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The Viability Kernel Algorithm for Computing Value Functions of Infinite Horizon Optimal Control Problems

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Abstract

We characterize in this paper the epigraph of the value function of a discounted infinite horizon optimal control problem as the viability kernel of an auxiliary differential inclusion. Then the viability kernel algorithm applied to this problem provides the value function of the discretized optimal control problem as the supremum of a nondecreasing sequence of functions iteratively defined. We also use the fact that an upper Painlevé-Kuratowski limit of closed viability domains is a viability domain to prove the convergence of the discrete value functions.

Introduction

The concept of viability kernel of a closed subset under a differential inclusion has been introduced in [2, Aubin] in the 1985 meeting in honor of Professor Ky Fan. It is the largest closed subset contained in a given set which is viable under this differential inclusion. Furthermore, Pierre Cardaliaguet, Hélène Frankowska, Marc Quincampoix, Patrick Saint-Pierre and their collaborators found algorithms allowing us to compute the viability kernels, which run on personal computers for small dimensions. (see [22,23, Frankowska & Quincampoix], [28, Quincampoix & Saint-Pierre], [29, Saint-Pierre], [14,15,16, Cardaliaguet, Quincampoix & Saint-Pierre]). On the other hand, the concept of viability kernel happened to be a very useful tool for studying other problems, such as the construction of absorbing sets and attractors, reformulating Lyapunov stability, solving the target problem, characterizing and constructing Lyapunov functions, devising the Montagnes Russes Algorithms for finding a global minimum of a lower semicontinuous function, characterizing and constructing minimal time functions, invariant manifolds of a system of differential inclusions, constructing feedback maps dumping chattering controls, deriving the differential equation governing heavy solutions, constructing cascades in hierarchical dynamical games, etc., without mentioning in details their economic and biological applications which motivated this concept in the first place (see the papers by the preceding authors and [6,7,8, Aubin & Frankowska], [9, Aubin & Najman], [10, Aubin & Seube], [13, Bonneuil & Müllers], [17, Cartelier & Müllers], [19,18, Clément-Pitiot & Doyen], [24, Gorre], [25,26, Müllers], [27, Quincampoix], [30, Seube] for instance).

In this paper of the special issue of Journal of Mathematical Analysis and Applications in honor of Professor Ky Fan, we illustrate this point by characterizing the epigraph of the value function of a discounted infinite horizon optimal control problem as the viability kernel of an auxiliary differential inclusion. Then the viability kernel algorithm applied to this problem provides the value function of the discretized optimal control problem as the supremum of a nondecreasing sequence of functions iteratively defined. We also use the fact that an upper Painlevé-Kuratowski limit of closed viability domains is a viability domain to prove the convergence of the discrete value functions.

The dynamics (U, f) of a control system with a priori feedbacks are described by

$$\begin{cases} i) & x'(t) = f(x(t), u(t)) \\ ii) & u(t) \in U(x(t)) \end{cases} \quad (1)$$

where the state space X and the control space Z are finite dimensional vector spaces, $U : X \rightsquigarrow Z$ associates with each state x the set $U(x)$ of feasible controls (in general state-dependent) and $f : \mathrm{Graph}(U) \mapsto X$ describes the dynamics of the system.

Observe that state constraints are implicitly taken into account in the definition of the domain of the feedback map U.

Let us denote by $\mathcal{S}(x_0)$ the set of state-control pairs $(x(\cdot), u(\cdot))$ solutions to the control problem (1) starting from x_0 at time 0.

Under adequate assumptions on U and f, Viability Theorem 6.1.4 of VIABILITY THEORY, [3, Aubin] provides necessary and sufficient conditions for the nonemptiness of the solution map $\mathcal{S}$ for every initial state $x_0 \in \mathrm{Dom}(U)$.

Let us introduce now a lower semicontinuous function

$$W : (x, v) \in \mathrm{Graph}(U) \mapsto W(x, v) \in \mathbf{R}_+$$

assumed to be convex with respect to v.

We consider the discounted optimal control problem

$$V_\propto(x_0) := \inf_{(x(\cdot,u(\cdot))\in\mathcal{S}(x_0)} \int_0^\infty e^{a\tau} W(x(\tau), u(\tau))d\tau \ \in \ [0, +\infty]$$

In this paper, we shall prove that the value function $V_\propto$ of the optimal control problem **is the smallest of the lower semicontinuous nonnegative extended functions V satisfying the following monotonicity property: From any initial state $x_0 \in \mathrm{Dom}(V)$ starts a solution to control problem (1) satisfying**

$$\forall\, t \geq 0, \ \ e^{at}V(x(t)) - V(x_0) + \int_0^t e^{a\tau} W(x(\tau), u(\tau))d\tau \ \leq \ 0 \quad (2)$$

which is equivalent to

$$\inf_{v\in U(x)} (D_\uparrow V(x)(f(x, v)) + W(x, v)) + aV(x) \ \leq \ 0$$

We refer to [1, Aubin] or to Chapter 6 of DIFFERENTIAL INCLUSIONS, [4, Aubin & Cellina], and to [21, Frankowska & Plaskacz] for an exposition of the consequences of such an inequality and of generalized solutions (both contingent and viscosity) to Hamilton-Jacobi-Bellman equations.

Actually, we shall characterize the epigraph of the value function as the **viability kernel** of an auxiliary differential inclusion. This being done, we apply the **viability kernel algorithm** to this epigraph, which provides the value function of the discretized problem. In this way, we obtain an algorithm for computing the discrete value function. This (approximated) discrete value function is then used for approximating an optimal solution. Finally, we shall prove the convergence of the discrete value functions in an adequate sense.

The authors acknowledge personal communications of Daniel Gabay for pointing out in particular the relations between the viability kernel algorithm applied for approximating the value function of an infinite horizon optimal control problem and algorithms obtained in [11, Bertsekas] for computing the value function of stochastic optimal control problems. They thank him warmly.

1 Decreasing Cost Functions

The evolution of a control system (U, f) with a priori feedback map of controls is governed by

$$\left\{ \begin{array}{lll} i) & x'(t) & = & f(x(t), u(t)) \\ ii) & u(t) & \in & U(x(t)) \end{array} \right. \tag{3}$$

Observe that state (viability) constraints are implicitly taken into account in the definition of the domain of the feedback map U by setting $K := \mathrm{Dom}(U)$.

