TOWARD A GLOBAL MODEL: A METHODOLOGY FOR CONSTRUCTION AND LINKAGE OF LONG-RANGE NORMATIVE DEVELOPMENT MODELS

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I. Introduction

In recent years an increasing activity in the field of modelling of development processes on the regional, national, and global level can be observed.

In global models a descriptive approach is commonly used while in the regional and national models an attempt is being made to employ the normative concepts which are regarded as rather useful tools for long-range planning purposes.

It should be also noted that the global modellers usually disregard, for the sake of simplicity, the national differences in a similar way as the national macro-model builders avoid the regional aspects. As a result, the projections derived by the use of global models are usually too general or not accurate enough to be useful on a regional or national level, while the regional and national models use a great number of unknown outside (i.e. exogenous) variables, and as a result they are not useful for the improvement of global projection accuracy.

Speaking of projection accuracy, it is necessary to note that the model accuracy can be determined only ex post, when the model projections, which were derived in the past, can be compared with present statistical data. It became customary, however, to speak about model accuracy ex ante. The model is regarded as good if it is accurate in the so-called "historical runs". For that purpose, it is necessary to compare the model past projections with past statistical data. In particular, that approach is being used in the econometrical models, where the well-known measures of accuracy (in the statistical sense) such as the mean square error, correlation coefficient, etc., has proved to be useful. The existing econometric models, constructed on the global, national and regional levels use the simple analytic structures (the relation between exogenous and endogenous variables is being described by linear equations or a polynomial) and are accurate enough for short projection intervals. In order to use these models for longer projection intervals, it is, generally speaking, necessary to employ a more sophisticated structure, based, e.g. on the macro-economic growth models, environmental models, population and social models, etc. In addition, in order to increase the projection accuracy it is also necessary to incorporate into the model structure the decision centers with the corresponding development goals or utility functions on the global, national or regional levels. Since in that sense it is necessary to derive the optimum decision strategies, the modelling problem becomes more complicated and hardly can be solved in the explicit form without using concepts based on decentralization or decomposition of goals and decisions theory.

Another useful approach in complex modelling is based on the linkage of already existing sub-models. For example, the world trade econometric models can be constructed by using the national econometric models [1].

It is a general belief that the long-range planning and forecasting on the regional, national and (probably in the future)global level could be much improved if the existing models could be linked and fitted together. That requires, however, that the model aggregation and decomposition techniques or linkage methodology exists and can be used effectively, employing the available data bank.

In the present paper, an effort has been made to show that such a technique exists and can be used effectively for the construction of a class of models on the regional, national and global levels. Using that technique, we have already been able to construct (at the Institute of Organization and Management of the Polish Academy of Sciences) a class of models called the MR series. The core model (MRI) enables us to make long-range projections of national development in fifteen sectors of the Polish economy. Using the core model, a number of more specialized models (used for regional, environmental, education, R&D etc. - projections) could be constructed. It is hoped that the model can be linked to the international trade model. As the data base, the Statistical Year Book published each year by the Polish Main Statistical Office has been used. The model accuracy, tested in historical runs has proved to be encouraging to further development.

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II. Production and Decentralization of Decisions

The basic problem in normative modeling is to choose the proper structure of production and decision processes. The structure chosen should enable one to take into account:

a) the decentralization of decisions,
b) the aggregation of sectors or the subsequent sector decomposition in the regional and administrative aspects as well,
c) the impact of technical and organizational progress on the production growth (resulting from the government expenditures in the field of R&D, education, medical care, etc.),
d) the estimation of model parameters by using the available statistical data.

It is especially difficult to realize the a), b), requirements. In the case of an aggregated, one-sector economy, it is convenient to use the neo-classical growth model, in which the G.N.P. Y depends on capital K and labor L used, as well as the disembodied technological progress coefficient μ:

$$Y = Ae^{\mu t}K^{\beta}L^{1-\beta} \quad , \qquad /0 < \beta < 1/ \quad , \tag{1}$$

where A, β, μ - given positive constants.

When one wants to get a better understanding of production processes, it is natural to start with the n-sector production model, shown in Fig.1, where X_{ii} represents the output production of sector S_i, and X_{ji} - the amount of goods which S_j is selling to S_i. The net output X_i, $i=1,\ldots,n$ can be used for consumption purposes. When one adopts the n-sector structure of Fig.1, it is necessary to describe the input-output relation for each sector (i.e. the sector production functions). At that stage, one is tempted to use the linear model (e.g. the Leontief input-output model) which is attractive from the point of view of analytic simplicity and an easy way of estimating technological coefficients. However, the linear models are not so attractive from the point of view of macro-economic growth theory and can hardly be used effectively to describe the long-range development. In order to describe the technological change and substitution among the production factors (and first of all the substitution between capital and labor) one would rather adopt the C.E.S. sector production functions. That approach has been used for MRI models. It will be explained in the present paper for a particular case of C.E.S. function, i.e. the Cobb-Douglas production functions:

$$X_{ii} = F_i^{q_i} \prod_{\substack{j=1\\j\neq i}}^{n} X_{ji}^{\alpha_{ji}} \quad , \qquad i = 1,\dots,n \quad ,$$

$$q_i = 1 - \sum_{\substack{j=1\\j\neq i}}^{n} \alpha_{ji} > 0 \quad , \qquad \alpha_{ji} \geq 0 \quad , \qquad i,j = 1,\dots,n \quad , \tag{2}$$

where F_i, α_{ji}, $i,j = 1,\dots,n$, given positive numbers.†

We shall assume at the present section that a set of sector prices p_i, $i = 1,\dots,n$ is given so the model of Fig.1 can be described in monetary terms by the Eqs.

$$Y_{ii} - \sum_{\substack{j=1\\j\neq i}}^{n} Y_{ij} = Y_i \quad , \tag{3}$$

$$Y_{ii} = K_i \prod_{\substack{j=1\\j\neq i}}^{n} Y_{ij}^{\alpha_{ij}} \tag{4}$$

where $Y_{ij} = p_j X_{ij}$, $K_i = p_i F_i^{q_i} \prod_{\substack{j=1\\j\neq i}}^{n} p_j^{-\alpha_{ji}}$, $i,j = 1,\dots,n$.

Now it is possible to introduce the decision structure. We shall assume that each sector S_i, $i = 1,\dots,n$ can decide how much of input Y_{ji}, $j = 1,\dots,n$ to buy in order to maximize the net profit (value added)

$$D_i = Y_{ii} - \sum_{\substack{j=1\\j\neq i}}^{n} Y_{ji} \quad , \qquad i = 1,\dots,n \quad . \tag{5}$$

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†Since $q_i > 0$ a decreasing return to scale effect appears.

As shown in [3] there exists a unique strategy $Y_{ji} = \hat{Y}_{ji}$, $j = 1,\ldots,n$, for each sector $(i = 1,\ldots,n)$ which maximizes (5) and it can be derived by Eqs.

$$\hat{Y}_{ji} = \alpha_{ji}\hat{Y}_{ii} \quad , \qquad j,i = 1,\ldots,n \quad , \qquad j \neq i \quad , \tag{6}$$

$$\hat{Y}_{ii} = F_i \prod_{\substack{j=1 \\ j\neq i}}^{n} \left(\frac{\alpha_{ji}}{p_j}\right)^{\alpha_{ji}/q_i} p_i^{1/q_i} \quad , \qquad i = 1,\ldots,n \quad . \tag{7}$$

When one uses that strategy

$$D_i(\hat{Y}_{ji}, j = 1,\ldots,n, j \neq i) = \hat{D}_i = (1 - q_i)\hat{Y}_{ii} \quad , \qquad i = 1,\ldots,n \tag{8}$$

and the gross product becomes

$$Y = \sum_{i=1}^{n} Y_i = \sum_{i=1}^{n} \hat{D}_i \quad . \tag{9}$$

The two important results follow from the relations (6) ÷ (9) :

Result 1. As follows from (6) the normative n-sector non-linear model (3) ÷ (5) behaves, under optimum strategy, in a similar way to the linear Leontief model with the technological coefficients α_{ji}, $i,j = 1,\ldots,n$, $i \neq j$.† However, the outputs $\hat{Y}_{ii}$, $i = 1,\ldots,n$, in the nonlinear model are specified in a unique manner by prices p_j, $j = 1,\ldots,n$, and F_i coefficients.

Result 2. When prices are fixed the sector net profits $\hat{D}_i$, $i = 1,\ldots,n$ do not depend on sector interactions (in terms of Y_{ji}, $j,i = 1,\ldots,n$, $i \neq j$) and the gross net product is a linear function of F_i, $i = 1,\ldots,n$, coefficients.

† As shown in [4] a similar relation can be obtained for n-sector CES production function model.

Result 1 can be used for a simple estimation procedure of α_{ji} coefficients. Assume for that purpose that the input-output tables for an n-sector economy are given. Assume also that each sector optimizes the net profit (5) so the relation (6) is valid. Then it is possible to derive the numbers $Y_{ji}(t)/Y_{ii}(t) = \hat{Y}_{ji}(t)/\hat{Y}_{ii}(t)$, $j = 1,\dots,n$, for each sector $i = 1,\dots,n$, provided the input-output tables for the past time $t = 0, -1, -2,\dots,$ are known. Then using well-known statistical estimation methods it is possible to find the estimates α_{ji}* of α_{ji} which fit the statistical data.

In the same way the estimates F_i^* of F_i can be derived (using the relation (7) and estimates α_{ji}^* and p_j^*). That approach has been used successfully for the construction of MRI models, in which the 15x15 and 30x30 input-output tables of Polish economy for the last nine years were used.

