Working Paper

Optimal Operational Strategies for an Inspected Component – Statement of the Problem

U. Pulkkinen and S. Uryas'ev

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October 1990

IIASA

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Foreword

This is the second report on work done on time dependent probabilities initiated in cooperation between the International Atomic Energy Agency (IAEA) and IIASA in 1990. The treatment of the underlying mathematical model is rather theoretical, but the intent is to cover a broad range of applications. The advantage with the problem formulation is that it enables the inclusion also of monetary considerations connected to risks and the actions for decreasing them. The intent in formulating the model is that it will be used for a computerized optimization of selected decision variables. Originally, the formulation was initiated by the problem of optimization of test intervals at nuclear power plants. In this paper the non-destructive testing of major components has been approached. The main result of the paper is the formulation of an optimal rule for decision if continued operation can be considered safe enough. The decision rules integrates the earlier operational history, safety concerns and economic considerations. Also other applications are proposed to be treated within the modeling framework. One specific problem is the selection of the most suitable time instant for a major repair or retrofitting at a plant. The time horizon of the model can be selected either short-term, stretching only over a few weeks or long-term, to encompass the complete life time of a depository of spent nuclear fuel.

Comments or proposals for applications of this modeling approach are invited.

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OPTIMAL OPERATIONAL STRATEGIES FOR AN INSPECTED COMPONENT – STATEMENT OF THE PROBLEM

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1 Introduction

The failures of mechanical components often develop gradually. This fact should be taken into account, if we would like to predict failures and optimize operational procedures and strategies for preventive maintenance and repair of the system. The prediction of failures is usually based on probabilistic models, the simplest ones being the probability distributions for the time to fail with deterministic failure or hazard rate. More general models which may describe also the gradual development of the failures are based on the notion of stochastic intensity, which has been discussed by some authors (see, for example, [11],[15]). Mathematically, the models with stochastic intensities are much more difficult than the traditional models.

In many practical cases the gradual development of the failures cannot be observed directly but through more or less imperfect measurements or inspections. The models for imperfect observations are also probabilistic, and this leads to filtering problems in which the stochastic intensity is estimated on the basis of imperfect observations. This problem has been intensively considered in the literature (see, for example, [7],[10],[13]).

The optimization of the repair and maintenance strategies is usually made by minimization of the expected cost due to the failures and the maintenance and repairs. The practical means for controlling failure behavior are often very limited, the only possibilities are preventive maintenance or replacement of the component, and sometimes stopping the operation of the component. Since the controlling actions are made on the basis of imperfect measurements, it is also desirable to optimize the inspection or testing policies. In practice this means, for example, the selection of inspection intervals, methods and strategies.

Mathematically, the optimization of strategies for inspection, repair and preventive maintenance can be formulated as a stochastic optimization problem. As a rule, analytical solutions for such problems do not exist, and some numerical algorithms should be used. In principle, many problems of this type can be solved by using the dynamic programming algorithm (see [2] and others). Usually the dimension of the problem is very high and in practice one has to apply other approaches, for example the stochastic quasi-gradient techniques (see [4],[8], [12] and others) or scenario analysis (see [14]). We are going to use stochastic quasi-gradient algorithms with adaptive parameters control to solve the problem.

The models or approaches outlined above are useful in making decisions concerning the safety and economical operation of nuclear power plants. This is due to the stochastic nature of the failure phenomena and the high cost caused by accidents and extensive inspections of the plants.

The failures of nuclear power plant components which develop gradually are numerous. The growth of defects in the pressure vessel and in the pipings are good examples. A similar type of phenomena occur also in the pipings of the steam generators at PWR-plants. The inspections of the defects are usually very expensive, and the inspection costs depend on the ability of the inspection method to reveal the defects. The information obtained from the inspections is used in the decisions, for example, for stopping the plant or for preventive maintenance, which may lead to rather large costs. The problem is to make optimal decisions in order to gain economical profits and maintain a sufficient safety level of the plant.

This paper describes a model to evaluate and predict the gradual development of failures of a system which is inspected periodically. Further, an optimization problem is formulated in order to find the most appropriate inspection and maintenance policies.

2 General Description of the Model

2.1 Description of the Physical Phenomena

The manufacturing process of any mechanical component is more or less imperfect. When the component is taken into operation it has always some defect or faults, which do not cause the failure of the component or the system immediately. The number, the size and the type of the existing defects are usually unknown because they cannot be observed directly. The only possibility to evaluate these is to make good guesses on the basis of experiences obtained from similar manufacturing processes and components. The uncertainty about the defects can be modeled with suitable probability distributions.

The characteristics of each defect (e.g., size, shape) will change with time. The rate of growth of the defect may be dependent on several environmental conditions, and the number of defects may increase. Often the development of the defects is correlated with the shocks which occur in the system due to some external phenomena. For example, at a nuclear power plant typical shocks may be thermal transients due to spurious or inadvertent operation of safety systems or due to emergency shut down of the reactor. The empirical and theoretical research carried out on this subject is wide (see, for example, [3],[5],[6]).

The development of the defects is followed by making inspections periodically and possibly at any moment of time when there is reason to believe that the defects have increased. The probability of identifying and properly estimating characteristics of the defects depends on the properties of the inspection methods. In the literature, there are some rather reliable models for describing the effectivity of inspections, and a lot of experimental research has been made on the subject ([1],[9]). In the case of metallic components, the most popular inspection methods are based on radiographic or ultrasonic inspections or eddy current measurements.

The component fails if the size of defect exceeds some limit. In the case where several defects exist in the component, it is not easy to express the exact failure criteria. The component may fail even if all the sizes of the defects are under the failure limit. It is feasible to think that the failure probability can be expressed as a function of the failure rate or intensity. In our case, the failure intensity depends on the number and characteristics of the defects in the component. Since the defects may grow or change in time stochastically, the failure intensity is also a stochastic process.

In the following sections we will describe a probabilistic model for the phenomena described above. The model gives an idealized picture of the real degradation process of a (mechanical) component, and can be used to find optimal inspection and repair strategies.

2.2 Probability Distribution for the Number and Size of the Defects

The initial defects in a metallic structure may be classified according to their properties and growth mechanisms. The most important properties of a defect are its size and its orientation, which determine the probability with which they can be identified in an inspection. In principle, any defect can be characterized with some vector $z = (z_1, \dots, z_m)$, where each component of the vector z corresponds to some property of the defect. This kind of characterization would lead to more complicated models than what is needed in our example.