We recall that a control system is said to be a *Marchaud system* if

$$\left\{ \begin{array}{ll} i) & \mathrm{Graph}(U) \text{ is closed and the values } U(x) \text{ are convex} \\ ii) & f \text{ is continuous and affine with respect to the controls} \\ iii) & f \text{ and } U \text{ have linear growth} \end{array} \right. \tag{4}$$

In this paper, we shall assume once and for all that: the state space X and the control space Z are finite dimensional vector spaces, $U : X \rightsquigarrow Z$ associates with each state x the state-dependent set $U(x)$ of feasible controls and $f : \mathrm{Graph}(U) \mapsto X$ describes the dynamics of the system and that they satisfy

$$\left\{\begin{array}{ll} i) & (U,f) \text{ is a Marchaud control system} \\ ii) & W:(x,v)\in \mathrm{Graph}(U)\mapsto W(x,v)\in \mathbf{R}_+ \text{ is a nonnegative} \\ & \text{lower semicontinuous function convex with respect to } v \\ iii) & \exists\, c>0 \text{ such that } \forall\,(x,v)\in \mathrm{Graph}(U),\ \ W(x,v)\ \le\ c(\|x\|+1) \end{array}\right. \tag{5}$$

We denote by $L^1(0,\infty;X,e^{at}dt)$ the Lebesgue space of classes of measurable functions from $[0,\infty[$ to X integrable for the weighted measure $e^{at}dt$ and by $W_a^{1,1}(0,\infty;X)$ the space of functions $x(\cdot)\in L^1(0,\infty;X,e^{at}dt)$ such that their distributional derivative belongs to $L^1(0,\infty;X)$.

Denote by $\mathcal{S}(x_0)$ the set of state-control pairs $(x(\cdot),u(\cdot))\in W_a^{1,1}(0,\infty;X)\times L^1(0,\infty;Z,e^{at}dt)$ solutions to the control system (3) starting at x_0, i.e., such that $x(0)=x_0$.

Therefore, the discounted cost

$$\int_0^\infty e^{a\tau}W(x(\tau),u(\tau))d\tau\ \in\ [0,+\infty]$$

is well defined over the solutions of the control problem.

Theorem 1.1 *Assume that hypothesis (5) holds true and that $V:X\mapsto \mathbf{R}_+\cup\{+\infty\}$ is a nonnegative extended lower semicontinuous function (regarded as a cost function).*

We assume that there exists a positive constant c such that

$$\forall\, x\in \mathrm{Dom}(V),\quad \inf_{v\in U(x)} D_\uparrow V(x)(f(x,v))\ \ge\ -c(\|x\|+1) \tag{6}$$

Then the two following properties are equivalent:

1. *For any initial state $x_0\in \mathrm{Dom}(V)$, there exists a solution $(x(\cdot),u(\cdot))\in \mathcal{S}(x_0)$ to the control system (3) monotone in the sense that:*

$$\forall\, t\ge 0,\ \ e^{at}V(x(t))-V(x_0)+\int_0^t e^{a\tau}W(x(\tau),u(\tau))d\tau\ \le\ 0 \tag{7}$$

2. *V is a contingent solution to the Hamilton-Jacobi inequality*

$$\forall\, x\in \mathrm{Dom}(V),\quad \inf_{v\in U(x)}\left(D_\uparrow V(x)(f(x,v))+W(x,v)\right)+aV(x)\ \le\ 0 \tag{8}$$

Furthermore, if we denote by

$$R_V(x) \ := \ \{u \in U(x) \mid D_\uparrow V(x)(f(x,u)) + W(x,u) + aV(x) \ \leq \ 0\}$$

then the monotone solutions are governed by the optimal regulation law:

$$\text{for almost all } t \geq 0, \ \ u(t) \ \in \ R_V(x(t))$$

Remark — If we assume also that there exists a constant $\rho > 0$ such that

$$\forall \, (x,u) \in \text{Graph}(U), \ \ W(x,u) + aV(x) \ \geq \ \rho\|f(x,u)\|$$

then Ekeland's Variational Principle and property (8) imply that there exists an equilibrium of the control system. □

Proof of Theorem 1.1 — We introduce the set-valued map $G : X \times \mathbf{R} \leadsto X \times \mathbf{R}$ defined by

$$G(x,w) := \{(f(x,v),\lambda) \mid v \in U(x) \ \& \ \lambda + aw \in [-c(\|x\|+1), -W(x,v)]\} \quad (9)$$

1. It is clear that the graph of G is closed and its values are convex and nonempty by assumption (6). Its growth is linear by construction. Furthermore, the epigraph of V is a closed viability domain of G: take $v \in U(x)$ achieving the minimum of the lower semicontinuous function $D_\uparrow V(x)(f(x,\cdot)) + W(x,\cdot)$ on the compact subset $U(x)$. It satisfies

   $$D_\uparrow V(x)(f(x,v)) + W(x,v) + aV(x) \ \leq \ 0$$

   by assumption (8), so that the pair $(f(x,v), -aV(x) - W(x,v))$ belongs to the contingent cone to the epigraph of V at $(x, V(x))$. It also belongs to the contingent cone to the epigraph of V at (x,w) for every $w > V(x)$. Indeed, this assumption means that there exist sequences $h_n > 0$ converging to 0, v_n converging to $f(x,v)$ and d_n converging to $-W(x,v) - aV(x)$ such that $(x + h_n v_n, V(x) + h_n d_n) \in \mathcal{E}p(V)$. Therefore, for any $w > V(x)$, we obtain

   $$\begin{cases} (x + h_n v_n, w + h_n d_n) \ = \ (x + h_n v_n, V(x) + h_n d_n) + (0, (w - V(x))) \\ \in \ \mathcal{E}p(V) + \{0\} \times \mathbf{R}_+ \ = \ \mathcal{E}p(V) \end{cases}$$

   and consequently, the pair $(f(x,v), -aV(x) - W(x,v))$ belongs to the contingent cone to the epigraph of V at (x,w).

Hence $\mathcal{E}p(V)$ being a closed viability domain of $G(\cdot,\cdot)$, there exists a solution $(x(\cdot), w(\cdot))$ to differential inclusion

$$\text{for almost all } t \geq 0, \;\; (x'(t), w'(t)) \; \in \; G(x(t), w(t))$$

starting from $(x_0, V(x_0))$ at time 0 and viable in the epigraph of V. Inequalities

$$w'(\tau)+aw(\tau) \;\leq\; -W(x(\tau), u(\tau)) \;\&\; V(x(t)) \;\leq\; w(t), \;\; V(x_0) \;=\; w(0)$$

imply that

$$V(x(t)) \;\leq\; w(t) \;\leq\; e^{-at}V(x_0) + \int_0^t e^{-a(t-\tau)}W(x(\tau), u(\tau))d\tau$$

2. Conversely, let us consider a solution $(x(\cdot), u(\cdot))$ to the control system

$$x'(t) \;=\; f(x(t), u(t))$$

which is monotone:

$$\forall\, t \geq 0, \;\; e^{at}V(x(t)) - V(x_0) + \int_0^t e^{a\tau}W(x(\tau), u(\tau))d\tau \;\leq\; 0$$