Using the present methodology it is necessary to observe that the input-output tables deal with the aggregated data. This raises the problem of the economic meaning of sector prices p_i, $i = 1,\dots,n$, which are used as the main aggregating device within each sector. Suppose for example that the sector production is characterized by the vector

$$X_{ji} = (X_{ji1}, X_{ji2}, \dots, X_{jim}) \quad ,$$

and for each fixed j,ν the composition ratios are constant, i.e.

$$x_i = \frac{X_{ij\nu}}{\sum\limits_{\nu=1}^{m} X_{ij\nu}} = \text{const.} \quad , \qquad i = 1,\dots,n \quad . \qquad (10)$$

Suppose also that a set of $p_{i\nu}$, $\nu = 1,\dots,m$, prices for each $i = 1,\dots,n$ be given. In that case one can scalarize the intersector flows by introducting the averaged sector prices:

$$\tilde{p}_i \overset{\text{def}}{=} \frac{1}{X_{ij}} \sum_{\nu=1}^{m} p_{i\nu} X_{ij\nu} \quad , \qquad j = 1,\ldots,n \quad , \tag{11}$$

where

$$X_{ij} = \sum_{\nu=1}^{m} X_{ij\nu} \quad .$$

However, (10) in many cases may be too restrictive and instead of dealing with the weighted arithmetic mean (w.a.m.) prices (11) it is more convenient to use weighted geometric mean(w.g.m.) prices. That aggregation approach can be explained by way of the following example.

Suppose the n-sector model of Fig.1 should be aggregated to yield a two-sector model (in such a way that the sectors S_i, $i = 2,\ldots,n$ should be replaced by an aggregate $\bar{S}_2$) with the aggregated production functions:

$$Y_{11} = K_1 \bar{Y}_{21}^{\bar{\alpha}_{21}} \quad , \tag{12}$$

$$\bar{Y}_{22} = \bar{K}_2 \bar{Y}_{12}^{\bar{\alpha}_{12}} \quad , \tag{13}$$

where

$$\bar{Y}_{22} = \sum_{i=2}^{n} Y_{ii} \quad , \tag{14}$$

$$\bar{Y}_{12} = \sum_{j=2}^{n} Y_{1j} \quad , \tag{15}$$

$$\bar{Y}_{21} = \sum_{j=2}^{n} Y_{j1} \quad . \tag{16}$$

It is assumed that the original n-sector model and the aggregated two-sector model optimize their net profits so that the relations (6), (7) can be used. The production functions for $\bar{S}_1$, $\bar{S}_2$ can then be written in the form

$$\hat{Y}_{11} = F_1 p_1{}^{1/q_1}\left(\frac{\bar{p}_2}{\bar{\alpha}_{21}}\right)^{-\bar{\alpha}_{21}/q_1} , \qquad q_1 = 1 - \bar{\alpha}_{21} ,$$

$$\hat{Y}_{22} = \bar{F}_2\bar{p}_2{}^{\frac{1}{\bar{q}_2}}\left(\frac{p_1}{\bar{\alpha}_{12}}\right)^{-\bar{\alpha}_{12}/\bar{q}_2} , \qquad \bar{q}_2 = 1 - \bar{\alpha}_{12} ,$$

where the aggregated price $\bar{p}_2$ is defined as

$$\bar{p}_2 \underset{\text{def}}{=} \bar{\alpha}_{21} \prod_{j=2}^{n}\left(\frac{p_j}{\alpha_{ji}}\right)^{\alpha_{ji}/\bar{\alpha}_{21}} , \qquad \bar{\alpha}_{21} = \sum_{j=2}^{n}\alpha_{j1} . \tag{17}$$

Since by (6) $\sum_{j=2}^{n}\hat{Y}_{j1} = \hat{Y}_{11}\sum_{j=2}^{n}\alpha_{j1} = \hat{Y}_{11}\bar{\alpha}_{21}$ the condition (16) holds. In order to satisfy (15, (14), i.e.

$$\sum_{j=2}^{n}\hat{Y}_{1j} = \sum_{j=2}^{n}\alpha_{1j}\hat{Y}_{jj} = \bar{\alpha}_{12}\bar{Y}_{22} ,$$

and

$$\bar{Y}_{22} = \sum_{i=2}^{n}\hat{Y}_{ii} = \sum_{i=2}^{n}F_i\prod_{\substack{j=1\\ j\neq i}}^{n}\left(\frac{p_j}{\alpha_{ji}}\right)^{-\alpha_{ji}/q_i}p_i{}^{1/q_i} ,$$

it is necessary to assume

$$\bar{\alpha}_{12} = \frac{1}{\sum_{i=2}^{n}\hat{Y}_{ii}}\sum_{j=2}^{n}\alpha_{1j}\hat{Y}_{jj} ,$$

$$\bar{F}_2 = \sum_{i=2}^{n}\hat{Y}_{ii}\left(\frac{p_1}{\bar{\alpha}_{12}}\right)^{\bar{\alpha}_{12}/q_2}\bar{p}_2{}^{-1/\bar{q}_2} .$$

It should be observed that by (17) one easily obtains the relation for the w.g.m. price growth ratio:

$$\frac{\delta \bar{p}_2}{\bar{p}_2} = \sum_{j=2}^{n} \frac{\alpha_{j1}}{\bar{\alpha}_{21}} \cdot \frac{\delta p_j}{p_j}$$

The same relation can be obtained for the case of w.a.m. price

$$\tilde{p}_2 = \sum_{i=2}^{n} p_i \frac{x_{i1}}{\bar{x}_{21}} .$$

Indeed,

$$\delta \tilde{p}_2 = \sum_{i=2}^{n} \delta p_i \; x_{i1} / \bar{x}_{21} ,$$

and

$$\frac{\delta \tilde{p}_2}{\tilde{p}_2} = \sum_{i=2}^{n} \frac{\delta p_i}{p_i} \cdot \frac{p_i x_{i1}}{\tilde{p}_2 \bar{x}_{21}} = \sum_{i=2}^{n} \frac{\delta p_i}{p_i} \cdot \frac{\hat{Y}_{i1}}{\sum_{j=2}^{n} \hat{Y}_{j1}} = \sum_{i=2}^{n} \frac{\delta p_i}{p_i} \cdot \frac{\alpha_{i1}}{\bar{\alpha}_{21}} .$$

Then the effect of small price variations (δp_i) on the aggregated w.g.m. price ($\bar{p}_2$) is the same as in the case of the w.a.m. price ($\tilde{p}_2$).

The present method can be easily extended to the case when each sector can be represented by the given number of sub-sectors (commodities). Then using the concept of w.g.m. sector prices it is possible to reduce the model dimensionality and fit it to available input-output tables.

Now it is possible to consider the problem of decentralization of decisions and the optimization of development for a centrally planned economy. The model (2) ÷ (5) considered so far was static in the sense that the time and growth mechanisms were not introduced explicitly. A possible way of doing that is to assume that a past Z of gross net output (or the GNP) Y is used for productive investments (Z_1) and the rest to other government expenditures (Z_ν, $\nu = 2,\ldots,m$) in the field of education, R&D, medical care, etc., which can be generally represented by the vector of production factors $\underline{Z}$. It will be assumed that $\underline{Z}$ can influence the model performance by means of F_i, or in other words that $F_i(\underline{Z}_i)$ is an operator of the resources $Z_{\nu i}$, $\nu = 1,\ldots,m$, allocated among the sectors S_i, by a decision center DC (see Fig.1) in such a way that

$$\sum_{i=1}^{n} Z_{\nu i} = Z_\nu \quad ; \qquad \sum_{\nu=1}^{m} Z_\nu = Z \quad .$$

It is now also possible to deal with the variables which represent the intensities of output production $y_i(t)$, $z_{\nu i}(t)$, $i = 1,\ldots,n$, $\nu = 1,\ldots,m$ instead of aggregated (usually within one year) outputs Y_i and resources $Z_{\nu i}$, $i = 1,\ldots,n$, $\nu = 1,\ldots,m$. Using that notation, the intensity of sector production can be written as

$$y_i(t) = \prod_{\nu=1}^{m} \left\{f_{\nu i}(t)\right\}^{\beta_\nu} \quad , \qquad \sum_{\nu=1}^{m} \beta_\nu = 1 \quad , \tag{18}$$

where

$$f_{\nu i}(t) = \int_{-\infty}^{t} k_{\nu i}(t,\tau)\left[z_{\nu i}(\tau)\right]^{\alpha_\nu} d\tau \quad , \qquad 0 < \alpha_\nu < 1 \quad ,^{*} \tag{19}$$

$k_{\nu i}(t,\tau)$ = given non-negative functions which become zero for $t < \tau$.

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* The continuous variables are used here instead of discrete (changing once a year), which is a matter of convenience rather than general methodology.

α_ν, β_ν, $i = 1,\ldots,n$, $\nu = 1,\ldots,m$ = given positive numbers.

In the case when $k_{\nu i}(t,\tau)$ is stationary in time, $k_{\nu i}(t,\tau) = k_{\nu i}(t - \tau)$. A typical example of $k_{\nu i}(t)$ function is

$$k_{\nu i}(t) = \begin{cases} K_{\nu i} e^{-\delta_{\nu i}(t - T_{\nu i})}, & t > T_{\nu i} \\ 0 & t < T_{\nu i} \end{cases} \tag{20}$$

where

$$K_{\nu i} \quad , \quad \delta_{\nu i} \quad , \quad T_{\nu i} = \text{non-negative parameters.}$$

A more general model to describe the inertial effects of government expenditures uses the $k_{\nu i}(t)$ functions which can be characterized by the Laplace transforms:

$$K_{\nu i}(p) = \mathcal{L}\left\{k_{\nu i}(t)\right\} = \frac{A_{\nu i} e^{T_{\nu i} p}}{\prod\limits_{j=1}^{M} (p + a_{\nu ij})} , \tag{21}$$

where $A_{\nu i}$, $T_{\nu i}$, $a_{\nu ij}$, $\nu = 1,\ldots,m$, $i = 1,\ldots,n$, $j = 1,\ldots,M$ are given positive numbers.

Assuming that type of $k_{\nu i}(t)$ function, it is possible to take into account the delays $T_{\nu i}$ between expenditures and productive effects (for example, the construction delays in the case of productive investments, or delay between research and development, etc.). The parameters $\delta_{i\nu}$ specify the depreciation in time of capital, R&D, education, etc., while α_i are responsible for nonlinear saturation effects (there is usually a decreasing return to scale with respect to sector investments and other government expenditures).