We assume that a defect is completely characterized by its class, denoted by D with $D \in \{1, \dots, K\}$, and by its size, denoted by C with $C \in (0, C^D_{max}]$. Interval $(0, C^D_{max}]$ includes C^D_{max} and does not include 0. The class of the defect is needed because it is possible that all defects do not have similar growth mechanisms, or they cannot be identified with the same probability.

We suppose that at the beginning of the operation of the system (at time point $t = 0$) the initial number of defects is $M(0) \geq 0$. Each of the $M(0)$ defects are characterized with their class, $D_i(0)$, and their size, $C_i(0), i = 1, \dots, M(0)$, at $t = 0$.

The values of $M(0), D(0) = \left(D(0), \dots, D(0)_{M(0)}\right)$ and $C(0) = \left(C(0), \dots, C(0)_{M(0)}\right)$ are unknown, and they depend on the random variation of the manufacturing process. Thus it is possible to assume that they are random variables specified on the probability space $(P, \mathcal{F}, \Omega)$ [1]. The model for $M(0), D(0)$ and $C(0)$ is now simply their joint distribution:

$$P\big(M(0) = m, D(0) = d, C(0) \in [c, c + dc)\big) = \tag{1}$$

$$P\big(M(0) = m, D_1(0) = d_1, C_1(0) \in [c_1, c_1 + dc_1), \ldots,$$

$$D_m(0) = d_m, C_m(0) \in [c_m, c_m + dc_m)\big) \stackrel{\text{def}}{=} g_0^{MDC}(m, d_1, c_1, \ldots, d_m, c_m) dc_1 \times \ldots \times dc_m \,.$$

It should be noticed that the random variables $M(0)$ and $D(0)$ are discrete and the random variable $C(0)$ is continuous. If we assume that $\big(D_1(0), C_1(0)\big), \ldots, \big(D_m(0), C_m(0)\big)$ are independent given $M(0) = m$, and if the joint distribution of $\big(D_i(0), C_i(0)\big)$ is $g_0^{DC}(d_i, c_i)$, then the distribution (1) can be written in the form:

$$g_0^{MDC}(m, d_1, c_1, \ldots, d_m, c_m) = g_0^{M}(m) \prod_{i=1}^{m} g_0^{DC}(d_i, c_i) \,, \tag{2}$$

in which $g_0^{M}(m)$ is the distribution of $M(0)$.

If we finally assume that there is only one class of defects we obtain the distribution

$$g_0^{MC}(m, c_1, \ldots, c_m) = g_0^{M}(m) \prod_{i=1}^{m} g_0^{C}(c_i). \tag{3}$$

In practical situations the assumption of independency of the defect sizes may not be appropriate, and in that case we cannot write the above distributions in the product form. This may cause some calculational difficulties.

The functional form of the distribution (3) can be selected from rather wide family of distributions. In our case we restrict the distribution $g_0^{M}(m)$ on a set $m \in \{1, \ldots, M_{max}\}$ with M_{max} being rather small ($M_{max} \cong 50$). Further, it is probable that the size of any defect is small in the beginning of the operation and that its size doesn't exceed some upper value. Here we use the following discrete distribution for the number of the defects:

$$g_0^{M}(m) = \eta_m, \;\; 0 \leq m \leq M_{max} \tag{4}$$

with $\eta_m \geq 0$ and $\sum_{m=1}^{M_{max}} \eta_m = 1$.

Here we assume that the size of a defect will follow a truncated exponential distribution with the density function:

$$g_0^{C}(c) = Q_C \sigma_C \exp\{-\sigma_C c\} \tag{5}$$

in which $0 \leq c \leq c_{max}$, σ_C is a known parameter, and Q_C is a normalizing factor.

[1] We denote the random variables without the index $\omega \in \Omega$, i.e. the random variable $x(\omega)$ is denoted simply by x, whenever it is possible without confusion.

2.3 Probability Model for the Shock Occurrence

The defects of the component may grow due to the shocks which occur in the system. These shocks are caused by various external phenomena independently of the development of the defects. These kind of phenomena are usually described with random point process models. The most simple point process is the time homogeneous Poisson process which we shall apply here.

We assume that the shocks occur according to the homogeneous Poisson process model with constant intensity γ. Accordingly, the time points at which the shocks occur, τ_l, $l = 1, 2, \ldots$, can be expressed as sums of exponentially distributed random variables, i.e.,

$$\tau_l = \delta_1 + \delta_2 + \ldots + \delta_l \ , \tag{6}$$

in which the variables $\delta_1, \ldots, \delta_l$ are identically exponentially distributed random variables with density function

$$g^{\delta}(t) = \gamma \exp\{-\gamma t\} \ . \tag{7}$$

Generalizations of the above model can be easily developed, for example, the intensity γ may be assumed time dependent, or even stochastic and dependent on the other random variables of the model.

2.4 Model for the Defect Growth

In order to describe the random changes of the number and the sizes of the defects, we have to make some assumptions. First, we assume that the number of defects will be constant if the component is not repaired. Further, we assume that the size of the defects may increase only at the moments of the time where shocks occur (at the time points τ_l). If the component is repaired, all the old defects will be removed but some new defects may be introduced into the component. The repair may only be done after some shock or inspection point according to the control law which will be described later.

The above assumptions are not the only possible ones. For example, the defects may grow also between shocks due to some chemical phenomena. Further, it is possible that new defects may be introduced into the component also between repairs. Thus our assumptions must be considered as idealized approximations.

We denote the number of defects at time point t by $M(t)$ and the vector of sizes of the defects at the same time point by $C(t)$. The time points where the inspections are made are denoted by $t_1^I, t_2^I, \ldots$ We denote the ordered union of the points $t_1^I, t_2^I, \ldots$ and $\tau_1, \tau_1, \ldots$ by $V = \{t_1, t_2, \ldots\}$.

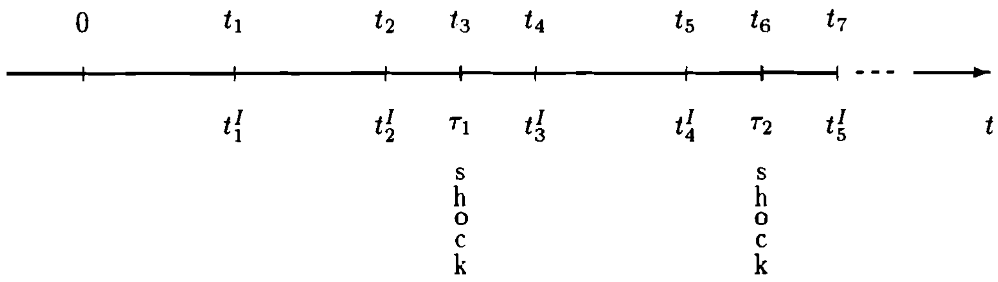

Figure 1: Set $V = \{t_1, t_2, \ldots\}$

At each t_j there either occurs a shock or an inspection (and possibly a repair) is made. The set V is illustrated in Figure 1.