We shall prove that there exists $u_0 \in U(x_0)$ such that

$$D_\uparrow V(x_0)(f(x_0, u_0)) + W(x_0, u_0) + aV(x_0) \;\leq\; 0 \tag{10}$$

The above monotonicity condition means that

$$(x(h), e^{-ah}V(x_0) - e^{-ah}\int_0^h e^{a\tau}W(x(\tau), u(\tau))d\tau) \;\in\; \mathcal{E}p(V)$$

Setting

$$u_h \;:=\; \frac{1}{h}\int_0^h x'(\tau)d\tau$$

and

$$\lambda_h \;=\; \frac{e^{-ah}-1}{h}V(x_0) - \frac{e^{-ah}}{h}\int_0^h e^{a\tau}W(x(\tau), u(\tau))d\tau$$

it can be rewritten in the form

$$(x_0 + hu_h, V(x_0) + h\lambda_h) \;\in\; \mathcal{E}p(V)$$

Since U is upper semicontinuous with compact values, we can associate with any $\varepsilon > 0$ an $\eta_0 \in]0, \varepsilon]$ such that $U(x) \subset B(U(x_0), \varepsilon)$ whenever $d(x, x_0) \leq \eta_0$. Since f is continuous, we deduce that

$$x'(\tau) \;=\; f(x(\tau), u(\tau)) \;\in\; f(x_0, U(x_0)) + \varepsilon B$$

for τ small enough. Hence, since f is affine with respect to u,

$$u_h \;\in\; f\left(x_0, \frac{1}{h}\int_0^h u(\tau)d\tau\right) + \varepsilon B \qquad (11)$$

Since W is lower semicontinuous, we can associate with any u_i an $\eta_i \in]0, \varepsilon]$ such that

$$\text{if } d(x, x_0) \;\leq\; \eta_i \;\&\; \|u - u_i\| \;\leq\; \eta_i, \;\text{ then }\; W(x_0, u_i) \;\leq\; W(x, u) + \varepsilon$$

Since $B(U(x_0), \varepsilon)$ is compact, it can be covered by a finite number r of such balls. Let us set $\eta := \min_{i=0,\ldots r} \eta_i$. Therefore, for any $u \in U(x_0)$, we can find a control u_i such that $\|u - u_i\| \leq \eta_i$, so that

$$\forall\, x \in B(x_0, \eta), \;\; (u_i, W(x, u) + \varepsilon) \;\in\; \mathcal{E}p(W(x_0, \cdot))$$

Hence, for every $\varepsilon > 0$, there exists η such that for every $x \in B(x_0, \eta)$ and every $u \in U(x)$,

$$(u, W(x, u) + \varepsilon) \;\in\; \mathcal{E}p(W(x_0, \cdot)) + \varepsilon B \times \{0\}$$

because

$$(u, W(x, u) + \varepsilon) \;=\; (u_i, W(x, u) + \varepsilon) + (u - u_i, 0)$$

Let us consider now $u(\tau) \in U(x(\tau))$ and α such that $d(x_0, x(\tau)) \leq \eta$ whenever $\tau \leq \alpha$. We deduce that

$$(u(\tau), e^{a\tau}W(x(\tau), u(\tau)) + \varepsilon) \;\in\; \mathcal{E}p(W(x_0, \cdot)) + 2\varepsilon B \times \{0\}$$

Since the epigraph of $W(x_0, \cdot)$ is closed and convex, this implies that

$$\left(\frac{1}{h}\int_0^h u(\tau)d\tau, \frac{1}{h}\int_0^h e^{a\tau}W(x(\tau), u(\tau))d\tau + 2\varepsilon\right) \;\in\; \mathcal{E}p(W(x_0, \cdot)) + 2\varepsilon B \times \{0\}$$

Let $h_n > 0$ be such that

$$\lim_{n\to\infty} \frac{1}{h_n} \int_0^{h_n} e^{a\tau} W(x(\tau), u(\tau)) d\tau \;=\; \liminf_{h\to 0+} \frac{1}{h} \int_0^{h} e^{a\tau} W(x(\tau), u(\tau)) d\tau$$

Since $u(x(\tau)) \in U(x(\tau)) \subset B(U(x_0), \varepsilon)$ for τ small enough, and since the values of U are convex and compact, we infer that

$$u_n \;:=\; \frac{1}{h_n} \int_0^{h_n} u(\tau) d\tau \;\in\; B(U(x_0), \varepsilon)$$

Since this set is compact, a subsequence (again denoted by) u_n converges to some $u_0 \in U(x_0)$.

Therefore, by taking the limit when $n \to \infty$, we deduce that

$$\left(u_0, \liminf_{h\to 0+} \frac{1}{h} \int_0^{h} e^{a\tau} W(x(\tau), u(\tau)) d\tau + \varepsilon \right) \;\in\; \mathcal{E}p(W(x_0, \cdot)) + 2\varepsilon B \times \{0\}$$

Since this is true for every ε, we infer that

$$\left(u_0, \liminf_{h\to 0+} \frac{1}{h} \int_0^{h} e^{a\tau} W(x(\tau), u(\tau)) d\tau \right) \;\in\; \mathcal{E}p(W(x_0, \cdot))$$

Therefore

$$W(x_0, u_0) \;\leq\; \liminf_{h\to 0+} \frac{1}{h} \int_0^{h} e^{a\tau} W(x(\tau), u(\tau)) d\tau$$

We thus conclude from (11) that u_{h_n} converges to $f(x_0, u_0)$ and that λ_{h_n} converges to some λ satisfying

$$\lambda \;\leq\; -aV(x_0) - W(x_0, u_0)$$

This implies that

$$D_{\uparrow} V(x_0)(f(x_0, u_0)) \;\leq\; -aV(x_0) - W(x_0, u_0)$$

so that property (10) is satisfied. The last statement translates the fact that for almost all $t \geq 0$, the derivative of the viable solution $(x'(t), w'(t))$ belongs to the contingent cone of the epigraph of V. □

Remark — We refer to Proposition 9.4.7 of VIABILITY THEORY, [3, Aubin] for conditions under which the optimal regulation map is lower semicontinuous, so that it can have continuous selections, minimal selections and other types of selection procedures. □

As a consequence, we deduce the following

Theorem 1.2 *We add to the hypothesis of Theorem 1.1 the assumption that the cost function V is continuous on its domain. Then the two following properties are equivalent:*

