Reconstruction of Boundary Sources through Sensor Observations

Kryazhimskiy, A.V., Maksimov, V.I. and Osipov, Y.S.

IIASA Working Paper

WP-96-097

August 1996

**Working Papers** on work of the International Institute for Applied Systems Analysis receive only limited review. Views or opinions expressed herein do not necessarily represent those of the Institute, its National Member Organizations, or other organizations supporting the work. All rights reserved. Permission to make digital or hard copies of all or part of this work for personal or classroom use is granted without fee provided that copies are not made or distributed for profit or commercial advantage. All copies must bear this notice and the full citation on the first page. For other purposes, to republish, to post on servers or to redistribute to lists, permission must be sought by contacting repository@iiasa.ac.at
Reconstruction of Boundary Sources through Sensor Observations

A. V. Kryazhimskii, V. I. Maksimov, and Yu. S. Osipov

WP-96-97
August 1996
Reconstruction of Boundary Sources through Sensor Observations

A. V. Kryazhimskii, V. I. Maksimov, and Yu. S. Osipov

WP-96-97
August 1996
Reconstruction of Boundary Sources through Sensor Observations

A. V. Kryazhimskii, V. I. Maksimov, and Yu. S. Osipov

1. Introduction. Problem Formulation.

The paper deals with a problem of reconstruction of extremal boundary disturbances in a parabolic system. The reconstruction is performed on the basis of inaccurate observations of linear signals on system’s states. Extremality of disturbances is understood with respect to a given linear functional. The paper joins theory of inverse problems for distributed systems (see, e.g., [Lavrentyev et al., 1980; Banks and Kunisch, 1982; Kurzhanski and Khapalov, 1989; Barbu, 1991; Osipov et al., 1991; Maksimov, 1993; Ainsba, 1994]) and theory of ill-posed problems (see, e.g., [Tikhonov and Arsenin, 1979], [Vasiliev, 1981]).

A variant of a substantial problem formulation is as follows. A flow of heat enters a solid body occupying space $\Omega$ through boundary domains $\gamma_1, \ldots, \gamma_n$. At time $t$ the instant velocity of the heat flow coming through $\gamma_k$ is represented as $v_k(t)\bar{\gamma}_k$ where $\bar{\gamma}_k$ is the square of $\gamma_k$. Values $v_k(t)$ are not available for direct observation. Instead, the average temperature of the body in spatial domains $\Omega_1, \ldots, \Omega_m \subset \Omega$ (as a variant, the temperature at points $\eta_1, \ldots, \eta_m \in \Omega$) is observed. It is required to reconstruct the minimum value of a linear functional $I = I(v(\cdot))$ over all $v(\cdot) = (v_1(\cdot), \ldots, v_n(\cdot))$ compatible with the observed data. A reconstruction algorithm should be robust to observation errors.

In our analysis we generalize the approach of [Kryazhimskii, and Osipov, 1993] based on the technique of adjoint equations (see [Lions, 1971] and [Marchuk, 1982]), and reduce the reconstruction problem to an extremal problem on the solutions of an appropriate finite-dimensional integral equation. To solve the latter problem, the method of convex optimization proposed in [Kryazhimskii, and Osipov, 1987] (see also [Kryazhimskii, 1994] and [Ermoliev et al., 1995]) is utilized. We design two iterative reconstruction algorithms and estimate rates of convergence.

Specify the problem. Let $\Omega$ be an open bounded domain in $\mathbb{R}^n$ with sufficiently smooth boundary $\Gamma$, $Ax$ be a second order partial differential operator in $\Omega$ with smooth real coefficients,

$$
(Ax)(\eta) = \sum_{i,j=1}^{n} \frac{\partial}{\partial \eta_i} (a_{ij}(\eta) \frac{\partial x(\eta)}{\partial \eta_j}) + \sum_{j=1}^{n} b_j(\eta) \frac{\partial x(\eta)}{\partial \eta_j} + c_0(\eta),
$$

and the principal part uniformly strongly elliptic in $\Omega$,

$$
\sum_{i,j=1}^{n} a_{ij}(\eta) \eta_i \eta_j \geq \gamma \sum_{j=1}^{n} \eta_j^2; \quad a_{ij} \equiv a_{ji}; \quad \gamma > 0.
$$

We consider the parabolic boundary control problem

$$
\dot{x}_i(t, \eta) = (Ax(t))(\eta) \quad \text{in} \quad T \times \Omega, \quad T = [0, \bar{t}],
$$

(1.1)
where
\[ g(t, \sigma) = \sum_{k=1}^{n} \omega_k(\sigma)v_k(t) + f(t, \sigma), \quad v = \{v_1, \ldots, v_n\} \in \mathbb{R}^n, \]

\( \omega_k \in L_2(\Gamma), \quad f(\cdot) \in L_2(T; \Gamma), \quad x_0 \in H = L_2(\Omega). \)