Due to our assumptions, the changes of $M(t)$ and $C(t)$ must be considered only at the time points t_j. $M(t)$ and $C(t)$ are constant in any interval $[t_j, t_{j+1})$, i.e.

$$M(t) = M(t_j), \quad \forall t \in [t_j, t_{j+1}), \quad j = 1, 2, \ldots \, , \tag{8}$$

and

$$C(t) = C(t_j), \quad \forall t \in [t_j, t_{j+1}), \quad j = 1, 2, \ldots . \tag{9}$$

If the component is repaired at the time point t_j then the old defects are removed from the component. However, some new defects are introduced into the component. The number and the sizes of these new defects follow the distributions

$$g_{t_j}^r(m) = \eta_m^r, \quad 0 \le m \le M_{max}^r \, , \tag{10}$$

with $\eta_m^r \ge 0$, $\sum_{m=1}^{M_{max}} \eta_m^r = 1$, and

$$g_{t_j}^{RC}(c) = Q_{RC} \sigma_{RC} \exp\{-\sigma_{RC} c\} \, , \tag{11}$$

where $0 \le c \le c_{max}^r$ and σ_{RC} is a parameter, and Q_{RC} is a normalizing factor.

The values of the process $M(t)$ are constant between repairs, or

$$M(t) = M(0), \quad \forall t \in [0, t_1^r) \, , \tag{12}$$

$$M(t) = M(t_j^r), \quad \forall t \in [t_j^r, t_{j+1}^r) \, , \tag{13}$$

in which the time point t_j^r is the point where the component is repaired, and $M(t_j^r)$ follows the distribution (10).

The values of the process $C(t)$ are constant between the shock points τ_l, but they may grow at each shock point. The increase of the size of the defect is a random variable which may depend on the size of the defect just before shock. We assume also that the defects grow independently. The conditional distribution of the size increase of a defect given the defect size, c, is assumed to be of the form,

$$g^{sc}\big(\Delta c \mid c\big) = \frac{\zeta}{c}\exp\Big\{-\frac{\zeta\Delta c}{c}\Big\}\,, \tag{14}$$

which means that the size increments are exponentially distributed random variables with the expected value $\frac{c}{\zeta}$.

The process $C(t)$ is constant between shocks (if there are no repairs between the shocks) or between successive repair and shock:

$$C(t) = C(0), \quad \forall t \in (0, \min\{t_1^r, \tau_1\})\,, \tag{15}$$

$$C(t) = C(\max\{\max_{\tau_l \le t} \tau_l\,,\ \max_{t_j^r \le t} t_j^r\})\,, \tag{16}$$

in which each component of the vector $C(0)$ follows the distribution (5), components of $C(t_j^r)$ follow the distribution (11) and at $t = \tau_i$ the increments of $C(t)$ components are random variables following the distribution (14). The increase of the size of one defect is illustrated in Figure 2.

The above model is one of the most simple possibilities. It is applied here in order to formulate the optimization problem. In literature (see [3],[6]) several different models have been considered. The model discussed here may be modified rather easily in order to make it more realistic.

2.5 Probability Model for Inspections

The inspections are made in order to measure the sizes of defects. However, the inspections are not complete and they will not identify all defects with probability one. Further, the measurements do not give exact information on the size of the defects. Berens (see [1]) discusses both the so called hit/miss model and the signal response model for modeling the reliability of inspection of metallic structures. We shall apply here the model based on the signal response approach.

The basic idea of the signal response model is that each defect causes a signal which can be measured. However, there is also measurement noise due to some external phenomena. Further, if the signal is too weak then the defect cannot be identified. We denote by $\theta_i(t_j)$ the signal caused by defect i with the size $C_i(t_j)$ at the time point $t = t_j$. If the signal $\theta_i(t_j)$ is below some limit, θ^{tr}, then the defect i is not identified. We assume that the signal $\theta_i(t_j)$ is related with the true size $C_i(t_j)$ according to

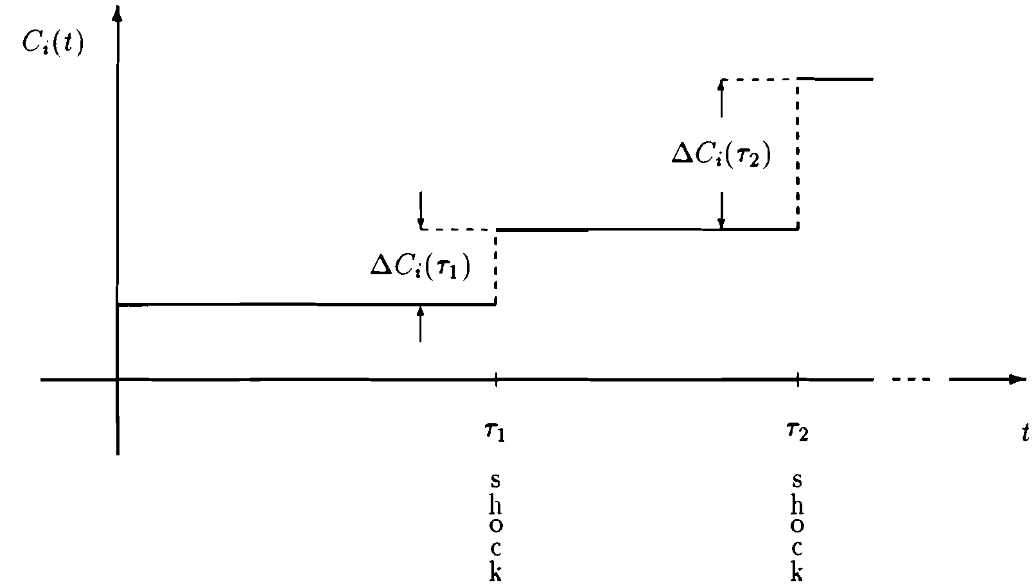

Figure 2: Graph $C_i(t)$.

$$\ln\theta_i(t_j) = \beta_0 + \beta_1 \ln C_i(t_j) + \xi \ , \tag{17}$$

in which β_0 and β_1 are parameters and ξ is a normally distributed random variable with zero mean and variance σ_ξ^2 (i.e. $\xi \sim N(0, \sigma_\xi^2)$). In other words, the conditional distribution of $\ln\theta_i(t_j)$ given $C_i(t_j)$ is normal with parameters $\beta_0 + \beta_1 \ln C_i(t_j)$ and σ_ξ. The random variable ξ describing the measurement noise could also follow any other distribution, but we use here the Gaussian distribution.