1. *For any initial state $x_s \in \mathrm{Dom}(V)$, there exists a solution to the control system (3) starting from x_s at time s and satisfying property:*

$$\forall\, t \geq s,\quad e^{at}V(x(t)) - e^{as}V(x(s)) + \int_s^t e^{a\tau}W(x(\tau), u(\tau))d\tau \;\leq\; 0 \quad (12)$$

2. *V is a contingent solution to the Hamilton-Jacobi inequality (8).*

Proof — We associate with $h \to 0+$ the grid jh, $(j = 1, \ldots)$ and we build a solution $x_h(\cdot) \in \mathcal{S}(x_0)$ to differential inclusion (1.1) by using Theorem 1.1 iteratively: for $j = 0$, we take $x_h(\cdot)$ on the interval $[0, h]$ satisfying (7). For $j > 0$, we consider the solution starting at $x_h(jh)$ and satisfying

$$e^{at}V(y_j(t)) - V(x_h(jh)) + \int_0^t e^{a\tau}W(y_j(\tau), v_j(\tau))d\tau$$

Setting $x_h(t) := y_j(t-jh)$ and $u_h(t) := v_j(t-jh)$ on the interval $[jh, (j+1)h]$, we see that the solution satisfies

$$\forall\, t \in [jh, (j+1)h],\;\; e^{at}V(x_h(t)) - e^{ajh}V(x_h(jh)) + \int_{jh}^{t-jh} e^{a\sigma}W(x_h(\sigma), u_h(\sigma))d\sigma \;\leq\; 0$$

Let $t > s$ be fixed. Since the Convergence Theorem implies that the image $\mathcal{S}(x_0)$ is compact in $\mathcal{C}(0, \infty; X) \times L^1(0, \infty, Z, e^{at}dt)$ when the first space is supplied with the compact topology and the second with the weak topology, a subsequence (again denoted) $(x_h(\cdot), u_h(\cdot))$ converges to some solution $(x(\cdot), u(\cdot)) \in \mathcal{S}(x_0)$ in $\mathcal{C}(s, t; X) \times L^1((s, t); Z)$.

An adaptation of the proof of the Convergence Theorem implies the following property:

Lemma 1.3 *Assume that* $W : (x,v) \in \mathrm{Graph}(U) \mapsto W(x,v) \in \mathbf{R}_+$ *is a lower semicontinuous function convex with respect to* v*. Then the functional*

$$(x(\cdot),u(\cdot)) \mapsto \int_s^t e^{a\tau} W(x(\tau),u(\tau))d\tau$$

from $\mathcal{C}(s,t;X) \times L^1(s,t,X)$ *to* $\mathbf{R} \cup \{+\infty\}$ *is lower semicontinuous when* $\mathcal{C}(s,t;X)$ *is supplied with the uniform convergence and* $L^1(s,t,X)$ *with the weak convergence.*

(We refer to Proposition 6.3.4 of DIFFERENTIAL INCLUSIONS, [4, Aubin & Cellina], for the proof of this Lemma.) Hence

$$\int_s^t e^{a\tau} W(x(\tau),u(\tau))d\tau \;\leq\; \liminf_{h\to 0+} \int_s^t e^{a\tau} W(x_h(\tau),u_h(\tau))d\tau$$

Let $t > s$ be approximated by $j_h h \geq k_h h$ so that

$$e^{aj_h h} V(x_h(j_h h)) - e^{ak_h h} V(x_h(k_h h)) + \int_{k_h h}^{j_h h} e^{a\tau} W(x_h(\tau),u_h(\tau))d\tau \;\leq\; 0$$

The function V being continuous on its domain, inequality (12) ensues. □

2 The Optimal Cost Functions

A cost function $V : X \to \mathbf{R}_+ \cup \{+\infty\}$ being given, there is no reason why the monotonicity property of Theorem 1.1 should hold true. However, we can construct the smallest lower semicontinuous cost function larger than or equal to V, i.e., the smallest nonnegative lower semicontinuous contingent solution $V_\propto$ to the Hamilton-Jacobi inequalities (8) larger than or equal to V.

Theorem 2.1 *Assume that hypothesis (5) holds true and that* $V : X \mapsto \mathbf{R}_+ \cup \{+\infty\}$ *a nonnegative extended lower semicontinuous function.*

Then there exists a smallest nonnegative lower semicontinuous solution $V_\propto : X \mapsto \mathbf{R}_+ \cup \{+\infty\}$ *to the contingent Hamilton-Jacobi inequalities (8) larger than or equal to* V *(which can be the constant* $+\infty$*), which thus enjoys the monotonicity property:* $\forall\, x_0 \in \mathrm{Dom}(V_\propto)$*, there exists one solution to (3) starting from* x_0 *at time* 0 *and satisfying*

$$\forall\, t \geq 0,\;\; V(x(t)) \;\leq\; V_\propto(x(t)) \;\leq\; e^{-at}V_\propto(x_0) - \int_0^t e^{-a(t-\tau)} W(x(\tau),u(\tau))d\tau \quad (13)$$

Proof — By Theorem 4.1.2 of VIABILITY THEORY, [3, Aubin], we know that there exists a largest closed viability domain $\mathcal{K} \subset \mathcal{E}p(V)$ (the viability kernel of the epigraph of V) of the set-valued map $(x,w) \rightsquigarrow G(x,w)$ defined by (9). If it is empty, it is the epigraph of the constant function equal to $+\infty$.

If not, we have to prove that it is the epigraph of the nonnegative lower semicontinuous function V_α defined by

$$V_\alpha(x) \ := \ \inf_{(x,\lambda)\in\mathcal{K}} \lambda$$

we are looking for. Indeed, since the viability kernel is the largest closed viability domain of the epigraph of V, the epigraph of any solution to the contingent inequalities (8) being a closed viability domain of the set-valued map G, is contained in the epigraph of V_α, so that V_α is smaller than any lower semicontinuous solution to (8) larger than V. Since

$$\mathcal{E}p(V_\alpha) = \mathrm{Graph}(V_\alpha) + \{0\} \times \mathbf{R}_+ \subset \mathcal{K} + \{0\} \times \mathbf{R}_+$$

it is therefore enough to show that $\mathcal{K} + \{0\} \times \mathbf{R}_+ \subset \mathcal{K}$.