The \( n \)-dimensional control \( v(\cdot) = (v_k(\cdot))_{k=1}^{n} \in L_2(T; \mathbb{R}^n) \) treated as a disturbance effects the system through the Dirichlet boundary conditions. The disturbance is uncontrolled and not available for observation. It is assumed (see [Lasiecka, 1980]) that the closed and densely defined operator \( A_1 : H \supset D(A) \to H \) determined by \( A_1 y = A y, y \in D(A) = \{h \in H : A h \in H, \ h\big|_{\Gamma} = 0 \} \) represents the infinitesimal generator of a strongly continuous semigroup of linear continuous operators \( Q(t), \ t \in \mathbb{R}. \)

An (abstract) solution \( x(\cdot) : T \mapsto L_2(\Omega) \) of problem (1.1) is defined by
\[
x(t) = Q(t)x_0 + \int_0^t A Q(t - \tau) \left\{ \sum_{k=1}^{n} v_k(\tau) D \omega_k + D f(\tau) \right\} d\tau
\]
(see [Lasiecka, 1980]). Here \( w_v(t) = \sum_{k=1}^{n} v_k(t) D \omega_k + D f(t), \ D \in L(L_2(\Gamma); H) \) is the Dirichlet map, i.e., \( D \omega = y_w \) is the generalized solution of the elliptic equation
\[
A y_w = 0 \text{ in } \Omega
\]
\[
y_w \big|_{\Gamma} = w \text{ in } \Gamma;
\]
equivalently, \( y_w \) is the unique element of \( H \) such that
\[
\int_{\Omega} y_w A \psi d\eta = \int_{\Gamma} w \frac{\partial \psi}{\partial \sigma} d\sigma \text{ for all } \psi \in D(A).
\]

It is known (see [Lasiecka, 1978, 1980]) that under the above conditions, for every \( x_0 \in H \) and \( v(\cdot) \in H_1 = L_2(T; \mathbb{R}^n) \) there exists a unique (abstract) solution \( x(\cdot) \) of problem (1.1); we denote it by \( x(\cdot; x_0, v(\cdot)). \)

Introduce the set of all admissible disturbances:
\[
V_* = \{v(\cdot) \in W^{1,2}(T; \mathbb{R}^n) : v(0) = 0, \ |\dot{v}(\cdot)|_{H_1} \leq M\};
\]
here and in what follows
\[
W^{1,2}(T; \mathbb{R}^n) = \{v(\cdot) \in H_1 : \dot{v}(\cdot) \in H_1\}.
\]

Let \( P : H \to \mathbb{R}^m \) be a linear continuous observation operator:
\[
Px = \{\langle p_i, x \rangle_H, \ldots, \langle p_m, x \rangle_H \}, \ p_i \in H, \ i \in [1 : m].
\]

We assume that, given a solution \( x(\cdot) = x(\cdot; x_0, v(\cdot)) \) corresponding to an (unknown) disturbance \( v(\cdot) \in V_* \), the \( m \)-dimensional vector
\[
z(t) = P x(t)
\]
is observed at each \( t \in T \). The observation result \( \xi(t) \) is, in general, inaccurate.
We suppose that $\xi^h(\cdot) \in H_2 = L_2(T; \mathbb{R}^n)$. Denote by $V_z$ the set of all admissible disturbances $v(\cdot)$ compatible with $z(\cdot)$:

$$V_z = \{ v(\cdot) \in V_* : Px(t; v(\cdot)) = z(t) \text{ for all } t \in T \}. $$

Let

$$I(v(\cdot)) = g'_1(v(\varphi)) + \int_0^\varphi g'_2(t)v(t)\,dt$$

be a linear functional on space $W^{1,2}(T; \mathbb{R}^n)$ ($g_1 \in \mathbb{R}^n$, $g_2 \in L_2(T; \mathbb{R}^n)$; prime stands for transposition),

$$I^0_z = \min \{ I(v(\cdot)) : v(\cdot) \in V_z \},$$

and

$$V_0(z) = \text{arg min} \{ I(v(\cdot)) : v(\cdot) \in V_z \};$$

note that

$$V_0(z) \neq \emptyset$$

(see below Theorem 2.1).

**Problem 1.** Given a family of observation results $\xi^h = \xi^h(\cdot) \in H_2$, $h > 0$, satisfying (1.5), build a family of admissible disturbances $v^h(\cdot) = v(\cdot; \xi^h)$, $h > 0$, such that

$$I(v^h(\cdot)) \to I(v_0(\cdot)) = I^0_z, \quad v^h(\cdot) \to v_0(\cdot) \in V_0(z) \text{ in } C(T; \mathbb{R}^n) \text{ as } h \to 0. \quad (1.9)$$

## 2. Problem reduction.

We reduce Problem 1 to another one more convenient for investigation. Further on, the following condition is assumed.

**Condition 2.1.** a) $\omega_j \in H^{2\beta-1}(\Gamma)$, $\beta = [0.25n + 1.75 + 0.5\varepsilon] + 1 (\varepsilon \in (0,1/2))$, $j \in [1:n]$; b) $x_0 \in H^{2\beta-1/2}(\Omega)$; c) $f(\cdot) \in H^{2\beta,\beta}(T; \Gamma)$.