We assume further that the defects are inspected independently, and thus the signal $\theta_i(t_j)$ corresponding to the defect i is stochastically independent of the other signals. The probability that a defect of the size c is not identified in the inspection is given by

$$P(\theta \le \theta^{tr} \mid c) = \Phi\left\{\frac{\ln\theta^{tr} - (\beta_0 + \beta_1 \ln c)}{\sigma_\xi}\right\} \stackrel{\text{def}}{=} \phi(c) \ , \tag{18}$$

in which $\Phi\{\cdot\}$ is the cumulative standard normal distribution function.

Let us denote by $\nu(t_j)$ the number of defects identified in an inspection at $t = t_j$. Since all defects are not identified with probability one, $\nu(t_j)$ may be smaller than the true number of defects $M(t_j)$, i.e. $\nu(t_j) \le M(t_j)$. The result of an inspection at $t = t_j$ is described with a random variable $\big(\nu(t_j), \theta^\nu(t_j)\big) = \big(\nu(t_j), \theta_1(t_j), \theta_2(t_j), \ldots, \theta_{\nu(t_j)}(t_j)\big)$.Define

$$\begin{pmatrix} M \\ \nu \end{pmatrix} = \frac{M!}{\nu!\,(M-\nu)!} \ .$$

The conditional probability distribution of $\big(\nu(t_j), \theta_1(t_j), \theta_2(t_j), \ldots, \theta_{\nu(t_j)}(t_j)\big)$ given $\big(M(t_j), C_1(t_j), C_2(t_j), \ldots, C_{M(t_j)}(t_j)\big)$ is now:

$$h\big(\nu(t_j),\theta(t_j)\mid M(t_j),C(t_j)\big) = h\big(\nu(t_j),\theta_1(t_j),\theta_2(t_j),\ldots,\theta_{\nu(t_j)}(t_j)\mid M(t_j),C(t_j)\big) =$$

$$\prod_{i=1}^{\nu(t_j)}\left[g^{tr}\big(\theta_i(t_j)\mid C(t_j),\theta_i(t_j)\geq\theta^{tr}\big)\right]\prod_{i=\nu(t_j)+1}^{M(t_j)}\phi\big(C_i(t_j)\big)\binom{M(t_j)}{\nu(t_j)} \tag{19}$$

in which

$$g^{tr}\big(\theta_i(t_j)\mid C(t_j),\theta_i(t_j)\geq\theta^{tr}\big) =$$

$$\begin{cases} Q^{tr}\frac{1}{\theta_i(t_j)\sigma_\xi\sqrt{2\pi}}\exp\left\{-\frac{[\ln\theta_i(t_j)-(\beta_0+\beta_1\ln C_i(t_j))]^2}{2\sigma_\xi^2}\right\}, & \text{if } \theta_i(t_j)\geq\theta^{tr}\ ; \\ 0\ , & \text{otherwise}\ ; \end{cases} \tag{20}$$

is the truncated density function of normal distribution with the parameters $\beta_0+\beta_1\ln C_i(t_j)$ and σ_ξ^2 , and normalizing constant Q^{tr} . The distribution (19) has the above form due to the independence of $\theta_i(t_j)$ given $\big(M(t_j),C(t_j)\big)$. It should be noticed that in (19) the defects are indexed in the order of identification (the defects $1,\ldots,\nu(t_j)$ are identified, the defects $\nu(t_j)+1,\ldots,M(t_j)$ are still latent after the inspection).

The above model describes a situation in which only one inspection is made. In practice, the component is inspected periodically and possibly by applying several inspection methods.

In order to describe this situation we have to extend the above model. First we assume that each inspection method can be described with the model given in the equations (17), (18). The only difference between the inspection methods are the parameters values of the respective normal distribution (the parameters $\beta_0,\beta_1,\sigma_\xi^2$). In the following we need these parameters for two inspection methods and we denote them by $\beta_{0_k},\beta_{1_k},\sigma_{\xi_k}^2$ with $k=1,2$.

The simplest way to model a series of successive inspections is to assume that the successive inspections are stochastically independent. This assumption has some practically unacceptable consequences. For instance, it is possible that a defect which has already been identified in earlier inspections will not be identified again. In practice, the known defects are usually identified at every inspection after the first identification. This means that the successive inspections are dependent which should be taken into account in the model. Here we apply the following simple approach. We assume that the result of an inspection depends on the result of the previous inspection such that if the defect was identified at the previous inspection (i.e. $\theta_i(t_{j-1})\geq\theta^{tr}$ for the defect i) then the result of the present inspection would also exceed the identification threshold θ^{tr}. Further, we assume that the conditional distribution of the result $\big(\nu(t_j),\theta^\nu(t_j)\big)$ of the present inspection j, given that the defect was identified at the previous inspection $j-1$ is the truncated normal distribution. More exactly

$$h\big(\nu(t_j),\theta^\nu(t_j)\mid M(t_j),C(t_j),\nu(t_{j-1}),\theta^\nu(t_{j-1})\big) =$$

$$h\big(\nu(t_j), \theta_1(t_j), \theta_2(t_j), \ldots, \theta_{\nu(t_j)}(t_j) \mid M(t_{j-1}), C(t_{j-1}), \nu(t_{j-1}), \theta^{\nu}(t_{j-1})\big) =$$

$$\prod_{i=1}^{\nu(t_j)} \left[g^{tr}\big(\theta_i(t_j) \mid C(t_j), \theta_i(t_j) \geq \theta^{tr}\big)\right] \prod_{i=\nu(t_j)+1}^{M(t_j)} \phi\big(C_i(t_j)\big) \begin{pmatrix} M(t_j) - \nu(t_{j-1}) \\ \nu(t_j) - \nu(t_{j-1}) \end{pmatrix} . \tag{21}$$

In the above equation (21) the first product describes the defects identified at the previous inspection at t_{j-1} and at the present inspection at t_j and the second product describes the defects which are still unidentified.

If there are several types of inspections we assume that each one follows the above model (with their own parameters), and that if a defect is identified at a previous inspection of any type then it will be identified at every future inspection of any type.