In fact, we prove *if* $\mathcal{M} \subset \mathrm{Dom}(U) \times \mathbf{R}_+$ *is a closed viability domain of* G, *then so is the subset*

$$\mathcal{M}_+ := \mathcal{M} + \{0\} \times \mathbf{R}_+$$

Obviously, $\mathcal{M}_+$ is closed. To see that $G(x,w) \cap T_{\mathcal{M}_+}(x,w) \neq \emptyset$, let

$$V_{\mathcal{M}}(x) \ := \ \inf_{(x,\lambda)\in\mathcal{M}} \lambda$$

By assumption, there exists $v \in U(x)$ such that $(f(x,v),d)$ belongs to the contingent cone to $\mathcal{M}$ at the point $(x, V_{\mathcal{M}}(x)) \in \mathcal{M}$ where $d \in -aw + [-c(\|x\|+1), -W(x,v)]$. Hence, there exist sequences $h_n > 0$ converging to 0, v_n converging to $f(x,v)$ and d_n converging to d such that

$$\begin{cases} (x + h_n v_n, w + h_n d_n) \\ = \ (x + h_n v_n, V_{\mathcal{M}}(x) + h_n d_n) + (0, (w - V_{\mathcal{M}}(x))) \ \in \ \mathcal{M}_+ \end{cases}$$

This proves that V_α is the smallest lower semicontinuous function satisfying (8) larger than or equal to V. □

3 Infinite Horizon Optimization Problems

Denote by $\mathcal{S}_t(x_0)$ the set of solutions $(x(\cdot), u(\cdot))$ to the control system (3) starting from x_0 at time t.

The smallest lower semicontinuous cost function $V_\propto$ is closely related to the value function

$$U^\flat(t, x_0) \; := \; \inf_{(x(\cdot),u(\cdot))\in\mathcal{S}_t(x_0)} \int_t^\infty e^{a\tau} W(x(\tau), u(\tau))d\tau \; \in \; [0, +\infty[\qquad (14)$$

of the intertemporal discounted optimization problem over the solutions $(x(\cdot), u(\cdot))$ to the control system starting at time t from x_0 of the discounted functional

$$\int_t^\infty e^{a\tau} W(x(\tau), u(\tau))d\tau$$

Theorem 3.1 *Assume that hypothesis (5) holds true and that the domain* $K := \mathrm{Dom}(U)$ *is closed. Then the* value function $U^\flat(t, x)$ *and the smallest lower semicontinuous function* $V_\propto$ *satisfying (7) larger than or equal to the indicator function* ψ_K *of* K *are related by the formula:*

$$U^\flat(t, x) \; = \; e^{at} V_\propto(x) \qquad (15)$$

Furthermore, a solution $(\widehat{x}(\cdot), \widehat{u}(\cdot)) \in \mathcal{S}(x_0)$ *to (3) satisfies inequality (7) for* $V_\propto$ *if and only if it is an optimal solution to the intertemporal optimization problem:*

$$\int_0^\infty e^{a\tau} W(\widehat{x}(\tau), \widehat{u}(\tau))d\tau = \inf_{(x(\cdot),u(\cdot))\in\mathcal{S}(x_0)} \int_0^\infty e^{a\tau} W(x(\tau), u(\tau))d\tau$$

In this case, it obeys the "optimality principle":

$$\forall \, t \geq 0, \;\; e^{at} V_\propto(\widehat{x}(t)) \; = \; \int_t^\infty e^{a\tau} W(\widehat{x}(\tau), \widehat{u}(\tau))d\tau \qquad (16)$$

and satisfies actually the equation

$$e^{at} V_\propto(\widehat{x}(t)) - e^{as} V_\propto(\widehat{x}(s)) + \int_s^t e^{a\tau} W(\widehat{x}(\tau), \widehat{u}(\tau))d\tau \; = \; 0 \qquad (17)$$

If we denote by

$$R_\propto(x) \; := \; \{u \in U(x) \mid D_\uparrow V_\propto(x)(f(x, u)) + W(x, u) + aV_\propto(x) \; \leq \; 0\}$$

then the optimal solutions to the intertemporal optimization problem are governed by the optimal regulation law:

$$\text{for almost all } t \geq 0, \;\; u(t) \; \in \; R_\propto(x(t))$$

Proof

1. Let $(x(\cdot), u(\cdot)$ be a monotone solution with respect to V_α starting from x_0. We deduce in particular that, V being nonnegative,

$$\forall\, t \geq 0, \quad \int_0^t e^{a\tau} W(x(\tau), u(\tau))d\tau \;\leq\; V_\alpha(x_0)$$

Since W is nonnegative, the sequence $\int_0^t e^{a\tau} W(x(\tau), u(\tau))d\tau$ is nondecreasing, so that, taking the limit when $t \rightarrow +\infty$, we obtain

$$U^\flat(0, x_0) \;:=\; \inf_{(x(\cdot),u(\cdot))\in\mathcal{S}(x_0)} \int_0^\infty e^{a\tau} W(x(\tau), u(\tau))d\tau \;\leq\; V_\alpha(x_0) \quad (18)$$

2. We observe that any solution $(\bar{x}(\cdot), \bar{u}(\cdot))$ monotone with respect to the value function in the sense that

$$U^\flat(s, \bar{x}(s)) - U^\flat(t, \bar{x}(t)) + \int_t^s e^{a\tau} W(\bar{x}(\tau), \bar{u}(\tau))d\tau \;\leq\; 0$$

is an optimal solution to the intertemporal optimization problem because

$$\int_t^\infty e^{a\tau} W(\bar{x}(\tau), \bar{u}(\tau))d\tau \;\leq\; U^\flat(t, \bar{x}(t)) \;:=\; \inf_{(x(\cdot),u(\cdot))\in\mathcal{S}_t(\bar{x}(t))} \int_t^\infty e^{a\tau} W(x(\tau), u(\tau))d\tau$$

Furthermore, property

$$\forall\, t \geq 0, \quad U^\flat(t, \bar{x}(t)) \;=\; \int_t^\infty e^{a\tau} W(\bar{x}(\tau), \bar{u}(\tau))d\tau \qquad (19)$$

holds true.

Conversely, if a solution $(\widehat{x}(\cdot), \widehat{u}(\cdot))$ is optimal, it is monotone with respect to the value function, because

$$U^\flat(t, \widehat{x}(t)) + \int_0^t e^{a\tau} W(\widehat{x}(\tau), \widehat{u}(\tau))d\tau \;\leq\; \int_0^\infty e^{a\tau} W(\widehat{x}(\tau), \widehat{u}(\tau))d\tau \;=\; U^\flat(0, x_0)$$

3. We also note that along optimal solutions, we have

$$U^\flat(t, \widehat{x}(t)) \;=\; e^{at} U^\flat(0, \widehat{x}(t))$$

Indeed, setting $\widehat{y}(\tau) := \widehat{x}(\tau + t)$ and $\widehat{q}(\tau) := \widehat{u}(\tau + t)$, we observe that the pair $(\widehat{y}(\cdot), \widehat{q}(\cdot))$ is a solution to the control system starting at $\widehat{x}(t)$ and that by a change of variables that

$$U^\flat(t, \widehat{x}(t)) \;=\; \int_t^\infty e^{a\tau} W(\widehat{x}(\tau), \widehat{u}(\tau))d\tau \;=\; e^{at} \int_0^\infty e^{a\tau} W(\widehat{y}(\tau), \widehat{q}(\tau))d\tau$$

The same change of variables shows that the pair $(\widehat{y}(\cdot), \widehat{q}(\cdot))$ is an optimal solution starting from $\widehat{x}(t)$ at time 0.