Symbol [a] denotes the integer part of $a$, $x^0(\cdot) = x(\cdot; x_0, 0)$, $x_j(\cdot) = x(\cdot; D\omega_j, 0)$. The algorithms described below solve Problem 1 under assumptions weaker than Condition 2.1. However, the estimates of the convergence rate (see (1.9)) provided in Sec.5 require functions $x(\cdot)$, $x^0(\cdot)$, $x_j(\cdot)$ to be sufficiently smooth (see below Lemma 2.1). The needed smoothness is ensured under Condition 2.1.

**Lemma 2.1** Let $u_j(\cdot) \in W^{1,2}(T; \mathbb{R})$, $u_j(0) = 0$, $j \in [1:n]$. Then $D\omega_j \in C(\Omega)$, $x_j(\cdot)$, $\dot{x}_j(\cdot)$, $x^0(\cdot)$, $\dot{x}^0(\cdot) \in C(T; C(\Omega))$, $x(\cdot) \in C(T; C(\Omega))$ and

$$x(t) = x^0(t) + \int_0^t \sum_{j=1}^n \left\{ x_j(t - \tau) - D\omega_j \right\} \dot{u}_j(\tau)\,d\tau, \quad t \in T. \quad (2.1)$$

**Proof.** If Condition 2.1 a) is true, then

$$D\omega_j \in C(\Omega), \quad x_j(\cdot), \dot{x}_j(\cdot) \in C(T; C(\Omega)), \quad \dot{x}^0(\cdot) \in L_2(T; C(\Omega)). \quad (2.2)$$
Indeed, as follows from [Lasiecka, 1980] (see also Theorem 7.4 in [Lions and Magenes, 1972]), we have $D \in L_2(H^{2\nu}(\Gamma); H^{2\nu + 1/2}(\Omega))$ for all $\nu \in \mathbb{R}$. Let $\nu = \beta - 0.5$. Theorem 6.1 from [Lasiecka, 1980] implies

$$x_j(\cdot) \in L_2(T; H^{2\beta+1/2-\varepsilon}(\Omega)), \quad \dot{x}_j(\cdot) \in L_2(T; H^{2(\beta-1)+1/2-\varepsilon}(\Omega)),$$

$$\ddot{x}_j(\cdot) \in L_2(T; H^{2(\beta-2)+1/2-\varepsilon}(\Omega)).$$

Therefore (see Theorem 9.8 in [Lions and Magenes, 1972]), changing, if necessary, $x_j(t, \eta), \ (t, \eta) \in T \times \Omega)$ on a set of zero measure we obtain that for an integer $\beta$ satisfying $2(\beta - 2) + 0.5 - \varepsilon > 0.5m$ the following is true:

$$H^{2\beta+1/2-\varepsilon}(\Omega) \subset H^{2(\beta-1)+1/2-\varepsilon}(\Omega) \subset H^{2(\beta-2)+1/2-\varepsilon}(\Omega) \subset C(\Omega),$$

$$D\omega_j \in C(\Omega), \quad \ddot{x}_j(\cdot) \in L_2(T; C(\Omega)).$$

The validity of (2.2) is established. Analogously, we deduce that if $f(\cdot) \in H^{2\beta}(T; \Gamma)$, $x_0 \in H^{2\beta-1}(\Omega)$, then

$$x_0^0(\cdot) \in C(T; C(\Omega)), \quad x_0^1(\cdot) \in C(T; C(\Omega)), \quad \ddot{x}_0^0(\cdot) \in L_2(= x(T; C(\Omega))).$$

To verify equality (2.1) it is sufficient to integrate by parts the term

$$\int_0^t AQ(t - \tau) \sum_{k=1}^n v_k(\tau) D\omega_k \, d\tau.$$ 

(see [Lasiecka, 1980]). Lemma is proved.

Introduce the linear operator $F : H_1 \to H_2$,

$$(Fu(\cdot))(\sigma) = \int_0^\sigma \varphi(t, \sigma) u(t) \, dt, \quad \sigma \in T; \quad (2.3)$$

where $\varphi(t, \sigma) = \varphi(\sigma - t)$ is the $m \times n$-dimensional matrice whose $k$-th row has the form

$$\varphi_k(t, \sigma) = \{\varphi_k(t, \sigma)\}_1, \{\varphi_k(t, \sigma)\}_2, \ldots, \{\varphi_k(t, \sigma)\}_n,$$

$$\{\varphi_k(t, \sigma)\}_j = \{\varphi_k(\sigma - t)\}_j = \begin{cases} \langle x_j(\sigma - t) - D\omega_j, p_k\rangle_H, & \sigma \geq t, \\ 0, & \sigma < t, \end{cases} \quad j \in [1 : n].$$

**Theorem 2.1** We have $v(\cdot) \in V_\alpha$ if and only if

$$\dot{v}(\cdot) \in U_\alpha, \quad (F \dot{v}(\cdot))(t) = b_\alpha(t) \quad \text{for all } t \in T.$$

