that the i^{th} column of K may be chosen equal to zero, i.e. component x_i does not appear in the law K.

Example. To illustrate Theorem 2, let Σ be formed from the matrices

$$F = \begin{bmatrix} -2 & -1 & 1 \\ -3 & -2 & 3 \\ -3 & -1 & 2 \end{bmatrix} , \qquad g = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} .$$

Then $\chi_F(z) = (z - 1)(z + 1)(z + 2)$ and

$$F = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$= TAV' \quad .$$

For this example, $p = 1$ and the form of T indicates that it is possible to stabilize F by a control law of the form $k = (0\ k_2\ k_3)$, i.e. with the component x_1 not appearing. To check this, we compute the characteristic polynomial of $(F + gk')$ obtaining

$$\chi_{F+gk'}(z) = z^3 + (2 - k_3)z^2 + (-4 - 3k_2 - 1)z + (5k_3 - 3k_2 - 2) \quad .$$

Application of the Hurwitz conditions shows that for stability it is necessary and sufficient that k_2 and k_3 satisfy

$$k_3 < 2 \quad , \quad 4k_3 + 3k_2 < -1 \quad , \quad 5k_3 - 3k_2 > 2$$

$$4k_3^2 - 3k_2k_3 - 12k_3 + 3k_2 > 0 \quad .$$

It is not hard to see that there are many solutions to this system, e.g. $k_2 = -2$, $k_3 = (6 - \sqrt{132})/8$ is one. It is also interesting to observe that the above inequalities have <u>no</u> solution if either $k_2 = 0$ or $k_3 = 0$, i.e. both x_2 and x_3 must appear in the control law to stabilize Σ.

IV. Discussion

The preceding results offer the possibility to precisely determine the "dimensions" of the minimal control field for many linear systems. As has been emphasized, the conditions presented are not simultaneously necessary and sufficient. It should be noted, however, that the necessary conditions of Theorem 1 may be combined with the usual Hurwitz conditions to form a single set of necessary and sufficient conditions. But, checking the sufficiency requires checking the consistency of a set of polynomial inequalities which, for large systems, is an operationally intractable situation. The most practical way to combine Theorems 1 and 2 is to use Theorem 2 to determine those components of x which may certainly be eliminated from the control law, then employ Theorem 1 to check the remaining components for possible elimination. This procedure also has the desirable side benefit of reducing the number of combinations needed for applications of Theorem 1 from $\sum_{k=1}^{n} \binom{n}{k}$ to $\sum_{k=1}^{n-m} \binom{n-m}{k}$, where m is the number of components of x which are eliminated by Theorem 2.

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