It should be observed here that (19) can be written as

$$f_{\nu i}(t) = \bar{f}_{\nu i}(t) + \int_0^t k_{\nu i}(t,\tau)[z_{\nu i}(\tau)]^{\alpha_\nu} d\tau \quad ,$$

where

$$\bar{f}_{\nu i}(t) = \int_{-\infty}^{0} k_{\nu i}(t,\tau)[z_{\nu i}(\tau)]^{\alpha_\nu} d\tau \quad ,$$

represents the contribution of past expenditures (i.e. for $\tau < 0$) to the present production capacity.

As shown in [2,4] the parameters $T_{\nu i}$, $\delta_{\nu i}$, $K_{\nu i}$, of (20) can be estimated by least r.m.s. method provided the past data concerning the expenditures $z_{\nu i}(\tau)$, $\tau = 0, -1, -2, \ldots$, $\nu = 1,\ldots,m$, $i = 1,\ldots,n$ are known as well as the values (estimates) of $F_i(t)$, $i = 1,\ldots,n$, $t \leq 0$.

It should be observed that a simpler version of the model (18) can be obtained when the government expenditures in certain fields (for example, medical care or education) are not assigned to a particular sector S_i.

In that case, the index i can be dropped in (19). A simpler version of the model (18) is obtained, assuming

$$y_i(t) = \int_{-\infty}^{t} k_i(t,\tau) \prod_{\nu=1}^{m} [z_{\nu i}(\tau)]^{\alpha_\nu} d\tau \quad . \tag{22}$$

That version (for m = 1) has been used in MRI.

In that case, however, $k_i(t,\tau)$ should incorporate the disembodied technological progress and one should assume

$$k_i(t,\tau) = e^{\mu_i t} k_i(t - \tau) \quad ,$$

where μ_i should be estimated using past statistical data.

Assuming that all the $F_i(\underline{Z}_i)$, $i = 1,\ldots,n$, functions have been identified and that the prices P_i, $i = 1,\ldots,n$, are chosen in such a way that they take care of adjusting the supply to the demand according to (3) (that will constitute the subject of consideration in Sec.4) we can turn to the problem of optimization of long-range development. The gross product generated within the planning interval [O,T] has the present value

$$Y = \sum_{i=1}^{n} \int_{O}^{T} W(t) y_i(t) dt \quad , \tag{23}$$

where $w(t) = (1 + \varepsilon)^{-t}$, ε - the given discount rate and $y_i(t)$ depend (as shown by (18)) on the expenditure intensities $z_{\nu i}(t)$, $t \varepsilon [O,T]$, $\nu = 1,\ldots,m$, $i = 1,\ldots,n$.

Assume that the amount of expenditures Z_ν spent in the planning interval [O,T] be given, i.e.

$$\sum_{i=1}^{n} \int_{O}^{T} w_\nu(t)\; z_{\nu i}(t) dt \leq Z_\nu \quad , \qquad \nu = 1,\ldots,m \quad . \tag{24}$$

where $w_\nu(t)$ = weight attached to the ν-th expenditure. When the expenditures are financed by bank loans which should be paid back not later than $t = T$, one can assume $w_\nu(t) = (1+\eta)^{T-t}$, where η - bank interest rate.

The development optimization problem can be formulated as follows. Find the non-negative functions $z_{\nu i}(t) = \hat{z}_{\nu i}(t)$, $\nu = 1,\ldots,n$, $i = 1,\ldots,n$, $t \varepsilon [O,T]$ such that the functional (23) attains its maximum value subject to the constraints (24). As shown in Appendix 1, such strategies exist for the class of production functions with $k_{\nu i}(t)$ of the type (21) and can be derived effectively from the set of equations:

$$f_{1i}(t) = c_{\nu i} f_{1i}(t) \quad , \qquad t \varepsilon [O,T] \quad ,$$

$$\nu = 2,\ldots,m \quad , \qquad i = 1,\ldots,n \quad , \tag{25}$$

where the index $\nu = 1$ corresponds to the most "inertial" activity among different government expenditures. The equations (25) express the following principle of "proportional expenditure effects".

In order to maximize the gross product the expenditure intensities in each sector of economy should be chosen in such a way that in each time instant $^{t}/$ the expenditures effects $f_{\nu i}(t)$ are proportional to the most inertial expenditure effect $f_{1i}(t)$.

According to that principle it does not pay to develop the production capacity if a corresponding level of education, R&D, etc. has not been achieved. In other words, if the education (or scientific level) is the most inertial processes one should coordinate the government expenditures in such a way that the increase of production capacity is proportional to the education (scientific) level.

It should be observed that the integral constraints (24) are regarded sometimes as not restrictive enough and are supplemented by amplitude constraints

$$\sum_{i=1}^{n} z_{\nu i}(t) \leq Z_{\nu}(t) \quad , \qquad t \in [O,T] \quad , \qquad \nu = 1,\ldots,m, \tag{26}$$

where $Z_{\nu}(t)$ given functions of time.

As shown in Appendix 1, it is possible to derive the optimum strategies $z_{\nu i}(t) = \hat{z}_{\nu i}(t)$, $\nu = 1,\ldots,m$, $i = 1,\ldots,n$, $t \in [O,T]$, which maximize (23), (18) subject to the constraint (26) and

$$\sum_{i=1}^{n} \int_{O}^{T} w_i(t) \; z_{\nu i}(t)dt \;:\; \sum_{\nu=1}^{m} \sum_{i=1}^{n} \int_{O}^{T} w_i(t) \; z_{\nu i}(t)dt$$

$$= \, ^{Z_\nu}/_{Z} = \delta_{\nu} \quad , \tag{27}$$

where δ_{ν}, $\nu = 1,\ldots,m$, given positive numbers $\sum_{\nu=1}^{m} \delta_{\nu} = 1$.

The gross product generated within [O,T] under optimum stragegies (A.19) becomes

$$Y = G^q \prod_{\nu=1}^{m} z_\nu^{\delta_\nu} = G^q \prod_{\nu=1}^{m} \left(\frac{\delta_\nu}{\delta}\right)^{\delta_\nu} z^\delta \quad , \tag{28}$$

$$\delta = \sum_{\nu=1}^{m} \delta_\nu \quad , \qquad \delta_\nu = \alpha_\nu \beta_\nu \quad .$$

The different optimization strategies for the integral and amplitude type of constraints have been derived for MRI model. For example, when one maximizes (23) where $y_i(t)$ are described by (22) and the constraints by (26) one gets [4]

$$\hat{z}_{\nu i}(\tau) = \frac{f_i(\tau)}{f(\tau)} Z_\nu(\tau) \quad , \qquad \begin{array}{l} \nu = 1,\ldots,m \\ i = 1,\ldots,n \quad , \end{array}$$

where

$$f(\tau) = \sum_{i=1}^{n} f_i(\tau) \quad , \qquad f_i(\tau) = \left\{\int_\tau^T k_i(t,\tau)\; w(t)dt\right\}^{1/q_i} \quad .$$

The value of gross product under optimum strategy (26) becomes

$$Y(\hat{\underline{z}}) = Y_o + \int_o^T f^q(\tau) \prod_{\nu=1}^{m} z_\nu^{\alpha_\nu}(\tau)\; dt \quad ,$$

where

$$Y_o = \int_o^T w(t) \sum_{i=1}^{n} \int_{-\infty}^{O} k_i(t,\tau) \prod_{\nu=1}^{m} y_{\nu i}^{\beta_\nu}(\tau)\; d\tau\; dt \quad ,$$

is the component of Y due to the past expenditures

When $T \to \infty$, $Y_o \to 0$ and $Y(\underline{Z})$ can be written as

$$Y(\hat{\underline{Z}}) = G \prod_{\nu=1}^{m} \bar{Z}_{\nu}^{\alpha_\nu} ,$$

where

$$\bar{Z}_{\nu} = \int_0^T Z_{\nu}(t)dt \quad , \qquad G = \int_0^T f^q(\tau) \prod_{\nu} [Z_{\nu}(\tau)/\bar{Z}_{\nu}]^{\alpha_\nu} d\tau \quad .$$

As a typical example it is also possible to consider the simple development model with two production factors: labor and capital only. In that case

$$y_i(t) = \left\{ \int_0^T k_i(t,\tau) \; z_i^{\alpha} \; (\tau)d\tau \right\}^{\beta} [x_i(t)]^{1 - \beta} \quad ,$$

where $z_i(\tau)$ - investment intensity (the expenditures $Z_{\nu i}$, $\nu > 1$ are neglected as well as the impact of the past investments on the present production), and

$$\sum_{i=o}^{n} \int_0^T z_i(t)dt \leq K \quad , \qquad \sum_{i=1}^{n} \int_0^T x_i(t)dt \leq L \quad .$$

As shown in [!]

$$Y(\hat{\underline{z}},\hat{\underline{x}}) = F^{\beta(1 - \alpha)} \; K^{\alpha\beta} \; L^{1 - \beta} \tag{29}$$

where

$$F = \sum_{i=1}^{n} F_i \quad , \qquad F_i = \int_0^T \left[\int_{\tau}^T w^{1/\beta}(t) \; k_i(t,\tau)dt \right]^{\frac{1}{1-\alpha}} d\tau$$

It is interesting to observe that the aggregated (as a result of optimum allocation of resources) production function (29) of the n-sector model is equivalent (for $\alpha = 1$) to the classical Cobb-Douglas production function (1). That equivalence can be shown also in the case of (28) if one assumes that

$$\mu t = \ln \prod_{\nu=3}^{m} z_\nu^{\delta_\nu} \quad , \tag{30}$$

i.e. that the technological progress is a result of expenditures z_ν, $\nu = 3,\ldots,m$.

Introducing the notion of the rate of growth $\rho_x = \dot{x}/_x$ for the differential function $x(t)$, it is also possible to find from (1) :

$$\rho_y = \mu + \beta\rho_k + (1 - \beta)\rho_L \quad , \tag{31}$$

and from (30)

$$\mu = \sum_{\nu=3}^{m} \delta_\nu \rho_{z_\nu} \quad . \tag{32}$$

The formulae (31) (32) can be used for the estimation of the role of government expenditures in the gross economic growth.