Here

$$b_\alpha(t) = \{b_\alpha(t)_1, \ldots, b_\alpha(t)_m\} \in \mathbb{R}^m,$$

$$b_\alpha(t)_k = z_k(t) - \langle x_0^0(t), p_k\rangle_H \in \mathbb{R}, \quad k \in [1 : m],$$

$$U_\alpha = \{u(\cdot) \in H_1 : |u(\cdot)|_{H_1} \leq M\}, \quad (2.4)$$

and $z(t) = \{z_1(t), \ldots, z_m(t)\}$ is defined by to (1.4).
The theorem follows from Lemma 2.1. Note that Theorem 2.1 implies (1.8). For any \( v(\cdot) \in V_\ast \) such that \( \dot{v}(t) = u(t) \) for a.a. \( t \in T \) the following equality
\[
I(v(\cdot)) = J(u(\cdot)) = \int_0^T g(t) u(t) \, dt
\]
is valid, provided \( g(\varphi) = g_1, \dot{g}(t) = -g_2(t) \) for a.a. \( t \in T \). Thus, Problem 1 is replaced by the following one. Let
\[
J_z^0 = \min \{ J(u(\cdot)) : u(\cdot) \in U_z \},
\]
\[
U_z = \{ u(\cdot) \in U_\ast : Fu(\cdot) = b_z \}, \quad U_0(z) = \arg \min \{ J(u(\cdot)) : u(\cdot) \in U_z \}.
\]

**Problem 2.** Given a family of observation results \( \xi^h = \xi^h(\cdot) \in H_1, \ h > 0, \) satisfying (1.5), build a family \( u^h(\cdot) = u(\cdot; \xi^h), \ h > 0, \) from \( U_\ast \) such that
\[
J(u^h(\cdot)) \to J(u_0(\cdot)) = J_z^0, \quad u^h(\cdot) \to u_0(\cdot) \in U_0(z) \text{ weakly in } H_1 \text{ as } h \to 0. \tag{2.7}
\]

It is easily seen that if family \( u^h(\cdot) \) solves Problem 2, then family \( v^h(\cdot) \)
\[
v^h(\tau) = \int_0^\tau u^h(\nu) \, d\nu, \quad \tau \in T,
\]
solves Problem 1, and
\[
I(v^h(\cdot)) = J(u^h(\cdot)). \tag{2.8}
\]
Further on we focus on Problem 2.

### 3. Solution algorithm. 1.

We use the approach of [Kryazhimskii and Osipov, 1987; Kryazhimskii, 1994]. By Theorem 2.1 and equality (2.8) we have
\[
J_z^0 = \min \{ J(v) : v \in U_\ast, \ Fu = b_z \}. \tag{3.1}
\]

Denote
\[
J_z^0[\gamma] = \min \{ J(v) : v \in U_\ast, \ |Fu - b_z|_{H_2}^2 \leq \gamma \}. \tag{3.2}
\]

It is clear that
\[
J_z^0[\gamma] \leq J_z^0, \quad \lim_{\gamma \to 0} J_z^0[\gamma] = J_z^0. \tag{3.3}
\]

Let \( U_z^0[\gamma, \beta] (\beta \geq 0, \ \gamma > 0) \) be the set of all \( u(\cdot) \in U_\ast \) such that
\[
|Fu - b_z|_{H_2}^2 \leq \gamma, \quad J(u) \leq J_z^0 + \beta.
\]

In what follows, we set \( N = \{0, 1, 2, \ldots\} \), and \( \langle \cdot, \cdot \rangle_{H_2} \) stands for the scalar product in \( H_2 \). Fix a family of mappings
\[
V_\alpha : \{i, y, \xi(\cdot)\} \mapsto V_\alpha(i, y, \xi) : N \times H_1 \times H_2 \to U_\ast, \quad \alpha > 0.
\]
For every $\delta, \alpha, h > 0$ define sequence $y_{\delta, h}(i), \ i \in N, \ in \ H_2,$ as follows:

$$y_{\delta, h}(i + 1) = y_{\delta, h}(i) + v_i^h \delta, \quad v_i^h = V_{\alpha}(i, y_{\delta, h}(i), \xi^h), \quad y_{\delta, h}(0) = 0, \quad i \in N. \quad (3.4)$$

It is easily seen that by virtue of the convexity and closedness of set $U_\ast$, we have

$$y_{\delta, h}(i)/(i \delta) \in U_\ast, \quad i > 0.$$  

Specify $V_{\alpha}$. We put

$$V_{\alpha}(i, y, \xi) = \arg \min \{ \ 2\langle Fy - i \delta b_i^h, \ Fu \rangle_{H_2} + \alpha J(u) : \ u \in U_\ast \},$$

or, explicitly,

$$V_{\alpha}(i, y, \xi) = u(v; i, y, \xi)(\nu) = u(\nu; i, y, \xi) \quad (\nu \in T),$$

$$u(\nu; i, y, \xi) = \begin{cases} -M \frac{D_{\alpha}(\nu; i, y, \xi)}{|D_{\alpha}(\nu; i, y, \xi)|_{H_1}}, & \text{if } |D_{\alpha}(\nu; i, y, \xi)|_{H_1} \neq 0, \\ 0, & \text{otherwise.} \end{cases} \quad (3.5)$$