4. Therefore, the function $x \mapsto U^\flat(0, x)$ satisfies property

$$e^{at} U^\flat(0, \widehat{x}(t)) - U^\flat(0, x_0) + \int_0^t e^{a\tau} W(\widehat{x}(\tau), \widehat{u}(\tau))d\tau \;\leq\; 0$$

Since the solution map $\mathcal{S}(\cdot)$ is upper semicontinuous with compact values from X to $\mathcal{C}(0, \infty; X) \times L^1(0, \infty, X)$ when $\mathcal{C}(0, \infty; X)$ is supplied with the compact convergence and $L^1(0, \infty, X)$ with the weak convergence by Theorem 3.5.2 of VIABILITY THEORY, [3, Aubin], the Maximum Theorem (see for instance Theorem 1.4.16 of SET-VALUED ANALYSIS, [5, Aubin & Frankowska]) and Lemma 1.3 imply that the value function $x \mapsto U^\flat(0, x)$ is lower semicontinuous.

Therefore, by Theorem 1.1, $U^\flat(0, \cdot)$ is a lower semicontinuous solution to the Hamilton-Jacobi inequality (8) and thus, larger than or equal to V_∞. By virtue of (18), it is then equal to V_∞. □

Remark — Monotone solutions to a control system enjoy asymptotic properties we just mention without proof:

Theorem 3.2 *We posit the assumptions of Theorem 3.1 and we assume that $K := \mathrm{Dom}(U)$ is compact. Then the optimal solution has "almost cluster points" $(\widehat{x}_\star, \widehat{u}_\star)$ satisfying*

$$f(\widehat{x}_\star, \widehat{u}_\star) \;=\; 0, \;\; \widehat{u}_\star \in U(\widehat{x}_\star) \;\;\&\;\; W(\widehat{x}_\star, \widehat{u}_\star) \;=\; 0$$

We refer to [1, Aubin] or to chapter 6 of DIFFERENTIAL INCLUSIONS, [4, Aubin & Cellina] for the precise definition of "almost convergence" and further asymptotic properties of monotone solutions. □

Example: Solutions with Minimal Length

Solutions with minimal length are obtained in the particular case when $W(x,u) = \|f(x,u)\|$ since

$$\int_0^\infty W(x(\tau),u(\tau))d\tau \;=\; \int_0^\infty \|x'(\tau)\|d\tau$$

measures the length of the solution to the differential equation $x'(t) = f(x(t),u(t))$. Then the optimal solutions are solutions with minimal length and the domain of the value function is the subset of initial states from which there exists at least one solution to the control system with finite length. Furthermore, any solution with minimal length converges when $t \to \infty$ to an equilibrium $x_\star$ of the system. i.e., a solution to

$$f(x_\star,u_\star) \;=\; 0 \;\&\; u_\star \;\in\; U(x_\star)$$

(see chapter 6 of [4, Aubin & Cellina]).

4 The Discrete Viability Kernel Algorithm

Let the discretization step $h \in]0, \frac{1}{a}[$ be fixed. We shall approximate the set-valued map U by set-valued maps U^h, the map f by maps $f^h : \text{Graph}(U^h) \mapsto X$ and the function W by nonnegative functions $W^h : \text{Graph}(U^h) \mapsto \mathbf{R}_+$.

The control system is replaced by the discrete control system

$$\forall\, s \geq 0,\;\; x_{s+1}^h \;:=\; x_s^h + hf^h(x_s^h,u_s^h) \;\text{ where }\; u_s^h \in U^h(x_s^h) \qquad (20)$$

and we denote by $\mathcal{S}^h(x_0)$ the set of solutions $(x^h,u^h) = ((x_s^h,u_s^h))_{s\geq 0}$ of the discrete system (20) starting from an initial state $x_0 \in \text{Dom}(U^h)$.

We define the discrete value function of the discrete optimal control problem by

$$U_h^\flat(0,x_0) \;:=\; \inf_{(x^h,u^h)\in\mathcal{S}^h(x_0)} h\sum_{r=0}^{\infty}(1-ah)^{-r-1}W^h(x_r^h,u_r^h)$$

As in the continuous case, we shall characterize the discrete value function as the smallest lower semicontinuous nonnegative function V^h which is not increasing along at least one solution to the discrete control problem in the following sense: from any $x_0 \in \text{Dom}(U^h)$, starts one solution $(x^h,u^h) \in \mathcal{S}^h(x_0)$ to the discrete control system (20) satisfying for every $0 < s < t$,

$$(1-ah)^{-t}V^h(x_t^h) - V^h(x_0) + h\sum_{r=0}^{t-1}(1-ah)^{-r-1}W^h(x_r^h,u_r^h) \;\leq\; 0 \qquad (21)$$

We shall associate with the function V^h its synthesis map R_V^h defined by

$$\begin{cases} R_V^h(x) \;:=\; \Big\{u \in U^h(x) \;\text{ such that }\; V^h(x + hf^h(x,u)) + hW^h(x,u) \\ \;=\; \inf_{v\in U^h(x)}(V^h(x + hf^h(x,v)) + hW^h(x,v))\Big\} \end{cases}$$

providing the solutions $(\bar{x}^h, \bar{u}^h)$ to the discrete control system (20) which are monotone with respect to V^h:

$$\bar{x}_{s+1}^h \;:=\; \bar{x}_s^h + hf^h(\bar{x}_s^h, \bar{u}_s^h) \;\text{ where }\; \bar{u}_s^h \in R_V^h(\bar{x}_s^h) \tag{22}$$

We set $V_0^h := \psi_{K^h}$, the indicator of the domain K^h of U^h and we define recursively the non decreasing sequence of functions V_n^h by the "viability kernel algorithm"

$$V_n^h(x) \;:=\; \max\left(V_{n-1}^h(x), \frac{1}{1-ah}\inf_{u\in U^h(x)}\left(V_{n-1}^h(x + hf^h(x,u)) + hW^h(x,u)\right)\right) \tag{23}$$

Theorem 4.1 *Assume that U^h is upper semicontinuous with compact images, that f^h is continuous and that W^h is lower semicontinuous with respect to the control. Let $V^h \geq \psi_{K^h}$ be a lower semicontinuous extended function.*

From any $x_0 \in \mathrm{Dom}(U^h)$, starts one solution $(x^h, u^h) \in \mathcal{S}^h(x_0)$ to the discrete control system (20) satisfying (21) if and only if V^h satisfies

$$\inf_{u\in U^h(x)}(V^h(x + hf^h(x,u)) + hW^h(x,u)) \;\leq\; (1-ah)V^h(x) \tag{24}$$

Furthermore, the discrete value function $x_0 \mapsto U_h^\flat(0, x_0)$ is the smallest lower semicontinuous function V_∞^h larger than or equal to ψ_{K^h} satisfying (21) and can be obtained through:

$$V_\infty^h(x) \;:=\; \sup_{n\geq 0} V_n^h(x) \tag{25}$$

where the functions V_n^h are defined recursively by the "viability kernel algorithm" (23).