It should be mentioned that the described methods of decomposition, optimization, and aggregation make it possible to solve one of the most difficult problems in the complex modelling which is an interrelation between the macro-models (of type (1)) and the sectorial models (2) ÷ (4). Each sectorial model can be decomposed into a number of submodels according to the production, regional, and management structure. There exist simple relations between the parameters of the higher and lower order submodels. Using those models there exists the possibility of investigation of investments and other government expenditures' impact on future development.

III. Allocation of Gross Product.

Consider the allocation of gross product and decision structure shown in Fig.2, which may be regarded as a simplified model of centrally planned economy. According to that model the gross product generated at the end of the year t - 1 by the n productive sectors $(P_1,\ldots,P_n)$ is divided by the I-level decision unit (D.C.) into m parts $(Z_1, Z_2, Z_3,\ldots,Z_m)$, i.e. productive investments, private consumption and government expenditures (education, health services, research and development, public utilities, etc.). The II decision level is concerned with allocation of z_i, $i = 1,\ldots,m$, among different sectors of production and consumption activities. The III and lower decision levels are concerned with the allocation of resources in the regional or administrative sense.

All the decisions on the production side are concerned with the maximization of the gross product while the decisions on the consumption side are concerned with social welfare which can be described by a utility function. Speaking about the decision goals more precisely, it is necessary to take into account the influence of government expenditures on the gross production and to formulate the long-range strategic goals and decisions as well as the short-range (operational) goals and decisions. The operational decision should result from the strategic decisions.

At the I-level of the hierarchical structure of Figure 2, one is concerned mainly with the aggregated model described by (1), or in the equivalent form (28) $\div$ (3), where Z_2 represents the cost of labor (equal to the private consumption)

$\sum_{\nu=3}^{m} Z_\nu$ - total government expenditures which contribute to the technological progress.

The usual approach to study the allocation of G.N.P. in macro-economic models for centrally planned economies is based on the so called imputation principle.

According to that principle, the gross product Y_o generated presently (or at the end of t-1 year) should be assigned to the immediate consumption (C_i) and the part which determines the future consumption (C_f) in such a way that the social utility function U (C_i, C_f), describing the consumer preferences with respect to the present (C_i) and future (C_f) consumption, attains maximum.

In our notation $C_i = Z_2 = \gamma_2 Y_o$, while the future consumption, according to (28) is

$$C_f = \gamma_2 Y = \gamma_2 G^q \prod_{\nu=1}^{m} Z_\nu^{\delta_\nu} \quad .$$

Then the problem boils down to maximization of

$$U = U\ (Z_2,\ \gamma_2 G^q \prod_{\nu=1}^{m} Z_\nu^{\delta_\nu})\ ,$$

with respect to Z_ν $\nu = 1,\ldots,n$, and subject to

$$\sum_{\nu=1}^{m} Z_\nu = Y_o \sum_{\nu=1}^{m} \gamma_\nu = Y_o \quad . \tag{33}$$

Assuming that the utility function has the known properties familiar to demand analysis, i.e. differentiable and strictly concave, we can introduce the lagrangean:

$$\emptyset = U\ (Z_1 \ldots Z_m) + \lambda \left[\sum_{\nu=1}^{m} Z_\nu - Y \right] \ ,$$

and obtain, together with (33), the necessary conditions of optimality:

$$U_f' \gamma_2 \delta_\nu \frac{Y}{Z_\nu} + \lambda = 0 \quad , \qquad \nu = 1,\ldots,m,\ \nu \neq 2 \tag{34}$$

$$U_i' + U_f' \delta_2 \gamma_2 \frac{Y}{Z_2} + \lambda = 0 \quad , \tag{35}$$

where U_i', U_f' are partial differentials of $U(C_i, C_f)$. Keeping in mind that $Z_{\nu}/_Y = \gamma_\nu$, $\nu = 1,\ldots,m$, and defining the discount rate r as

$$r_{df} = {U'_i}/{U'_f} - 1$$

one gets by (34)(35) the following conditions

$$\frac{\delta_\nu}{\gamma_\nu} = \frac{\delta_2}{\gamma_2} + \frac{(1+r)}{\gamma_2 \rho} \quad , \qquad \nu = 1,3\ldots,m \quad , \tag{36}$$

where

$$\rho = {Y}/{Y_o} = G^q \prod_{\nu=1}^{m} (\gamma_\nu)^{\delta_\nu} Y_o^{\delta_\nu - 1} \quad ,$$

and together with

$$\sum_{\nu=1}^{m} \delta_\nu = 1 \quad ,$$

can be used for computation of the optimum values $(\hat{\gamma}_\nu)$ of γ_ν , $\nu = 1,\ldots,m$. Since $U(C_i,C_f)$ is strictly concave, the eqs (36) are also sufficient for optimality. They can be also used for estimation of δ_ν, $\nu = 1,\ldots,m$ coefficients, when we have (as it usually happens) the data concerning the allocation of gross product in terms of γ_ν, $\nu = 1,\ldots,m$. If, for example, we are dealing with aggregated economy with labor and capital only and we know that γ_2 is (under optimum strategy) equal 0.15 (γ_1 = 0,85) and $\frac{1+r}{\rho} = 1$, it is possible to find out that $\delta_1 = 0.3$, $\delta_2 = 0,7$.

Now it is possible to find out how fast the economy will grow under optimum allocation strategy. Assuming $\hat{\gamma}_\nu$ = const, $\nu = 1,\ldots,m$, $t \in [O,T]$ one gets

$$Y = G^q \prod_{\nu=1}^{m} (\hat{\gamma}_\nu Y_o)^{\delta_\nu} \quad , \tag{37}$$

When

$$\sum_{\nu=1}^{m} \delta_\nu = 1 \quad ,$$

one gets

$$\rho = G^q \prod_{\nu=1}^{m} (\hat{\gamma}_\nu)^{\delta_\nu} \quad . \tag{38}$$

That growth can be achieved provided the decision concerning the allocation of resources at the II and lower levels of the structure shown in Fig.2 are optimum. As shown in Sec.2, these decisions can be derived explicitly when $Z_\nu = \gamma_\nu Y_o$, $\nu = 1,\ldots,m$ are known.

It should also be observed that in order to have an objective measure of growth it is necessary to know the price change in the optimization interval. The prices are, however, determined by the supply-demand relations with respect to the commodities generated by n sectors of economy. In order to investigate the price changes we have first of all to consider the consumer preference structure. In order to do that, assume that the total salary (Z_2) for the planning interval [O,T] is known. The (aggregated) consumer will spend that salary purchasing with the intensities $z_i(t)$ the n different commodities or services produced in sectors S_i $i = 1,\ldots,n$. We assume also that the utility level $x_i(t)$ is related to the expenditure intensity $z_i(t)$ in an inertial and nonlinear fashion:

$$x_i(t) = \int_o^t K_i(t,\tau)[z_i(\tau)]^{\alpha_i} d\tau \quad , \tag{39}$$

where $K_i(t,\tau)$ is of a similar nature as the function $k_i(t,\tau)$ introduced in Sec.2 (compare (20)(21).

In the case of $K_i(t,\tau)$ described by (20) the consumer expenditures create no utility before the delay time $t = T_i$.

Such a situation can be observed in the case of education, health, etc., expenditures. For example, in order to increase his utility level in education, the consumer must finish a school first and that requires tuition expenditures over T_i years of study. The benefits (in terms of increased salary) will not follow until he finishes school.

It should also be observed that due to the exponent exp $(-\delta_i t)$ the education level decreases along with time if no additional expenditures (re-education) follow.

Another example of inertial effects one can encounter is in the case of purchasing expensive durable goods (houses, motor cars, etc.) One must accumulate the money before a purchase is made and then the utility level decreases in time as a result of depreciation, aging, etc.

It should also be observed that due to $0 < \alpha_i < 1$ the decreasing return to scale effect follows in (39). Obviously one cannot increase the education level indefinitely increasing the expenditures.

It is assumed that consumer preferences concern the utility levels $x_i(t)$ rather than the expenditures $z_i(t)$ (if $x_i t)$ is not an immediate result of $z_i(t)$) and consequently we shall introduce the utility functional

$$U(\underline{z}) = \int_0^{T_i} w(t) \prod_{i=1}^{n} [x_i(t)]^{\beta_i} dt \quad ,$$

where w(t) = given discount function.

The present model can be easily extended to the case when there are p classes of consumers (e.g. white and blue collar workers, scientists, managers, etc.) having different salaries and utility functionals

$$U_j = \int_0^T w(t) \prod_{i=1}^{n} \left\{ \int_0^t k_{ij}(t-\tau)\,[z_{ij}(\tau)]^{\alpha_i}\, d\tau \right\}^{\beta_i} dt \quad , (40)$$

where

$z_{ij}(t)$ - the j-th consumer intensity of expenditures in i-th commodity category,

$k_{ij}(t)$ - given non-negative functions

$0 < \alpha_i < 1 \quad , \quad \sum_{i=1}^{n} \beta_i = 1 \quad , \quad j = 1,\ldots,p \quad ,$

$i = 1,\ldots,n \quad .$

The expenditures intensities should satisfy the following constraints

$$\sum_{j=1}^{p} \int_0^T w_i(t)\, z_{ij}(t)\, dt \leq Z_i \quad , \quad i = 1,\ldots,n \quad , \qquad (41)$$

where

$\sum_{i=1}^{n} Z_i \leq Z$ - total expenditures equal total salaries (Z_2).

The optimization of utility functional problem can be formulated as follows.

Find the non-negative strategies $z_{ij}(t) = \hat{z}_{ij}(t)$, $i = 1,\ldots,n$, $j = 1,\ldots,p$, such that the utility

$$U = \sum_{j=1}^{p} U_j \qquad (42)$$

attains maximum subject to constraint (41).

As shown in Appendix I, such a strategy can be derived in an explicit form.