Here

$$D_{\alpha}(\nu; i, y, \xi) = 2 \int \mu(\nu, \sigma; i, y, \xi) d\sigma + \alpha g(\nu), \quad \nu \in T, \quad (3.6)$$

$$\mu(\nu, \sigma; i, y, \xi) = \varphi(\nu, \sigma) v_0(\sigma; i, y, \xi), \quad v_0(\sigma; i, y, \xi) = \int_{\theta}^\sigma \varphi(\nu, \sigma) y(\nu) d\nu - i \delta b_\xi^h(\sigma),$$

$$b_\xi^h(\sigma) = \{ b_\xi^h(\sigma_1), \ldots, b_\xi^h(\sigma_m) \} \in \mathbb{R}^m, \quad b_\xi^h(\sigma)_k = \xi_k(\sigma) \in \{ x_0(t), p_k \}_{H_1} \in \mathbb{R}, \quad k \in [1: m].$$

Define constants $K_J$ and $K_F$ by $|J(v)| \leq K_J, |Fv - b_z|_{H_2} \leq K_F \quad (v \in U_\ast, \ z(\cdot) \in \{ z(\cdot) : z(t) = Px(t, v(t)) \text{ for all } t \in T, \ v(\cdot) \in U_\ast \}).$

**Theorem 3.1** It holds that

$$y_{\delta, h}(i)/(i \delta) \in U_z^0 \ [2K_J \alpha/(i \delta) + g_z(\delta, h)/(i \delta), \ g_z(\delta, h)/\alpha], \quad (3.7)$$

$$J_z^0 [2K_J \alpha/(i \delta) + g_z(\delta, h)/(i \delta)] - J_z^0 \leq J(y_{\delta, h}(i)/(i \delta)) - J_z^0 \leq g_z(\delta, h)/\alpha, \quad (3.8)$$

where

$$g_z(\delta, h) = K_F^2 \delta + 2K_F \sqrt{\delta} h i \delta, \quad i \geq 1. \quad (3.9)$$

**Proof.** We follow the proof pattern of [Kryazhinskiii, 1994]. Estimate recurrently the values

$$\Lambda_{\alpha}(i; y_{\delta, h}(\cdot)) = |Fy_{\delta, h}(i) - i \delta b_z|_{H_2} + \alpha \delta \sum_{j=1}^i J(v_{\delta, h}^h) - \alpha i \delta J_z^0, \quad i \geq 1.$$
Let

\[ v_z \in U_z^0 = \arg \min \{ J(v) : v \in U_z, \ Fv = b_z \}. \]

By Theorem 2.1 \( U_z^0 = U_0(z) \). Therefore

\[ \Lambda_o(i + 1; \ y_{\xi, h}^o(\cdot)) = \Lambda_o(i; \ y_{\xi, h}^o(\cdot)) + 2\delta \langle \lambda_i, \ (Fv_i^h - b_z) \rangle_{H_2} + \]

\[ + \left| Fv_i^h - b_z \right|_{H_2}^2 \delta^2 + \alpha J(v_i^h) - \alpha J_0^z \delta \leq \]

\[ \leq \Lambda_o(i; \ y_{\xi, h}^o(\cdot)) + [2\langle \lambda_i, \ Fv_i^h - b_z \rangle_{H_2} + \alpha J(v_i^h)] \delta + \]

\[ + [2\langle \lambda_i, \ Fv_z - b_z \rangle_{H_2} + \alpha J(v_z)] \delta + K_F^2 \delta^2, \quad \lambda_i = Fy_{\xi, h}^o(i) - i\delta b_z. \]

By virtue of (1.5)

\[ |\xi^h(\cdot) - z(\cdot)|_{H_2} \leq \sqrt{\vartheta h}. \]

Consequently,

\[ |b_z(\cdot) - b_{\xi}^h(\cdot)|_{H_2} \leq \sqrt{\vartheta h}, \quad (3.12) \]

\[ \langle \lambda_i, \ Fv_i^h - b_z \rangle_{H_2} = \langle Fy_{\xi, h}^o(i) - i\delta b_{\xi}^h + i\delta (b_{\xi}^h - b_z), \ Fv_i^h - b_z \rangle_{H_2} \leq \]

\[ \leq \langle Fy_{\xi, h}^o(i) - i\delta b_{\xi}^h, \ Fv_i^h - b_z \rangle_{H_2} + i\delta \sqrt{\vartheta h} \cdot K_F. \]

Similarly we obtain

\[ \langle \lambda_i, \ Fv_z - b_z \rangle_{H_2} \leq \langle Fy_{\xi, h}^o(i) - i\delta b_{\xi}^h, \ Fv_z - b_z \rangle_{H_2} + i\delta \sqrt{\vartheta h} \cdot K_F. \quad (3.13) \]