Proof — We introduce the set-valued maps $G^h : X \times \mathbf{R} \rightsquigarrow X \times \mathbf{R}$ defined by

$$G^h(x,w) := \{(x + hf^h(x,v), w - ahw - hW^h(x,v))\}_{v\in U^h(x)}$$

Assume that $\mathcal{M} \subset X \times \mathbf{R}_+$ is a closed viability domain of G^h in the sense that for any $(x, w) \in \mathcal{M}$, there exists $u \in U^h(x)$ such that

$$(x + hf^h(x, u), (1 - ah)w - hW^h(x, u))$$

belongs to $\mathcal{M}$. As in the continuous case, we observe that

$$\mathcal{M}_+ := \mathcal{M} + \{0\} \times \mathbf{R}_+$$

is also a closed viability domain of G^h.

If a function V^h satisfies (24), its epigraph is a closed viability domain of G^h: take $v \in U^h(x)$ achieving the minimum of the lower semicontinuous function $V^h(x + hf^h(x, \cdot)) + hW^h(x, \cdot)$ on the compact subset $U^h(x)$. It satisfies $V^h(x + hf^h(x, v)) + hW^h(x, v) \leq (1 - ah)V^h(x)$ by assumption (24), so that the pair $(x + hf^h(x, v), (1 - ah)w - hW^h(x, v))$ belongs to the epigraph of V^h at (x, w).

Hence $\mathcal{E}p(V^h)$ being a closed viability domain of $G^h(\cdot, \cdot)$, there exists a solution to the discrete set-valued dynamical system

$$\forall\ s \geq 0,\ \ (x^h_{s+1}, w^h_{s+1})\ \in\ G^h(x^h_s, w^h_s)$$

starting from $(x_0, V^h(x_0))$ and viable in the epigraph of V^h.

This implies that $w^h_{r+1} = (1 - ah)w^h_r - hW^h(x^h_r, u^h_r)$. Multiplying both sides by $(1 - ah)^{-r-1}$ and summing from s to $t - 1$, we deduce that

$$(1 - ah)^{-t} w^h_t\ =\ (1 - ah)^{-s} w^h_s - h\sum_{r=s}^{t-1}(1 - ah)^{-r-1} W^h(x^h_r, u^h_r)$$

Taking $s = 0$, we infer that inequality (21) is satisfied since $w^h_0 = V^h(x_0)$ and since $V^h(x^h_t) \leq w^h_0$.

Conversely, it is obvious that a function V^h such that from any $x_0 \in \text{Dom}(U^h)$, starts one solution $(x^h, u^h) \in \mathcal{S}^h(x_0)$ to the discrete control system (20) satisfying (21) satisfies inequalities (24): it is enough to take $t = 1$ in (21).

If the epigraph of ψ_{K^h} does not satisfy (24), one can prove as in the continuous case that its viability kernel is the epigraph of a function denoted by V^h_α. It is then the smallest of the lower semicontinuous functions larger than or equal to ψ_{K^h} which satisfies either (21) or (24).

The solutions along which the function V^h is monotone are obviously given by $x^h_{s+1} = x^h_s + hf^h(x^h_s, u^h_s)$ where $u^h_s \in R^h_V(x^h_s)$.

Since the function V^h is nonnegative, inequalities

$$h\sum_{r=0}^{t-1}(1-ah)^{-r-1}W^h(x_t^h,u_t^h) \;\leq\; V^h(x_0)$$

imply that $U_h^\flat(0,x_0)\leq V^h(x_0)$ and in particular that

$$U_h^\flat(0,x_0) \;\leq\; V_\alpha^h(x_0)$$

On the other hand, the value function $x_0\mapsto U_h^\flat(0,x_0)$ is lower semicontinuous and satisfies along an optimal solution $(\widehat{x}_s^h,\widehat{u}^h)$ inequalities

$$\left\{\begin{array}{l}\inf_{u\in U^h(x_0)}(U_h^\flat(x_0+hf^h(x_0,u))+hW^h(x_0,u)) \;\leq\; U_h^\flat(0,\widehat{x}_1^h)+hW^h(x_0,\widehat{u}_0^h)\\ \leq\; h\sum_{r=0}^{\infty}(1-ah)^{-r-1}W^h(\widehat{x}_{r+1}^h,\widehat{u}_{r+1}^h)+hW^h(x_0,\widehat{u}_0^h) \;\leq\; (1-ah)U_h^\flat(0,x_0)\end{array}\right.$$

because

$$U_h^\flat(0,x_0) \;=\; \frac{h}{1-ah}\left(W^h(\widehat{x}_0^h,\widehat{u}_0^h)+\sum_{r=1}^{\infty}(1-ah)^{-r}W^h(\widehat{x}_r^h,\widehat{u}_r^h)\right)$$

Hence property (24) is satisfied, so that $x^0\mapsto U_h^\flat(0,x_0)$ is larger than or equal to $x_0\mapsto V_\alpha^h(x_0)$. Therefore, they are equal.