It should be observed that in the present consumption model a decentralized system of decisions can be used. According to that system the government is concerned with the best allocation of Z_2 among the p consumer classes while the consumers are concerned with the best allocation of their salaries. In that model it is also possible to take into account the government expenditures if they contribute to the consumer welfare. For example, Z_i, $i = 1,\dots,N$, may represent the consumer private expenditures (which are financed out of his salary) while Z_i, $i = n + 1,\dots,N$, the government expenditures which contribute to the consumer welfare.

Assuming that the allocation structure for gross product is known, one can derive the demands Y_i, $i = 1,\dots,n$, confronting the production sectors:

$$Y_i = \sum_{\nu=1}^{m} \lambda_{\nu i} Z_\nu = Y_o \sum_{\nu=1}^{m} \lambda_{\nu i} \gamma_2 \quad , \qquad i = 1,\dots,n \qquad (43)$$

where

$$\sum_{i=1}^{n} \lambda_{\nu i} = 1 \quad , \qquad \nu = 1,\dots,m \quad ,$$

$\lambda_{\nu i}$ = given non-negative coefficients determining the ν-th expenditure contribution to the demand confronting the i-th production sector.

IV. Prices.

The net sector productions $\hat{Y}_i$, $i = 1,\dots,n$, of the system of Fig.1, under optimum strategy (6) can be written in the form

$$\hat{Y}_i = \hat{Y}_{ii} - \sum_{\substack{j=1 \\ j \neq i}}^{n} \hat{Y}_{ij} = \hat{Y}_{ii} - \sum_{\substack{j=1 \\ j \neq i}}^{n} \alpha_{ij} \hat{Y}_{jj} , \qquad i = 1,\ldots,n \quad . \tag{44}$$

Generally speaking the supplies (44) differ when compared to Y_i, $i = 1,\ldots,n$, values (43) on the demand side. In order to achieve an equilibrium it is necessary that the outputs $\hat{Y}_{ii}$, $i = 1,\ldots,n$, satisfy the equations

$$\hat{Y}_{ii} - \sum_{\substack{j=1 \\ j \neq i}}^{n} \alpha_{ij} \hat{Y}_{jj} = Y_i , \qquad i = 1,\ldots,n \quad . \tag{43}$$

Since the technological coefficients α_{ij}, $i,j = 1,\ldots,n$ are determined (estimated) from the real economy it is natural to assume the unique and non-negative solution of (43), exists This solution assumes the following form.

$$\hat{Y}_{ii} = l_i(\underline{\lambda},\underline{\gamma}) \; Y_o , \qquad i = 1,\ldots,n \quad , \tag{44}$$

where $l_i(\underline{\lambda},\underline{\gamma})$ are known functions of $\lambda_{\nu i}\gamma_\nu$, $i = 1,\ldots n$, $\nu = 1,\ldots,m$.

In order to satisfy (44) it is necessary to choose the sector prices in the proper way. For that purpose, consider the sector output (2) which can be written in the form

$$X_{ii} = [F_i(Z_1/w_1, Z_2/w_2, \ldots Z_m/w_m)]^{q_i} \prod_{\substack{j=1 \\ j \neq i}}^{n} x_{ji}^{\alpha_{ji}} , \tag{45}$$

where w_ν = prices attached to investment, labor and government activities, $\nu = 1,\ldots,m$.

Since F_i is by assumption a homogenous function of Z_ν (compare (28)) one can write:

$$X_{ii} = [F_i(Z_1,\ldots Z_m)]^{q_i} \prod_{\nu=1}^{m} w_\nu^{-\delta_\nu q_i} \prod_{\substack{j=1\\ j\neq i}}^{n} p_j^{-\alpha_{ji}} \prod_{\substack{j=1\\ j\neq i}}^{n} Y_{ji}^{\alpha_{ji}} \quad ,$$

$$i = 1,\ldots,n \quad .$$

In order to derive the equilibrium prices, it is necessary that a tangent hyperplane (which describes the demand side) to the production isocline exists. If, for example, the aggregated economy can be described by the production function

$$x = A\, K^{\beta}\, L^{1-\beta} \quad , \tag{46}$$

and the input cost

$$C = w_1\, K + w_2\, L \leq Y_o \quad , \tag{47}$$

where w_1, w_2 = given prices of capital and labor, it is possible to find the capital $K = \hat{K}$ and labor $L = \hat{L}$, which maximizes (46) subject to (47).

It is easy to show that the tangent point of the line $w\ \ k + w_2\, L = Y_o$ to the isocline (46) is

$$\hat{K} = \frac{\beta}{w_1}\, Y_o \quad , \qquad \hat{L} = \frac{1-\beta}{w_2}\, Y_o \quad ,$$

and the resulting price of the good X is

$$p = \frac{C(\hat{K},\hat{L})}{x(\hat{K},\hat{L})} = \bar{A}\, w_1^{\beta}\, w_2^{1-\beta} \quad , \qquad \bar{A} = A^{-1}\, \beta^{-\beta}\, (1-\beta)^{\beta-1} \quad ,$$

The similar method can be used for the derivation of sector prices. We assume that the government expenditures have been allocated in an optimum manner, i.e. $Z_\nu = \hat{Z}_\nu$, $\nu = 1,\ldots,m$ and that $Y_{ji} = \hat{Y}_{ji}$, $j,i = 1,\ldots,n,\ j \neq i$. Then

$$\hat{X}_{ii} = [F_i(\hat{Z}_1,\ldots\hat{Z}_m)]^{q_i} \prod_{\nu=1}^{m} w_\nu^{-\delta_\nu q_i} \prod_{j}^{n} \left(\frac{\alpha_{ji}}{p_j}\right)^{\alpha_{ji}} \hat{Y}_{ii}^{1-q_i} ,$$

$$i = 1,\ldots,n ,$$

while the input costs become

$$\hat{C}_i = \sum_j \alpha_{ji} \hat{Y}_{ii} + q_i \sum_{\nu=1}^{m} \delta_\nu \hat{Y}_{ii} = \hat{Y}_{ii} , \qquad i = 1,\ldots,n ,$$

Then taking the logarithms of $\hat{C}_{i/\hat{x}_{ii}}$ one gets

$$\ln p_i - \sum_{\substack{j=1 \\ j \neq i}}^{n} \alpha_{ji} \ln p_j = q_i[\ln \frac{\hat{Y}_{ii}}{a_i F_i} + \sum_{\nu=1}^{m} \delta_\nu \ln w_\nu] , \qquad (48)$$

$$i = 1,\ldots,n$$

where

$$a_i = \prod_{\substack{i=1 \\ j \neq i}}^{n} \alpha_{ji}^{\alpha_{ji}/q_i} ,$$

$$\hat{Y}_{ii} = l_i(\underline{\lambda},\underline{\gamma})\ Y_o .$$

Since the technological coefficients α_{ij} are "productive" the determinant

$$\begin{vmatrix} 1, & -\alpha_{21} & \cdots & -\alpha_{n1} \\ \vdots & & & \vdots \\ -\alpha_{1n}, & -\alpha_{2n} & \cdots & 1 \end{vmatrix} \neq 0 \quad ,$$

and there exists a unique solution to the linear set of eqs (48). It yields the set of positive prices p_i , $i = 1,\ldots,n$.

The present system can easily be extended to the open economy by introducing additional foreign trade sector S_o. The domestic production functions (45) should be supplemented now by the factor $Y_{oi}^{\alpha_{oi}}$, while q_i becomes $\bar{q}_i = q_i - \alpha_{oi} > 0$, $i = 1,\ldots,n$. In the eqs (48) we should add the term $\alpha_{oi} \ln p_{oi}$, where the price for the foreign trade can be also written as

$$p_{oi} = \bar{p}_{oi} M = p_i/T_{oi} \tag{49}$$

where

$\bar{p}_{oi}$ = the price of imported commodity in the foreign currency,

M = rate of exchange

$T_{oi} = \bar{p}_{io}/\bar{p}_{oi}$ = terms of trade (export to import price ratio).

Then the eq.(48) can be written

$$(1 - \alpha_{oi})\ln p_i - \sum_{\substack{j=1\\j\neq i}}^{n} \alpha_{ji}\ln p_j = q_i[\ln \frac{l_i Y_o}{q_i F_i} + \sum_{\nu=1}^{m} \delta_\nu \ln w_\nu]$$

$$- \alpha_{oi}\ln T_{oi} = 0 \quad ,$$

$$q_i = 1 - \sum_{\substack{j=1\\j\neq i}}^{n} \alpha_{ji} - \alpha_{oi} \quad ,$$

$$i = 1,2,\ldots,n \quad . \qquad (50)$$

The prices w_i, $i = 1,\ldots,n$, and T_{oi} should be regarded as exogenous. The cost of labor (or average salary) w_2 can be derived, e.g. from the condition

$$w_2 = Z_{2/L_o} = \gamma_2 \frac{Y_o}{L_o} \quad , \qquad (51)$$

where L_o = total employment.

Since the most important, for modelling purposes, are the relative price changes with respect to the previous year, one can introduce the so-called price indices †

$$p_i^t \underset{df}{=} \frac{p_i(t)}{p_i(t-1)} \quad , \qquad i = 1,\ldots,n$$

and rewrite the eqs.(50) in an equivalent form

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†As shown in Sec.2 for small price variations the model is good for aggregated w.a.m. prices as well as for w.g.m. prices.

$$(1-\alpha_{oi})\ln p_i^t - \sum_{\substack{j=1\\ j\neq i}}^{n} \alpha_{ji} \ln p_j^t = q_i\left[\ln \frac{k_i^t\, Y^{t-1}}{F_i^t} + \sum_{\nu=1}^{m} \delta_\nu \ln w_\nu^t\right]$$

$$- \alpha_{oi} \ln T_{oi}^t = 0 \quad ,$$

$$i = 1,\ldots,n \qquad (52)$$

where l_i^t , Y^{t-1}, F_i^t, w_ν^t, T_{oi}^t are defined as ratios of present to past values.

As follows from (52) the price indices p_i^t increase along with the GNP (y^{t-1}) and price indices for capital, labor, and government services (w_ν^t). They go down, however, when capital capacity (F_i^t) or terms of trade (T_{oi}^t) increase. They depend also on the consumption structure change (l_i^t).