Then the following equality is true

\[ 2\langle Fy - i\delta b_{\xi}^h, \ Fu \rangle_{H_2} + \alpha J(u) = 2 \int_0^\vartheta \nu_0(\sigma; i, y, \xi^h) \int \varphi(\nu, \sigma) u(\nu) \, d\nu \, d\sigma + \]

\[ + \alpha \int_0^\vartheta g(\sigma) u(\sigma) \, d\sigma = \int_0^\vartheta D'(\nu; i, y, \xi^h) u(\nu) \, d\nu = \langle D_\alpha, \ u \rangle_{H_1}. \quad (3.14) \]

From (3.4), (3.5), (3.10)–(3.14) we derive that

\[ \Lambda_o(i + 1; \ y_{\xi, h}^o(\cdot)) \leq \Lambda_o(i; \ y_{\xi, h}^o(\cdot)) + 4i\delta \sqrt{\vartheta K_F} h + K_F^2 \delta^2 \leq g_i(\delta, h)(i + 1) \delta. \quad (3.15) \]

Note that

\[ \lambda_a/(\cdot)(y_{\xi, h}^o(i)/(i\delta)) \leq \Lambda_o(i; \ y_{\xi, h}^o(\cdot))/(i\delta)^2 \leq g_i(\delta, h)/(i\delta). \quad (3.16) \]

Here

\[ \lambda_a(y) = |Fy - b_z|_{H_2}^2 + aJ(y) - aJ^0_z, \quad a > 0. \]
Besides, if \( v \in U_* \), \( \lambda_v(y) \leq \varepsilon \), then the following relations hold:

\[
y \in U_z^0 [2KJa + \varepsilon, \varepsilon/a],
\]

\[
J_z^0 [2KJa + \varepsilon] - J_z^0 \leq J(y) - J_z^0 \leq \varepsilon/a.
\]

The estimates (3.7)–(3.9) follow from (3.15)–(3.18). The Theorem is proved.

Theorem 3.1 yields the following.

**Theorem 3.2** Let \( \alpha(h) > 0 \), \( \delta(h) > 0 \), and \( i(h) \in N \) satisfy

\[
\alpha(h) \to 0, \quad \delta(h) \to 0, \quad \delta(h)i(h) \to \infty, \quad (hi(h)\delta(h) + \delta(h))/\alpha(h) \to 0 \text{ as } h \to 0.
\]

Then the family

\[
u^h(\cdot) = g_0(i(h))/((i(h)\delta(h)), \quad h > 0,
\]

solves Problem 2, i.e. convergences (2.7) take place.

4. Solution algorithm. 2.

Let us describe another solution algorithm. Introduce the mapping \( p : (i, y, \xi) \mapsto p(i, y, \xi^h(\cdot)) : T \times H_1 \times H_2 \to H_1 \):

\[
p = p(\nu; i, y, \xi) = \int_0^\varphi \mu(\nu, \sigma; i, y, \xi) d\sigma, \quad \nu \in T
\]

(see (3.6)). For an arbitrary triple \((i, y, \xi) \in T \times H_2 \times H_2\), we represent element \( g \in H_1 \) (see (2.5)) in the form

\[
g = \mu p + c_1,
\]

\[
\mu = \langle g, p \rangle_{H_1}, \quad |p|_{H_1}^2, \quad c_1 = g - \mu p,
\]

and denote

\[
\mu = \mu(i, y, \xi), \quad p = p(i, y, \xi), \quad c_1 = c_1(i, y, \xi).
\]

Note that \( c_1 \) is orthogonal to \( p \), i.e. \( \langle c_1, p \rangle_{H_1} = 0 \).

Let the mapping \( V : (i, y, \xi) \mapsto V(i, y, \xi) \in N \times H_2 \times H_2 \to U_* \) be defined by

\[
V(i, y, \xi)(\nu) = u(\nu; i, y, \xi) \quad (\nu \in T),
\]

where

\[
u(\nu) = u(\nu; i, y, \xi) = \begin{cases} -Mg(\nu)|g|_{H_1}^{-1}, & \text{if } p = 0 \\ \nu_1 p(\nu) + v_1(\nu), & \text{otherwise}, \end{cases}
\]

\[
(4.5)
\]

\[
(4.4)
\]
and \( p = p(v) = p(v; i, y, \xi), \) \( v \in T, \) is defined by (4.1). If \( c_1 = c_1(i, y, \xi) \neq 0, \) then we have

\[
v_1 = -k_sc_1, \quad k_s = \left( (M^2 - v_1^2) |p|_{H_1}^2 / |c_1|_{H_1}^2 \right)^{1/2},
\]

\[
\nu_1 = \min \{ \nu_0, g_2 \}, \quad \nu_0 = -\text{sign} \left\{ a_0 \frac{a_0 \sqrt{a_1}}{\sqrt{a_0^2 + a_2 \sqrt{a_2}}} \right\}, \quad (4.6)
\]

\[
a_0 = \mu |p|_{H_1}^2, \quad a_1 = M^2 |c_1|_{H_1}, \quad a_2 = |p|_{H_1}^2 \cdot |c_1|_{H_1},
\]

\[
g_1 = -M|p|_{H_1}^{-1} = -\sqrt{a_1 a_2^{-1}}, \quad g_2 = \min \{ -g_1, \beta |p|_{H_1}^{-2} \},
\]

\[
\beta = \beta(i, y, \xi) = \langle Fy - i \delta b_\xi^h, b_\xi^h \rangle_{H_2} + \varrho_1(h) = \int_0^\varphi \left( \int_0^\varphi \varphi(t, \sigma) y(t) dt - i \delta b_\xi^h(\sigma) \right) b_\xi^h(\sigma) d\sigma + \varrho_1(h),
\]

\[
\varrho_1(h) = 2K_F \sqrt{\vartheta h} + \vartheta h^2.
\]