The above model for inspections is rather simple. In a real situation it is possible that the results of future inspections will be dependent also on the value of the previous inspections (not only on the fact that the defect was identified as in our model). In some cases the inspections may be so noisy that they indicate the existence of a defect although there is no defect.

2.6 Probability Model for Failures

A component with many large defects will obviously fail with higher probability than a component without any defects. However, also a component without any defects from the described class will fail after a rather long period of time. Here we assume that the increase of the failure intensity due to a defect will be proportional to the size of the defect. Thus the failure intensity of the component can be written in the form

$$\lambda(t) = \lambda^1(t) + \mu \sum_{i=1}^{M(t)} C_i(t) , \tag{22}$$

where $\lambda^1(t)$ is the deterministic part of the failure intensity and the sum describes the stochastic contribution due to the defects.

In the above equation $M(t)$ (the number of the defects at time t) develops stochastically according to the model given in the equations (8,10,12,13) and $C(t)$ (the sizes of the defects) behaves according to the equations (9,11,14,15,16). The development of the variables $M(t)$ and $C(t)$ depend also on the occurrence of shocks and the decisions made to control the failure intensity.

The deterministic part of the failure intensity, $\lambda^1(t)$, is assumed to be of the form

$$\lambda^1(t) = \alpha_0 + \alpha_1 t , \tag{23}$$

in which α_0 and α_1 are nonnegative constants. Figure 3 describes the process $\lambda(t)$. The increase of the failure intensity, $\Delta\lambda(\tau_i)$, in Figure 3 is given by

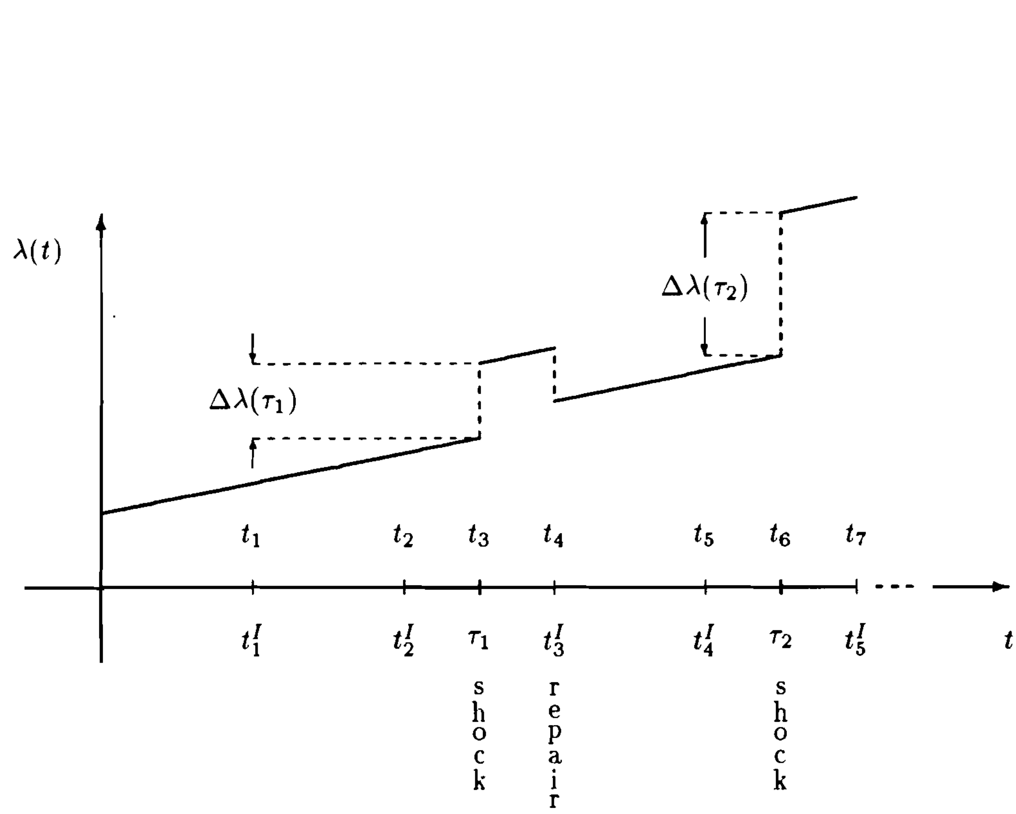

Figure 3: Graph $\lambda(t)$.

$$\Delta\lambda(\tau_i) = \mu \sum_{i=1}^{M(t)} \Delta C(\tau_i) \, .$$

The conditional survival function (i.e. the probability that the component will survive over the time period $[0, t_j]$) is (see, for example [15])

$$S(t_j) \stackrel{\text{def}}{=} P\big(T_* > t_j \mid \lambda(t), 0 \leq t \leq t_j\big) = \exp\left\{ - \int_0^{t_j} \lambda(t)\, dt \right\} , \tag{24}$$

in which $\lambda(t)$ is defined in the equation (22) and T_* is the failure time.

2.7 Estimation of the Failure Intensity

The inspections made in time points t_j give information on the number and the size of the defects in the component. Since the failure intensity is related to the size and the number of the defects this information can be used in the estimation of failure intensity at each inspection point.

In fact, the joint conditional distribution of the variables $M(t)$ and $C(t)$ given the results of inspections until t (i.e. $\big(\nu(0), \theta(0)\big), \ldots, \big(\nu(t_j), \theta(t_j)\big), t_j \leq t < t_{j+1}$) and the time points $(\tau_1, \ldots, \tau_l, \tau_l \leq t < \tau_{l+1})$ where shocks have occurred, can be determined recursively by using the law of the of the defect size increase, the law of the shock occurrence and Bayes rule. By using this conditional distribution we could also determine the respective conditional failure probability. In this study we do not consider these distributions, but we 'estimate' the failure intensity directly using the information from inspections.

Let us assume that the information up to the time t is the following set:

$$\Im(t) = \left\{ \left(\nu(0), \theta^{\nu}(0)\right), \ldots, \left(\nu(t_j), \theta^{\nu}(t_j)\right), \tau_1, \ldots, \tau_l, t \right\},$$

in which $t_j \leq t < t_{j+1}, \tau_l \leq t < \tau_{j+1}$. The estimate of the failure intensity is a function of the variable listed in the above set, for instance of the form:

$$\hat{\lambda}(t) = \lambda^1(t) + \Lambda\left(\left(\nu(0), \theta^{\nu}(0)\right), \ldots, \left(\nu(t_j), \theta^{\nu}(t_j)\right), \tau_1, \ldots, \tau_l, t\right), \tag{25}$$

in which $\Lambda(\cdot)$ is a function satisfying some measurability requirements.