Using the price model base on eqs.(50)(52)it is possible to derive the value of gross production in constant prices assuming that w_ν^t, T_{oi}^t, L_o^t are given (e.g. predicted by econometric and population growth models). However, the estimation of w_ν^t, in the general case $\nu = 1,2,\ldots,m$, is difficult. In the MRI model we have used therefore two prices w_1^t, w_2^t only. Since the p_i^t, T_{oi}^t, $i = 1,\ldots,n$ and w_2^t can be derived from the past statistical data an effort has been made to estimate the change of aggregated price w_1^t, which represents capital and government price change. In the case of integrated (n=1) closed economy one gets from (52)

$$w_1^t = \left[\frac{p_i^t F_1^t}{l_1^t\; Y^{t-1}\,(w_2^t)^{1-\beta}}\right]^{1/\beta} \qquad (53)$$

Setting in (53) the statistical integrated data for the Polish economy in 1972:

$$Y^{t-1} = 1{,}101 \quad , \qquad F_1^t = 1{,}059 \quad , \qquad p_1^t = 1{,}000 \quad ,$$

$$w_2^t = 1{,}067 \quad , \qquad \ell_1^t = 1$$

and assuming $\beta = 0.32$, one gets $w_1^t = 0{,}81$.

Assuming that $w_1^t = 0{,}81$ will not change in 1973 it is possible to derive the value

$$p_1^t = 0{,}81^{\beta} \frac{l_1^t \, Y^{t-1} (w_2^t)^{1-\beta}}{F_1^t} \quad ,$$

for the next (1973) year.

That technique can be extended for the general n-sector case. It can be observed that when one computes the price indices by (52) it is possible to use the delayed by $t = 1$ values of

$$Z_\nu^t = \gamma_\nu^t \, Y^{t-1} \quad , \qquad \nu = 1,\ldots,m \quad ,$$

and consequently derive the $F_i^t(Z_\nu^t)$ values.

The most difficult problem is to derive the consumption structure change l_i^t. For that purpose in the MRI a model of consumption structure change has been used. The model assumes that the structure of consumption changes when the GNP per capita $\bar{Y} = Y/p_{op}$, and integrated price index p^t in the last year change, i.e.

$$l_i^t = [\bar{Y}^{t-1}]^{\varepsilon_i} [p^{t-1}]^{E_i} \tag{54}$$

where ε_i, E_i have been estimated using past statistical data. Using (54) in (52) it is possible to see that the price changes p_i^t, $i = 1,\ldots,n$, $t = 1,2,\ldots$ can be derived in an iterational way starting with the $t = o$ data base.

As follows form (50)(52) the national price model depends much on the international trade and world prices. In order to get projections for world prices it is necessary to construct a world trade model. Such a model can be constructed using the general methodology described in the present paper. In order to demonstrate that consider the n-sector model shown in Fig.1. Each sector S_i, $i = 1,\ldots,n$ represents now a national economy. The flow Y_{ij}, $j = 1,\ldots,n$, $j \neq i$, represent the exports to $S_j (j \neq i)$ sectors in S_i currency. In the same way Y_{ji}, $j = 1,\ldots,n$, $j \neq i$ represent the import of goods from $S_j (j \neq i)$ in S_j currency.

It should be assumed here, however, that the balance of payment (D_i) is observed at each sector, i.e.:

$$\sum_{\substack{j=1 \\ j \neq i}}^{n} [\hat{Y}_{ij} - \hat{Y}_{ji}] = \sum_{\substack{j=1 \\ j \neq i}}^{n} \alpha_{ij} \hat{Y}_{jj} - \alpha_i \hat{Y}_{ii} = D_i \quad , \qquad i = 1,\ldots,n \tag{55}$$

where

$$\alpha_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} \alpha_{ji} \quad , \qquad \sum_{i=1}^{n} D_i = 0 \quad .$$

Since the eqs. (55) are linearly dependent, i.e.

$$\sum_i \left[\sum_j \alpha_{ij} \hat{Y}_{jj} - \alpha_i \hat{Y}_{ii} \right] = \sum_j \hat{Y}_{jj} \sum_i \alpha_{ij} - \sum_i \alpha_i \hat{Y}_{ii} = 0 \quad ,$$

in order to get a solution introduce the new variables

$\hat{Y}_{jj}/\hat{Y}_{11} \underset{df}{=} x_j$, $j = 2,\ldots,n$. Then instead of (55) one gets

$$\sum_{\substack{j=2 \\ j \neq i}}^{n} \alpha_{ij} x_j - \alpha_i x_i = D_i/\hat{Y}_{11} , \quad i = 1,\ldots,n . \qquad (56)$$

When that system is being solved one can also find $\hat{Y}_{11}$ from the eq.

$$\sum_{j=2}^{n} \alpha_{ij} x_j - \alpha_1 \hat{Y}_{11} = D_1 .$$

The price model (48) can now be rewritten in terms of $p_j/p_1 \underset{df}{=} M_j$, $j = 2,\ldots,n$:

$$\ln M_i - \sum_{\substack{j=2 \\ j \neq i}}^{n} \alpha_{ji} \ln M_j = \ln \left[\frac{F_1^{q_1} \prod_{\nu=1}^{m} w_\nu^{\delta_\nu q_i} \prod_j \alpha_{j1}^{\alpha_{j1}}}{F_i^{q_i} \prod_{\nu=1}^{m} w_\nu^{\alpha_\nu q_1} \prod_j \alpha_{ji}^{\alpha_{ji}}} \right] , \qquad (57)$$

$i = 2,\ldots,n$.

From the eqs. (57) it is possible to derive the exchange ratios M_j, $j = 1,\ldots,n$. Using that methodology it is also possible to construct the more general models of international trade in which n countries exchange N different products.

V. Conclusions.

The present paper presents a number of decomposition, optimization, and aggregation methods which are useful for the construction of long-range, normative development models. The usefulness of these methods has been demonstrated by considering a development model of a centrally planned economy. That economy includes, however, a market for consumer goods and as a result the sector prices can be regarded as equilibrium prices.

A similar approach can be used for a market economy assuming that the decisions concerning investment are taken at the lower (private producer) level. Such a model was investigated in Ref. [4] . Another difference concerns the labor and capital prices, which were regarded as exogenous (or decision) variables in our model. In the market economy the labor and capital prices may be regarded as equilibrium prices according to the well-known Keynesian philosophy.

In the model discussed, we have used a directed rather than the disembodied technological progress and as a result we were able to obtain the explicit relations between the growth rate (31) and the government expenditure rates in education, science, health, etc. That approach can be used effectively when government institutions or expenditures can be assigned to specific sectors of economy.

The main advantage of the present approach is that it derives the optimum decisions at each level of the decision structure in an explicit form. The model can easily be extended to include the lower-level subsystems. In that way, we are able to aggregate the micro-production submodels up to one sector macro-model (1), (28). Due to the explicit relations it is possible to derive at each decision level the corresponding production function or utility function.

The model assumes that all the decisions are optimum. In the case when the decision taken at a particular level is not optimum, it is possible to show that the resulting aggregated performance factors (such as G in (28) or F, F_i in (29)) decrease. Since these factors are derived by using the estimated values of $K_{\nu i}$ (see (20)) the model will work as well with the sub-optimal decisions but the growth achieved will be less than the maximum growth possible.

The optimum decision, which can be derived by our long-range normative (core) model should be regarded first of all as strategic decisions. One of the important strategic decisions is concerned with choosing the areas of national production specialization which enable the fastest economic growth. For that purpose, one can investigate (by using price model (52)) the impact of the projected terms of trade on the allocation of sector investments and corresponding government expenditures. For example, when the terms of trade are favorable for the national specialization in coal production the corresponding capital investments should be implemented. The sector investments change the relative prices in the whole economy. However, using the price model together with the production model, one is able to find that strategy which maximizes the growth per capita in constant prices. In that way, the model takes care of the consumer and producer benefits. In other words, the derived by the model strategy takes into account the so-called opportunity costs while the optimum strategy enables the achievement of the comparative advantages.

The strategic decisions derived by the use of the core model can be employed for operational (i.e. sectional and regional) decisions. For that purpose one can use a number of specialized regional or sectorial submodels. For example, in order to implement the strategic decision regarding the development of coal production, it is helpful to investigate the development processes at the regional level (find the best location of coal mines and power stations which produce electric power by burning coal, find the necessary expenditures in labor training, research and development costs, structural changes in employment and consumption, pollution of the environment, etc.).

Another strategic decision is concerned with long-range impact of world prices in food on national development.

Investigating the (projected) terms of trade change on the national economy as well as the consumption structure change, it is possible to determine the best structure of long-range food production. In the case of a country like Poland, the strategic problem is the following. Should the economy import grain in order to export meat, or should it strive toward self-sufficiency? The answer to that question is not simple, because the food production depends on the supply of fertilizers, labor and capital, which are needed as well in other sectors of the economy. In order to solve the present complex problem, the core model should be used with the possible co-operation of more specialized submodels dealing with employment, environment, regional, etc. submodels.

These two examples show how important it is (for the purpose of long-range planning and development of the economy) to have a family of submodels cooperating with the national core model. Since the development decision depends much on the world prices, it is important as well to have a good long-range global model. There is, of course, a long way to go in the construction of good regional, national, and global models.

The results obtained so far should be regarded as encouraging,and stimulate further research. However, much still remains to be done along these lines.

APPENDIX 1

Consider the single sector (consumer) model with the production (utility) functional

$$Y = \int_0^T \prod_{\nu=1}^{m} f_\nu(t)\, dt \quad , \tag{A.1}$$

where

$$f_\nu(t) = \left\{ \int_0^t k_\nu(t-\tau)\, x_\nu(\tau)\, d\tau \right\}^{\beta_\nu} \quad , \qquad \sum_{\nu=1}^{m} \beta_\nu = 1 \quad ,$$

$$x_\nu(\tau) = \left[Z_\nu(t) \right]^{\alpha_\nu} \quad , \qquad 0 < \alpha_\nu < 1 \quad , \qquad \nu = 1,\ldots,m \quad ,$$

$k_\nu(t)$ = given functions, $k_\nu(t) = 0$, $t < 0$, having "inertial" Laplace transforms $K_\nu(p)$. We assume that $\nu = 1$ corresponds to the most inertial term. When, e.g. $K_\nu(p) = e^{T_\nu p}$, $\nu = 1,\ldots,n$ one can set $T_1 \geq T_2 \geq \ldots \geq T_m$.