If \( c_1 = c_1(i, y, \xi^h) = 0, \) then

\[
v_1 = \begin{cases} 
  g_2, & a_0 > 0 \\
  g_1, & a_0 < 0 \\
  v_s \in [g_1, g_2], & a_0 = 0,
\end{cases} \quad (4.7)
\]

and \( v_1 = v_1(v), \) \( v \in T, \) is defined as an arbitrary element in space \( H_1 \) such that

\[
\langle v_1, p \rangle_{H_1} = 0, \quad v_1 \left| p \right|_{H_1}^2 + \left| v_1 \right|_{H_1}^2 \leq M^2.
\]

Below, in the proof of Theorem 4.1, we shall show that set \( V(i, y, \xi) \) (4.4) is that of all solutions of the problem

\[
\min v \in U_s, \langle Fy - i \delta b_\xi^h, Fv - b_\xi^h \rangle_{H_2} \leq \varrho_1(h).
\]

Introduce the sequence \( y_{\delta, h}(i) \in H_2 \) (\( \delta > 0, h \geq 0), \) \( i \in N, \) by

\[
y_{\delta, h}(i + 1) = y_{\delta, h}(i) + v_i \delta, \quad v_i = V(i, y_{\delta, h}(i), \xi^h).
\]

Let

\[
\varrho_s(\delta, h) = K_F^2 \delta + 4\varrho_1(h).
\]

**Theorem 4.1** It holds that

\[
y_{\delta, h}(i),/i/\delta \in U_s^0[\varrho_s(\delta, h)/i/\delta, 0], \quad (i > 0),
\]

\[
J_s^0[\varrho_s(\delta, h)/i/\delta]) \leq J(y_{\delta, h}(i)/i/\delta)) \leq J_s^0.
\]
Proof. It is sufficient to show that
\[ |F(y_{\delta, h}(i)/(i\delta)) - b_z^2_{H_2}| \leq \varrho_*(\delta, h)/(i\delta), \quad J(y_{\delta, h}(i)/(i\delta)) \leq J_0^0. \]

It is easily seen that these inequalities are ensured by
\[ |Fy_{\delta, h}(i\delta) - i\delta b_z^2_{H_2}| \leq \varrho_*(\delta, h)i\delta, \quad \delta \sum_{j=1}^{i} J(v_{j-1}) \leq i\delta J_0^0. \quad (4.9) \]

Let us verify (4.9). First, we prove that \( V(i, y, \xi^h) \) is the set of all solutions of problem (4.8). Using (4.1)–(4.3), transform problem (4.8) into
\[
\min \mu \nu |p|_{H_1}^2 + \langle c_1, v_1 \rangle_{H_1}, \\
\{ \nu, v_1 \} \in \mathbb{R}^+ \times H_1, \quad \langle v_1, p \rangle_{H_1} = 0, \\
\nu^2 |p|_{H_1}^2 + |v_1|_{H_1}^2 \leq M^2, \quad \nu \leq \beta |p|_{H_1}^{-2}, \quad \mathbb{R}^+ = \{ z \in \mathbb{R} : z \geq 0 \}. \quad (4.10)
\]

For a fixed \( \nu \) the minimum in (4.10) is reached at
\[ v_1 = -k_0 c_1, \quad k_0 = ((M^2 - \nu^2 |p|_{H_1}^2) |c_1|_{H_1}^2)^{1/2} \]
if \( c_1 \neq 0 \). In the opposite case \( v_1 \) is an arbitrary element such that
\[ \langle v_1, p \rangle_{H_1} = 0, \quad \nu^2 |p|_{H_1}^2 + |v_1|_{H_1}^2 \leq M^2. \]

In this case problem (4.10) is reduced to
\[
\min \mu \nu |p|_{H_1}^2 + ((M^2 - \nu^2 |p|_{H_1}^2) |c_1|_{H_1}^2)^{1/2}, \\
\nu^2 |p|_{H_1}^2 \leq M^2, \quad \nu \leq \beta |p|_{H_1}^{-2} \quad (4.11)
\]
if \( c_1 \neq 0 \), and
\[
\min \mu \nu |p|_{H_1}^2, \\
\nu^2 |p|_{H_1}^2 \leq M^2, \quad \nu \leq \beta |p|_{H_1}^{-2} \quad (4.12)
\]
if \( c_1 = 0 \). Rewrite problem (4.11) in the form
\[
\min \{ a_0 \nu - (a_1 - a_2 \nu^2)^{1/2} : \nu \in [g_1, g_2] \}, \quad (4.13)
\]