Here we take into account only the result of the most recent inspection, i.e. $\left(\nu(t_j), \theta^{\nu}(t_j)\right)$. Since the result of the inspection and the variables $M(t), C(t)$ are related in a very simple way (see the equation (17)) and the failure intensity is a simple function of $M(t)$ and $C(t)$ (see the equations (22), (23)), we have the following estimate

$$\hat{\lambda}(t) = \lambda^1(t) + \mu \sum_{i=1}^{\nu(t_j)} \exp\left\{ \frac{\ln \theta_i(t_j) - \beta_0}{\beta_1} \right\}, \tag{26}$$

where μ and $\lambda^1(t)$ are defined in the equations (22), (23) and β_0 and β_1 are defined in the equation (17).

3 Formulation of the Optimization Problem

3.1 General Description

The component described above is used to give profit to the operator. Thus it is preferable to use it as long as possible. If the system is used over a long time, then the probability of failure increases. The losses due to the failure may be very high, especially in the case of nuclear power plants. The operator or the decision maker (DM) has the possibility to avoid high costs due to the failure by stopping the operation of the component or by repairing it. The early stopping will lead to loss of profit and the repair may be very expensive.

The decisions either to stop the component or to repair it may be done on the basis of the information obtained from the inspections. However, the inspections are also expensive and they cannot be made very often. The DM has to decide how often an inspection should be made and which type of inspection should be used.

Since we have described the stochastic behavior of the component we may formulate the selection of the best decision as a stochastic optimization problem. For that purpose we have to determine the objective function, give the model for the dynamics of the component, and establish the decision rules.

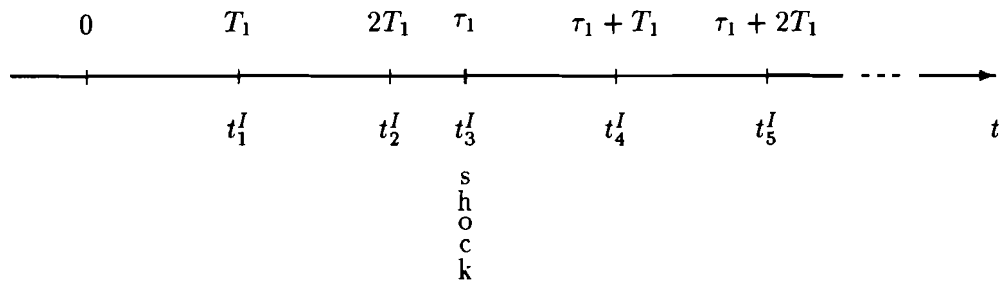

Figure 4: Shifted inspections schedule.

3.2 Decision Alternatives and Decision Rules

We assume that there are two types of inspections which follow the models described in Section 2.5. The most extensive inspection, 'the large inspection' is made regularly or periodically with fixed interval T_1. The cost of the large inspection is G_1. The other type of inspection, 'the small inspection', is made after every shock. The cost of the small inspection is G_2. Depending on the result of the small inspection also a large inspection is made after shock, in that case the time schedule of the large inspections is shifted. This means that if the shock occurred at the time point τ_l and the decision was to make also a large inspection at τ_l, then the next regular large inspection will be made at $t = \tau_l + T_1$ (see Figure 4).

After obtaining the result of a large inspection at some time point $t_j \in V$ (see Chapter 2.4 for definition of V), the DM has to choose between the following alternatives:

- continue the operation of the component;
- repair the component;
- stop the operation of the component.

The decision after a large inspection is denoted by $u_1(t_j)$ with the following values:

$$u_1(t_j) = \begin{cases} 0\,, & \text{continue the operation without repair}\,; \\ 1\,, & \text{repair the component}\,; \\ 2\,, & \text{stop the operation of the component}\,. \end{cases} \tag{27}$$

After obtaining the result of a small inspection, the DM has to choose between alternatives:

- continue the operation of the component;
- make a large test and shift the time schedule.

We denote the decision after a small inspection by $u_2(t_j)$ which has the following values:

$$u_2(t_j) = \begin{cases} 0\,, & \text{continue the operation}\,; \\ 1\,, & \text{make a large inspection, shift the time schedule}\,. \end{cases} \tag{28}$$

The selection between the above alternatives at each time point t_j should be based on the information obtained until the time point t_j. Further, the decisions should be such that they minimize the expected value of total costs.

We assume here that the decision is made on the basis of the most recent failure intensity estimate described in the equation (26). At the time point t_j, the DM measures $\big(\nu(t_j), \theta^{\nu}(t_j)\big)$ and uses this value in selecting the best decision. In order to be able to find the best decision, the DM has to follow some decision rules which are of fixed form.

Here we consider only one class of decision rules. In principle, any other rule could be chosen, and possibly they would lead to better solutions (to better value of the objective function). We assume that the decisions are made on the basis of the following rules:

$$u_1(t_j) = \begin{cases} 0\,, & \text{if} \quad \varphi_1\big(\hat\lambda(t_j), v_1(t_j)\big) < 0\,, \\ 1\,, & \text{if} \quad \varphi_1\big(\hat\lambda(t_j), v_1(t_j)\big) \geq 0\,,\ \varphi_2\big(\hat\lambda(t_j), v_2(t_j)\big) > 0\,, \\ 2\,, & \text{otherwise}\,, \end{cases} \tag{29}$$

and

$$u_2(t_j) = \begin{cases} 0\,, & \text{if} \quad \varphi_3\big(\hat\lambda(t_j), v_3(t_j)\big) < 0\,, \\ 1\,, & \text{otherwise}\,. \end{cases} \tag{30}$$

where $\varphi_1 : R^2 \to R$, $\varphi_2 : R^2 \to R$, $\varphi_3 : R^2 \to R$ are monotone and continuous functions with respect to both variables. and $v_1 : R \to R$, $v_2 : R \to R$, $v_3 : R \to R$ are monotone and continuous functions.

The interpretation for the above decision rules is rather simple: the decisions are made if the recent failure intensity estimate exceeds some thresholds. The sense of the functions φ_1, φ_2, φ_3 can be explain as follows:

- if $\varphi_1\big(\hat\lambda(t_j), v_1(t_j)\big) < 0$, then continue the operation of the component;
- if $\varphi_2\big(\hat\lambda(t_j), v_2(t_j)\big) > 0$, then make a repair of the component (in case of $\varphi_1\big(\hat\lambda(t_j), v_1(t_j)\big) \geq 0$) ;
- if $\varphi_3\big(\hat\lambda(t_j), v_3(t_j)\big) < 0$ then continue operation after small inspection without large inspection.