It is necessary to find non-negative strategies $x_\nu(t) = \hat{x}_\nu(t)$, $\nu = 1,\ldots,m$, $t \in [0,T]$ such that (A.1) attains maximum subject to the constraints

$$\int_0^T w_\nu(\tau) [x_\nu(t)]^{1/\alpha_\nu}\, d\tau \leq Z_\nu \quad , \qquad \nu = 1,\ldots,m \quad . \tag{A.2}$$

In order to solve that problem one can apply the generalized Hölder inequality

$$Y = \int_0^T \prod_{\nu=1}^{m} f_\nu(t)\, dt \leq \prod_{\nu=1}^{m} \left\{ \int_0^T |f_\nu^{\frac{1}{\beta_\nu}}(\tau)|\, d\tau \right\}^{\beta_\nu} \quad ,$$

which becomes an equality if (almost everywhere)

$$f_1^{1/\beta_1}(\tau) = c_\nu f_\nu^{1/\beta_\nu}(\tau) \quad , \quad \tau \in [0,T] \quad , \quad \nu = 2,\ldots,m \tag{43}$$

c_ν = const, $\nu = 2,\ldots,m$. In the present case one obtains

$$Y \leq \prod_{\nu=1}^{m} c_\nu^{-\beta_\nu} \int_0^T dt \int_0^t k_1(t-\tau)x_1(\tau)\, d\tau$$

$$= \prod_{\nu=1}^{m} c_\nu^{-\beta_\nu} \int_0^T x_1(\tau)\, d\tau \int_\tau^T k_1(t-\tau)\, dt \quad .$$

Applying again Hölder inequality one gets

$$Y \leq \prod_{\nu=1}^{m} c_\nu^{-\beta_\nu} \left\{ \int_0^T w_1(\tau)\,[x_1(\tau)]^{1/\alpha_1}\, d\tau \right\}^{\alpha_1}$$

$$\left\{ \int_0^t \left[w_1(t)^{-\alpha_1} \int_\tau^T k_1(t-\tau)\, dt \right]^{\frac{1}{1-\alpha_1}} d\tau \right\}^{1-\alpha_1}$$

where the equality appears if (almost everywhere)

$$x_1(\tau) = c_1 \bar{x}_1(\tau) \quad , \quad \bar{x}_1(\tau) = \left[w_1^{-1}(\tau) \int_\tau^T k_1(t-\tau)\, dt \right]^{\frac{\alpha_1}{1-\alpha_1}}$$

The value of c_1 can be derived by (A.2) yielding

$$c_1 = \left\{ \frac{Z_1}{\int_0^T w_1(\tau)\,[\bar{x}_1(\tau)]^{1/\alpha_1}\, d\tau} \right\}^{\alpha_1} \quad .$$

Then

$$Y(\hat{\underline{x}}) = \prod_{\nu=1}^{m} c_\nu^{-\beta_\nu} Z_1^{\alpha_1} \left\{ \int_0^T w_1(\tau) \left[w_1^{-1}(t) \int_\tau^T k_1(t-\tau)dt \right]^{\frac{1}{1-\alpha_1}} d\tau \right\}^{1-\alpha_1}$$

$$= \prod_{\nu=1}^{m} \left(\frac{c_1}{c_\nu} \right)^{\beta_1} \int_0^T w_1(t) [\bar{x}_1(t)]^{1/\alpha_1} d\tau \qquad (A.4)$$

In order to solve equations (A.3) in an explicit way it is convenient to use Laplace transforms of (A.3) :

$$K_1(p)X_1(p) = c_i K_i(p) X_i(p) \quad , \quad i = 2,\dots,m$$

Since

$$\hat{x}_1(p) = c_1 K_1^*(p) \quad , \quad K_1^*(p) = \mathscr{L} \left\{ w_1^{-1}(\tau) \int_\tau^T k_1(t-\tau)dt \right\}^{\frac{\alpha_1}{1-\alpha_1}}$$

$$\hat{x}_\nu(t) = \frac{c_1}{c_\nu} \bar{x}_\nu(t) \quad , \quad \nu = 2,\dots,m \quad ,$$

where

$$\bar{x}_\nu(t) = \mathscr{L}^{-1} \left\{ \frac{K_1^*(p) K_1(p)}{K(p)} \right\}$$

The numerical values of c_ν, $\nu = 2,\dots,m$ can be derived by (A.2) and one gets

$$\hat{x}_\nu(t) = \left(\frac{Z_\nu}{X_\nu} \right)^{\alpha_\nu} \bar{x}_\nu(t) \quad , \quad X_\nu = \int_0^T w_\nu(\tau) [\bar{x}_\nu(\tau)]^{1/\alpha_1} dt \quad ,$$

$$\nu = 1,\dots,m \qquad (A.5)$$

Then

$$Y(\hat{\underline{x}}) = G^q \prod_{\nu=1}^{m} Z_\nu^{\delta_\nu} \quad , \qquad \delta_\nu = \alpha_\nu \beta_\nu \quad , \qquad \nu = 1,\ldots,m \quad , \tag{A.6}$$

when

$$G^q = X_1 \prod_{\nu=1}^{m} X_\nu^{-\delta_\nu} \quad , \qquad q = 1 - \sum_{\nu=1}^{m} \delta_\nu$$

Example

Maximize

$$Y = \int_0^T \left\{ \int_0^t e^{-\lambda(t - \tau)} [x(\tau)]^{\frac{1}{2}} d\tau \right\}^\beta [y(t)]^{1-\beta} dt \quad , \tag{A.7}$$

subject to

$$\int_0^T x(t)dt \leq X \quad , \tag{A.8}$$

$$\int_0^T y(t)dt \leq Y \quad , \tag{A.9}$$

By (A.5) one obtains

$$\bar{x}(\tau) = \int_\tau^T e^{-\lambda(t - \tau)} dt = \frac{1}{\lambda}\left[1 - e^{-\lambda(T - \tau)}\right] \quad , \tag{A.10}$$

$$\bar{y}(t) = \int_0^t \bar{x}(\tau) e^{-\lambda(t-\tau)} d\tau = \frac{1}{\lambda^2}\left[1 - e^{-\lambda t} - e^{-\lambda T}(1 - e^{-\lambda t})\right] \tag{A.11}$$

It is interesting to observe that $\bar{x}(\tau)$ is monotonously decreasing to zero when $t \to T$. At the same time $y(t)$ increases (starting from zero) monotonously along with t. When T is large enough $\bar{x}(t)$, $\bar{y}(t)$ attain saturation levels when $t \to \infty$, or $t \to o$ respectively. It can be explained assuming (A.7) describes the integrated economy with limited supply of capital stock (A.8) and labor (A.9). The best investment strategy (A.10) should decrease in time when capital stock increases while the labor intensity (A.11) should increase in order to use the increasing capacity.

It should be explained why it is necessary to assume that $K_1(p)$ should be most "inertial" among $K_\nu(p)$, $\nu = 1,\ldots,m$. For that purpose assume $K_\nu(p) = e^{T_\nu p}$, $\nu = 1,\ldots,m$. Then

$$\bar{X}_\nu(p) = d^{p(T_1 - T_\nu)} K_1^*(p) \quad ,$$

and when $T_1 < T_\nu$, the function $X_\nu(p)$ is nonanalytic. For the same reason when $K_1(p)/_{K_\nu(p)}$ is a ratio of polynomials with numerator of higher order than denumerator the $\hat{x}_\nu(t)$ contains $\delta(t)$ functions and the integral in (A.2) will not converge. It is now obvious also why it is important to have $0 < \alpha_\nu < 1$ when one wants to have a unique solution.

Since the amount of resources (A.2) is bounded, the solutions obtained are very sensitive to the time horizont T. As a result the model (A.1),(A.2) is especially useful for the analysis of concrete projects, where the expenditures and revenues terminate in time.

This is the usual situation also at the macro-level where one takes into account only these investment projects which incur the benefits (i.e. which contribute to the output production) within the planning interval. Then the planning should be organized as a continuous process with the moving (each year) time horizont while the optimization strategies should be readjusted each year according to the changing objectives.

For a continuous development process the integral constraints (A.2) may be replaced by the amplitude constraint

$$\sum_{\nu=1}^{m} [x_\nu(t)]^{1/\alpha_\nu} = Z(t) \quad , \quad t \in [O,T] \quad . \tag{A.12}$$

Since the equations (A.3) can be solved for $\hat{x}_\nu(t)$, $\nu = 2,\ldots,m$, so that the operators $x\ (t) = R_\nu[x_1]$ exist one should try to solve the eq.

$$[x_1(t)]^{1/\alpha_1} + \sum_{\nu=2}^{m} [R_\nu(x_1)]^{1/\alpha_\nu} = Z(t) \quad , \quad t \in [O,T]$$

for $\hat{x}_1(t)$. The constants c_ν, $\nu = 2,\ldots,m$ should be determined by the conditions :

$$\int_O^T w_\nu(\tau)[\hat{x}_\nu(\tau)]^{1/\alpha_\nu} d\tau = \gamma_\nu \int_O^T w(t)Z(t)dt \quad , \quad \nu = 1,\ldots,m \tag{A.13}$$

where

$$\sum_{\nu=1}^{m} \gamma_\nu = 1 \quad , \quad \gamma_\nu \text{ - given positive numbers} \quad .$$

The present consumption model (A.1), (A.12). (A.13) corresponds to a situation when the existing resources $Z(t)$ should be completely utilized at each $t \in [O,T]$ with the given consumption structure determined by γ_ν, $\nu = 1,\ldots,m$.

The single sector (consumer) model (A.1)(A.2) can be extended easily to the n sector case (18) ÷ (24) and (40) ÷ (42).