It remains to prove formula (25). Since V_α^h satisfies property (24) and is larger than or equal to ψ_{K^h}, we see that we can associate with any x an element $u\in U^h(x)$ such that $(x+hf^h(x,u),(1-ah)V_\alpha^h(x)-hW^h(x,u))$ belongs to $\mathcal{E}p(\psi_{K^h})=\mathrm{Dom}(U^h)\times\mathbf{R}_+$, so that

$$\frac{h}{1-ah}\inf_{u\in U^h(x)}W^h(x,u) \;\leq\; V_\alpha^h(x)$$

Therefore, $V_1^h(x)\;\leq\;V_\alpha^h(x)$. We thus check recursively that if $V_n^h(x)\leq V_\alpha^h(x)$, then, by (24),

$$\left\{\begin{array}{l}\inf_{u\in U^h(x)}(V_n^h(x+hf^h(x,u))+hW^h(x,u)) \;\leq\\ \inf_{u\in U^h(x)}(V_\alpha^h(x+hf^h(x,u))+hW^h(x,u)) \;\leq\; (1-ah)V_\alpha^h(x)\end{array}\right.$$

and thus,

$$\sup_{n\geq 0}V_n^h(x) \;\leq\; V_\alpha^h(x)$$

Finally, we have to prove that $V^h := \sup_{n\geq 0}V_n^h$ satisfies property (24). If so, it will be larger than or equal to V_α^h, and thus, equal to it. By construction of the functions V_n^h, we can associate with any x an element $u_n \in U^h(x)$ such that

$$V_n^h(x + hf^h(x, u_n)) + hW^h(x, u_n) \ \leq \ (1 - ah)V_{n+1}^h(x) \ \leq \ (1 - ah)V^\sharp(x)$$

Since $U^h(x)$ is compact, there exists a subsequence (again denoted by) u_n converging to some $\widehat{u} \in U^h(x)$. The sequence of functions V_n^h being nondecreasing, we deduce that

$$V^\sharp(x + hf^h(x, \widehat{u})) \ \leq \ \liminf_{n \to \infty} V_n^h(x + hf^h(x, u_n))$$

Therefore, since W^h is lower semicontinuous with respect to u, we infer that

$$\begin{cases} V^\sharp(x + hf^h(x, \widehat{u})) + hW^h(x, \widehat{u}) \\ \leq \ \liminf_{n \to \infty} V_n^h(x + hf^h(x, u_n)) + h \liminf_{n \to \infty} W^h(x, u_n) \\ \leq \ \liminf_{n \to \infty} (V_n^h(x + hf^h(x, u_n)) + hW^h(x, u_n)) \\ \leq \ (1 - ah) \liminf_{n \to \infty} (V_{n+1}^h(x) \ \leq \ (1 - ah)V^\sharp(x) \end{cases}$$

This implies that

$$\inf_{u \in U^h(x)} (V^\sharp(x + hf^h(x, u)) + hW^h(x, u)) \ \leq \ (1 - ah)V^\sharp(x)$$

which proves the claim. □

5 Convergence of Discrete Value Functions

The next question we may ask is the following: *Is the limit of a sequence of discrete functions V_∞^h a lower semicontinuous function satisfying (8)?*

It depends on what we understand as "limit": the appropriate concept is the one of *lower epilimit* defined in the following way:

Definition 5.1 *The epigraph of the lower epilimit of a sequence of extended functions $V_n : X \mapsto \mathbf{R} \cup \{+\infty\}$ is the upper limit of the epigraphs:*

$$\mathcal{E}p(\lim^\sharp{}_{\uparrow n \to \infty} V_n) \ := \ \mathrm{Limsup}_{n \to \infty} \mathcal{E}p(V_n)$$

One can check that

$$\lim^\sharp{}_{\uparrow n \to \infty} V_n(x_0) \ = \ \liminf_{n \to \infty, x \to x_0} V_n(x)$$

and that if the sequence is increasing, that

$$\lim{}^{\sharp}_{\uparrow n \to \infty} V_n(x_0) \; = \; \sup_{n \geq 0} V_n(x_0)$$

We refer to Chapter 7 of SET-VALUED ANALYSIS, [5, Aubin & Frankowska] for further details on *epigraphical convergence*.

We deduce from Proposition 4.5.2 of VIABILITY THEORY, [3, Aubin] that

Theorem 5.2 *We posit the assumptions of Theorem 1.1 and we take* $U^h := U$, $f^h := f$ *and* $W^h := W$. *The lower epilimit of the sequence of discrete smallest functions* V_∞^h *satisfying (24) is a lower semicontinuous function satisfying (8) larger than or equal to* ψ_K.

Proof — Indeed, Proposition 4.5.2 of VIABILITY THEORY, [3, Aubin] states that if the discrete set-valued maps G^h satisfy

$$\forall\, \varepsilon > 0,\ \exists\, h_\varepsilon > 0 \mid \forall\, h \in]0, h_\varepsilon],\ \text{Graph}\left(\frac{G^h - \mathbf{1}}{h}\right) \subset \text{Graph}(G) + \varepsilon B \tag{26}$$

and if for every h the epigraph of the function V^h is a viability domains of the set-valued map G^h, then the Painlevé-Kuratowski upper limit of the epigraphs of V^h is a viability domain of the set-valued map G defined by (9).

We observe that

$$\frac{G^h(x,w) - (x,w)}{h} \; = \; \{(f^h(x,u), -aw - W^h(x,u))\}_{u \in U^h(x)}$$

Assumption (26) is obviously satisfied when we take $U^h := U$, $f^h := f$ and $W^h := W$ since in this case $\frac{G^h(x,w)-(x,w)}{h} \subset G(x,w)$.

By Theorem 4.1, the epigraph of V^h is a viability domain of G^h if and only if V^h satisfies (24) and by Theorem 1.1, the epigraph of V is a viability domain of G if and only if V satisfies (8). Since the upper limit of the epigraphs of V^h is the epigraph of their lower epilimit, we infer that the lower epilimit of functions V^h satisfying (24) does satisfy (8). In particular, the lower epilimit of V_∞^h satisfies (8), so that Theorem 5.2 ensues. □

Remark — Under Lipschitzianity conditions implying that G is Lipschitz, we deduce from [28, Quincampoix & Saint-Pierre] that the upper limit of the discrete viability kernels is the viability kernel. In this case, we deduce that V_∞ is the lower epilimit of V_∞^h. □

Remark — In the general case, assumption (26) amounts to checking that U^h, f^h and W^h satisfy the following condition: for every $\varepsilon > 0$, there exists $h_\varepsilon > 0$ such that, for every $h \in]0, h_\varepsilon[$, for every $(x^h, u^h) \in \text{Graph}(U^h)$, there exists $(x, u) \in \text{Graph}(U)$ such that

$$\max(\|x - x^h\|, \|f(x,u) - f^h(x^h, u^h)\|, W(x,u) - W^h(x^h, u^h)) \leq \varepsilon$$

because in this case, we have

$$\begin{cases} \text{Graph}\left(\frac{G^h - \mathbf{1}}{h}\right) = \{(x, w, f^h(x,u), -aw - W^h(x,u))\}_{(x,u)\in\text{Graph}(U^h)} \subset \\ \{(x, w, f(x,u), -aw + [-c(\|x\|+1), -W(x,u)])\}_{(x,u)\in\text{Graph}(U)} + \varepsilon B_{(X\times\mathbf{R})^2} \\ = \text{Graph}(G) + \varepsilon B_{(X\times\mathbf{R})^2} \end{cases}$$ □

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