We shall consider only the following forms for the functions φ_1, φ_2, φ_3:

$$\varphi_p\big(\hat\lambda(t_j), v_p(t_j)\big) = \hat\lambda(t_j) - v_p(t_j)\,, \quad p = 1, 2, 3\,. \tag{31}$$

Functions $v_1(t), v_2(t), v_3(t)$ are the controls in the model. Here we consider that they are linear:

$$v_p(t) = x_1^p + x_2^p t, \quad p = 1,2,3 . \tag{32}$$

Define $x = (x_1^1,\, x_2^1,\, x_1^2,\, x_2^2,\, x_1^3,\, x_2^3)$. Now the optimization problem is to find the vector x which leads to the smallest expected costs. In more general cases the respective problem could consist of finding the optimal forms of the functions $\varphi_p,\ v_p,\ p = 1,2,3,$ and of finding the optimal failure intensity estimate.

3.3 Objective Function

The objective of the decisions described above is to minimize the costs due to the use of the component. The total costs depend on the stochastic development of the defects in the component, and on the choice of the threshold functions $v_p(t_j)$. The costs will be very high, if the component has failed, and the profits gained from the use of the component will be larger if the component is used over a longer time. The total cost will be different for each trajectory of the stochastic process consisting of the values of the variables $M(t), C(t), \nu(t), \theta^\nu(t)$, etc.

We denote the profit of the operation of the component per unit time by G_p and the cost of failure by G_f. The cost of repair is denoted by G_r. The operation of the component will be terminated at the time point T_{max}, if it is not terminated earlier (the time interval $[0, T_{max}]$ is the time horizon of the optimization problem). We designate by T_{stop} the stop time for the component due to decision rule $u_1(t_i) = 2$:

$$T_{stop} = \min_{u_1(t_j)=2} t_j \ .$$

Denote by T_{end} the termination point

$$T_{end} = \min\{T_* \ , \ T_{stop}\} \ ,$$

where T_* is the failure time of the component. The termination point is a random variable depending on control vector x and the random state of the nature, $\omega \in \Omega$, i.e. $T_{end} = T_{end}(x, \omega)$.

Let the number of the large inspections (the small inspections) during the time interval $[0, T_{end}]$ be N_1 (N_2 , respectively) and let the number of repairs during the same interval be N_r . Now the total cost is

$$f(x,\omega) = \begin{cases} G(x,\omega)\,, & \text{if the component has not failed ;} \\ G_f + G(x,\omega)\,, & \text{otherwise ,} \end{cases} \tag{33}$$

where

$$G(x,\omega) = G_1 N_1 + G_2 N_2 + G_r N_r - G_p T_{end} \ .$$

We consider that the objective function $F(x)$ is the expected value of the cost function $f(x,\omega)$

$$F(x) = E\left[f(x,\omega)\right], \tag{34}$$

and $f(x,\omega)$ is defined in the equation (33).

The objective function (34) is not easily evaluated. The expected value cannot be determined analytically due to the complicated structure of the stochastic processes involved. However, the expected value can be evaluated in the form

$$F(x) = E\Big[\, E\left[\, f(x,\omega) \mid \mathcal{F}_\lambda \,\right]\Big], \tag{35}$$

in which the first expectation is evaluated with respect to σ-algebra generated by the the processes $\lambda(t)$, $\hat{\lambda}(t)$, $0 \le t \le T_{max}$. It should be noticed that time points t_j and decisions $u_1(t_j), u_2(t_j)$ are random variables measurable with respect to $\mathcal{F}_\lambda$. The conditional expectation in the formula (35) may be calculated analytically.

The costs due to inspections or repairs may increase only at the time points $t = t_j$, where the decisions are made. The failure intensity develops independently on the decisions between the time points where repairs are made. At repair points the failure intensity changes, and if the decision at some time point is to stop the operation of the component, then the failure intensity will be equal to zero. Thus the failure intensity is a stochastic process which depends on the decisions made at time points t_j in a simple way. Given the trajectories of the processes $\lambda(t)$, $\hat{\lambda}(t)$, $0 \le t \le T_{stop}$, the conditional expectation failure probability in the time interval $[0, T_{stop}]$ is calculated as follows (see (24))

$$1 - S(T_{stop}) = 1 - \exp\left\{ - \int_0^{T_{stop}} \lambda(t)\, dt \right\}, \tag{36}$$

in which the failure intensity $\lambda(t)$ is a random function defined in the equation (26). Consequently the conditional expectation of the failure cost is equal to $G_f\big(1 - S(T_{stop})\big)$.

The expected costs given $\lambda(t)$, $\hat{\lambda}(t)$, $0 \le t \le T_{stop}$, due to the inspections and repairs can be written in analogues form. Define j_{stop} as follows

$$t_{j_{stop}} = T_{stop} \,.$$

The costs due to the large and small inspections are given by

$$E\left[G_1 N_1 \mid \mathcal{F}_\lambda\right] = G_1 \sum_{j=1}^{j_{stop}} \left[\chi_1(t_j)\, S(t_j)\right], \tag{37}$$

and

$$E\left[G_2 N_2 \mid \mathcal{F}_\lambda\right] = G_2 \sum_{j=1}^{j_{stop}} \left[\chi_2(t_j)\, S(t_j)\right],$$

where $\chi_1(t_j), \chi_2(t_j)$ are random variables depending on the failure intensity estimates and they are defined by the equations

$$\chi_1(t_j) = \begin{cases} 0\,, & \text{if } t_j = \tau_l \text{ for some } l\,, \text{ and } u_2(\tau_l) = 0\,; \\ 1\,, & \text{otherwise}\,, \end{cases} \tag{38}$$

and

$$\chi_2(t_j) = \begin{cases} 1\,, & \text{if } t_j = \tau_l \text{ for some } l\,; \\ 0\,, & \text{otherwise.} \end{cases} \tag{39}$$

The respective expected cost due to repairs are given by

$$E\left[G_r N_r \mid \mathcal{F}_\lambda\right] = G_r \sum_{j=1}^{j_{stop}} \left[\chi_r(t_j)\, S(t_j)\right]$$

where $\chi_r(t_j)$ is a random variable defined with

$$\chi_r(t_j) = \begin{cases} 1\,, & \text{if } \chi_1(t_j) = 1 \text{ and } u_1(t_j) = 1\,; \\ 0\,. & \text{otherwise}\,. \end{cases} \tag{40}$$

The random variables χ_1, χ_2, χ_r in the above equations are indicators, which determine the control action (continue the operation, make an inspection ets.) according to the rules given in equations (29),(30). These random variables depend on the vector x, which is to be chosen optimally.