Suppose the n production (utility) functions (A.6) :

$$Y_i = G_i^q \prod_{\nu=1}^{m} Z_{\nu i}^{\delta_\nu} \quad , \quad i = 1,\ldots,n \quad ,$$

$$q = 1 - \sum_{\nu}^{m} \delta_\nu > O \quad ,$$

be given. Assume also that the expenditures in each ν-th sphere of activity (Z_ν) be given, i.e.

$$\sum_{i=1}^{n} Z_{\nu i} \leq Z_\nu \quad , \qquad \nu = 1,\ldots,m \quad . \tag{A.14}$$

The problem of optimum allocation of Z_ν among the n sectors (consumers) can be formulated as follows. Find the non-negative $Z_{\nu i} = Z_{\nu i}$, $\nu = 1,\ldots,m$, $i = 1,\ldots,n$, such that

$$Y = \sum_{i=1}^{n} G_i^q \prod_{\nu=1}^{m} Z_{\nu_i}^{\delta_\nu} \quad , \tag{A.15}$$

attains maximum subject to (A.14).

Since Y is a strictly concave in the convex set (A.14) a unique solution exists and(as can be easily shown)becomes

$$\hat{Z}_{\nu i} = \frac{G_i}{G} Z_\nu \quad , \qquad \nu = 1,\ldots,m \quad , \qquad i = 1,\ldots,n \tag{A.16}$$

where

$$G = \sum_{i=1}^{n} G_i \quad ,$$

and

$$Y(\hat{Z}_{\nu i}) = G^q \prod_{\nu=1}^{m} Z_\nu^{\delta_\nu} \tag{A.17}$$

Now it is also possible to solve the problem of optimum allocation of the total amount of resources Z among the m different spheres of activity. In other words, it is necessary to find $Z_\nu = \hat{Z}_\nu$, $\nu = 1,\ldots,m$, such that maximize (A.17) subject to the constraint

$$\sum_{\nu=1}^{m} Z_\nu \leq Z \quad .$$

The optimum strategy is unique and (as can be shown) becomes

$$\hat{Z}_\nu = \frac{\delta_\nu Z}{\delta} \quad , \qquad \delta = \sum_{\nu=1}^{m} \delta_\nu \quad , \qquad \nu = 1,\ldots,m \quad , \tag{A.18}$$

while

$$Y(\hat{Z}_\nu) = G^q \prod_{\nu=1}^{m} \left({}^{\delta_\nu}/_{\delta} \right)^{\delta_\nu} Z^\delta \quad . \tag{A.19}$$

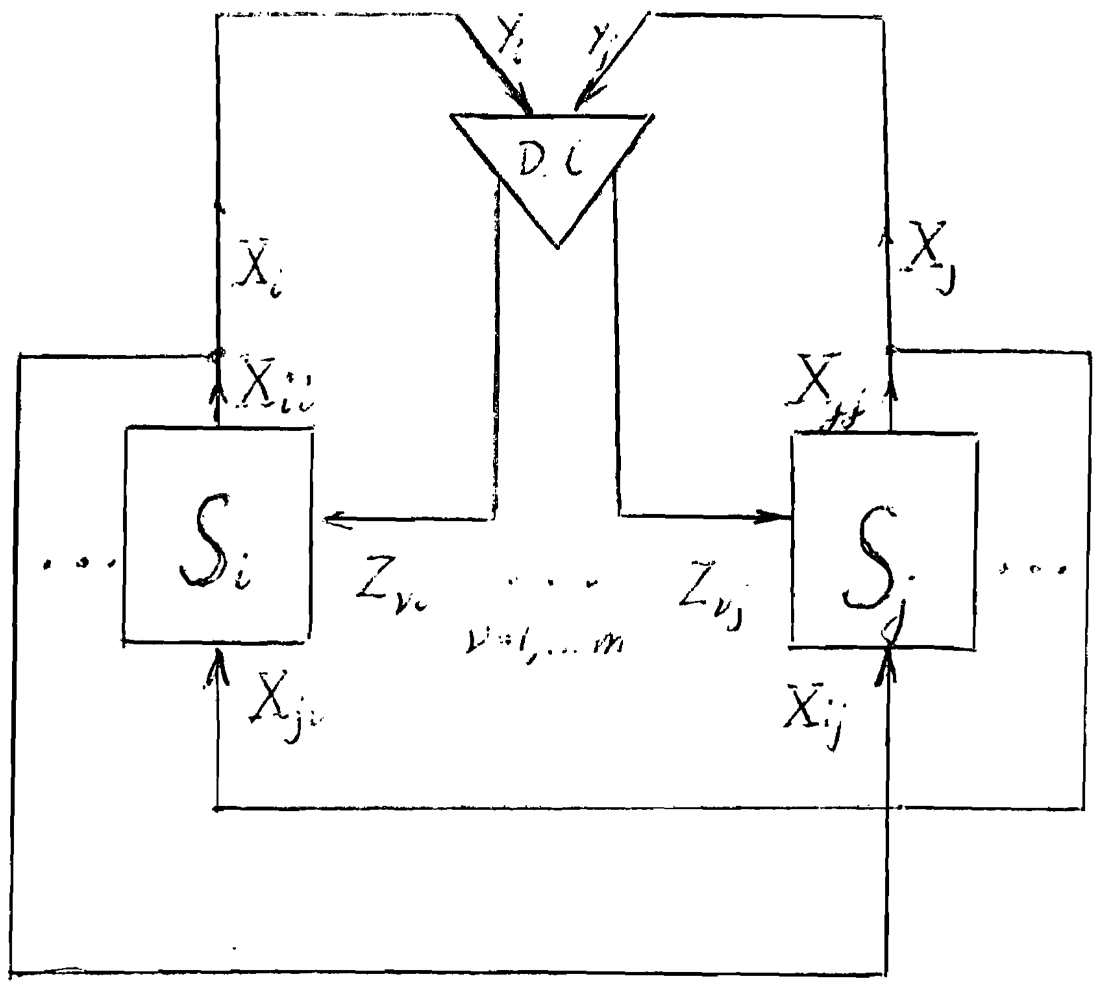
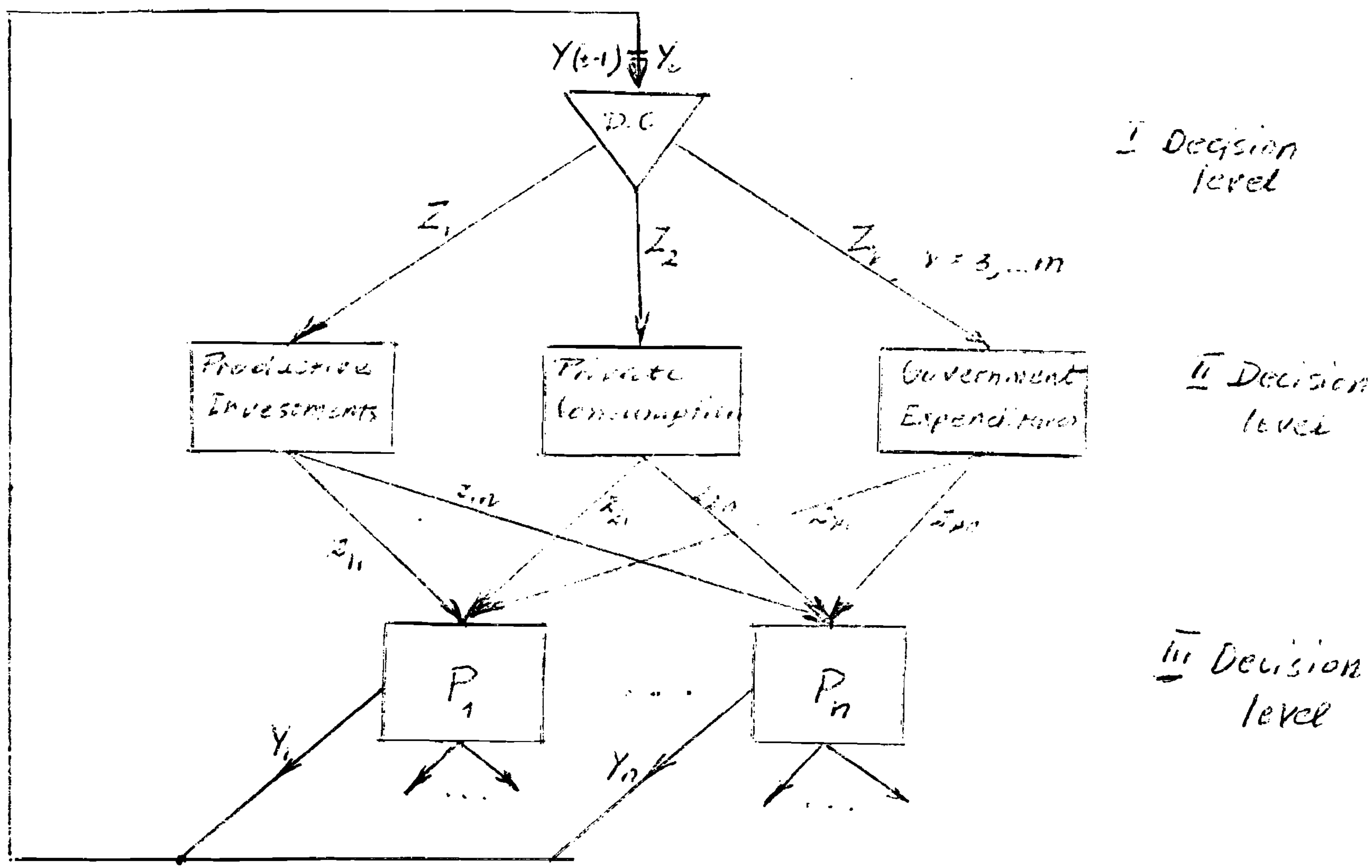


Fig 1



Y(t-1) = Y0
D.C
Z1
Z2
Zν ν = 3,...m
Productive Investments
Private Consumption
Government Expenditures
z11
z1n
z21
z2n
zν1
zνn
P1
Pn
Y1
Yn
I Decision level
II Decision level
III Decision level

Fig 2

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