The respective conditional expectation must be determined also for $-G_p T_{end}$, the expected profit due to the operation of the component. The conditional expectation is of the form:

$$E\left[-G_p T_* \mid \mathcal{F}_\lambda\right] = -G_p\left[\int_0^{T_{stop}} t\,\lambda(t) S(t)\,dt \;+\; T_{stop}\, S(T_{stop})\right],$$

in which

$$\int_0^{T_{stop}} t\,\lambda(t) S(t)\,dt$$

is the conditional expectation of the failure time, given that the component failed during interval $[0, T_{stop}]$.

By collecting the above formulae we obtain the conditional expectation of the cost function $f(x,\omega)$ given $\lambda(t)$, $\hat{\lambda}(t)$, $0 \le t \le T_{max}$ in the form:

$$\begin{aligned} &E\left[f(x,\omega) \mid \mathcal{F}_\lambda\right] = \\ &G_f\big(1 - S(T_{stop})\big) \;+\; \sum_{j=1}^{j_{stop}} \left\{S(t_j)\left[G_1\chi_1(t_j) + G_2\chi_2(t_j) + G_r\chi_r(t_j)\right]\right\} - \\ &- G_p\left[\int_0^{T_{stop}} t\,\lambda(t) S(t)\,dt \;+\; T_{stop}\, S(T_{stop})\right]. \end{aligned} \tag{41}$$

From the above equation and (35) we finally obtain the objective function.

3.4 Approximate Calculation of the Objective Function

We consider that in the model described above, the fault probability of the component is small value and

$$\int_0^{T_{stop}} \lambda(t)\, dt \;\ll\; 1\,. \tag{42}$$

Consequently

$$S(t_j) = \exp\left\{ -\int_0^{T_{stop}} \lambda(t)\, dt \right\} \;\approx\; 1\,, \tag{43}$$

$$\int_0^{T_{stop}} t\,\lambda(t) S(t)\, dt \;+\; T_{stop}\, S(T_{stop}) \;\approx\; T_{stop}\,, \tag{44}$$

$$1 - S(T_{stop}) = 1 - \exp\left\{ -\int_0^{T_{stop}} \lambda(t)\, dt \right\} \;\approx\; \int_0^{T_{stop}} \lambda(t)\, dt\,. \tag{45}$$

Substitution of the (43 - 45) in the expression (41) gives

$$E\left[f(x,\omega) \mid \mathcal{F}_\lambda \right] \;\approx\; \tilde{f}(x,\omega) \stackrel{\text{def}}{=}$$

$$G_f \int_0^{T_{stop}} \lambda(t)\, dt \;+\; \sum_{j=1}^{j_{stop}} \left[G_1\chi_1(t_j) + G_2\chi_2(t_j) + G_r\chi_r(t_j) \right] \;-\; G_p T_{stop}\,.$$

Further we use function $\tilde{f}(x,\omega)$ to formulate stochastic optimization problem with respect to vector x.

3.5 Statement of the Optimization Problem

The stochastic behavior of the component (see Section 2), the possible decisions and the corresponding decision rules (see Section 3.2) and the objective function are now defined. Let us designate by X a feasible set for decision vector x

$$X = \{ x \in R^6 \colon\ \underline{x}_l \le x_l \le \overline{x}_l\ ,\ \text{for } l = 1,\ldots,6 \}\,,$$

here $\underline{x}_l$, $\overline{x}_l$, $l = 1,\ldots,6$ are low and upper bounds for variables x_l, $l = 1,\ldots,6$. Now we are ready to state the optimization problem. It can be given in the standard form

$$F(x) = E\left[\, E\left[\tilde{f}(x,\omega) \mid \mathcal{F}_\lambda \right] \right] \;\longrightarrow\; \min_{x \in X}\,,$$

subject to the dynamics of the process $\lambda(t)\,, \hat{\lambda}(t)$ and the decision rules. The dynamics of these processes are described in Sections 2 and 3.2.

4 Conclusions

The aim of this paper is to formulate an optimization model for finding good operational strategies for an inspected component. In the formulation of the model, the costs of inspections, repairs and failures and the profit earned from using the component were taken into account. The purpose of the optimal operational strategy was assumed to be the minimization of the costs accumulated during the operation of the component.

We suppose that the component fails randomly and that the failure intensity of the component depends on the number of initial defects in the component. Further, we assume that the defects grow stochastically. The models for the stochastic failure intensity and the defect growth are rather simple, they would need further development before they are used for practical applications.

The model for the inspection of the component was adopted from the theory developed for evaluating the reliability of non–destructive testing of metallic structures. The inspections are not complete, and therefore the model was probabilistic.

The above mentioned elements of the problem led to a very complicated (from a numerical point of view) stochastic optimization problem. Although the models applied to describing of the phenomena were simple, the optimization problem appears to have several difficult and interesting features. In this preliminary paper, we did not try to solve all this problems.

The analytical solution of the problem is not possible, since the objective function is a very complicated integral (mathematical expectation). For this reason we are going to apply stochastic quasi-gradient algorithms. Due to the nature of the algorithm, it is useful to express the objective function as a double expected value; first we evaluate the conditional expectation of some function and finally calculate the unconditional expectation. The solution techniques will be discussed in future papers on this subject.

The optimal operational strategies described in this paper are the optimal stopping and repair time of the component and the inspection strategies. The interesting problem connected with the inspection strategy is to find conditions for gathering more information on the latent defects with the component. Principally, this problem may be solved by using the model described above.

The control rules applied here are some sort of threshold control rules. The actions were assumed to be made on the basis of the failure intensity estimate, evaluated from the results of inspections. In our opinion, this kind of rules may be easily implemented in practical situations. The optimal values of the threshold parameters depend upon the cost structure and on the parameters of the failure models.

Practical applications of the model are possible if there exist reliable data for the parameters used. In this respect the most problematic part of the model is the probability model for the defects growth and failure intensity. Some data are available, but it is not evident that they are enough to estimate parameters of the model with sufficient accuracy. The data for the inspection model seem to be rather reliable.

The model discussed here is designed mainly for applications in nuclear power plant safety. However, there are a lot of other applications which can be even more fruitful. One possible area for this kind of model is the condition monitoring of mechanical components.

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