It is easily seen that the minimum in (4.13) is reached at \( \nu = v_1 \) (4.6). In turn, a solution of problem (4.12) is of the form (4.7). Thus, \( V(i, y, \xi) \) is the set of all solutions of problem (4.8). Let us verify inequalities (4.9). Using (3.12), we obtain
\[
|\langle Fg - b_z, Fv - b_{\xi} \rangle_{H_2} - \langle Fg - b_{\xi}^h, Fv - b_{\xi}^h \rangle_{H_2}| \leq \\
\leq |\langle Fg - b_z, b_{\xi}^h - b_z \rangle_{H_2}| + |\langle Fg - b_z, Fv - b_{\xi}^h \rangle_{H_2} - \langle Fg - b_{\xi}^h, Fv - b_{\xi}^h \rangle_{H_2}| \leq \\
\leq K_F \sqrt{\delta h} + \{ |Fv - b_z|_{H_2} + |b_{\xi}^h - b_z|_{H_2} \} \sqrt{\delta h} \leq g_1(h). \]

10
Hence by the definition of $V(i, y, \xi^h)$ we have
\[ J(v_i) \leq J_2^0. \] (4.14)

Let $S(i) = F y_{\delta, h}(i) - i \delta$. Then by virtue of the inclusion
\[ v_i \in \{ v \in U : (S(i), F v - b_z)_{H_2} \leq 2 \varrho_1(h) \} \]
the following inequality holds:
\[ |S(i + 1)|_{H_2}^2 = |S(i)|_{H_2}^2 + 2\langle S(i), F v_i - b_z \rangle_{H_2} \cdot \delta + \]
\[ + |F v_i - b_z|_{H_2}^2 \delta^2 \leq |S(i)|_{H_2}^2 + K^2 \delta^2 + 4 \varrho_1(h) \delta. \] (4.15)

Besides,
\[ |S(0)|_{H_2} = 0. \] (4.16)

The inequality (4.9) follows from (4.14)–(4.16). The Theorem is proved.

Theorem 4.1 yields the following.

**Theorem 4.2** Let $\delta(h) > 0$ and $i(h) \in N$ satisfy
\[ \delta(h) \to 0, \quad \delta(h)i(h) \to \infty \text{ as } h \to 0. \]
Then the family
\[ u^h(\cdot) = y_{\delta(i(h)), h}(i(h)) / (i(h)\delta(h)), \quad h > 0, \]
solves Problem 2, i.e. convergences (2.7) take place.

**5. Estimates of convergence rate.**

In this Section we provide estimates of the convergence rate for the above described algorithms. Assume that the following condition is satisfied.

**Condition 5.1** Functions $\varphi(\cdot)$ and $b_z(\cdot)$ are Lipshitz.

By Lemma 2.1 Condition 5.1 is ensured by Condition 2.1.

Fix a partition of interval $T$:
\[ \{ t_j \}_{j=0}^{E(\Delta)}, \quad t_0 = 0, \quad t_{j+1} = t_j + \Delta, \quad t_{E(\Delta)} = \vartheta. \]
Introduce the operator $F_\Delta : H_1 \to H_2$ (a “$\Delta$ - approximation” to $F$):
\[ (F_\Delta u(\cdot))(t) = \int_0^t \varphi_{\Delta}(t - \sigma) u(\sigma) d\sigma \quad (t \in T), \] (5.1)
where
\[ \varphi_{\Delta}(t - \sigma) = \varphi(t_j - \sigma) \quad (t \in [t_j, t_{j+1})], \quad j \in [0 : E(\Delta) - 1], \quad (\sigma \in T). \]
Denote by $K$ a Lipschitz constant for functions $\varphi(t)$ and $b_z(t)$ on $T$. Let

$$b_z^\Delta(t) = b_z(t_j), \quad t \in [t_j, t_{j+1}),$$

$$J_z^\Delta[\varepsilon] = \min \{ J(v) : v \in U_\ast, \ |F_\Delta v - b_z^\Delta|_{H_2} \leq \varepsilon \}. \quad (5.2)$$

Note that the following inequalities are true:

$$|b_z - b_z^\Delta|_{H_2} \leq K \sqrt{\theta} \Delta,$$

$$|F_\Delta u - Fu|_{H_2} \leq \Delta KM \theta \quad \forall u \in U_\ast. \quad (5.3)$$

By virtue of (5.3) we obtain

$$J_0^0 \geq J_0^0 [\gamma^{1/2}] \geq J_z^\Delta [\gamma + \Delta k_\ast] \quad (\gamma > 0), \quad k_\ast = K(M \theta + \sqrt{\theta}),$$

hence

$$J_0^0 \geq J_0^0 [(\varepsilon - k_\ast \Delta)^{1/2}] \geq J_z^\Delta [\varepsilon] \quad (\varepsilon > k_\ast \Delta). \quad (5.4)$$

The next theorem provides an estimate for the difference between $J_0^0$ and $J_z^\Delta[\varepsilon]$ (recall that $J_0^0$, $J_z^\Delta$, $U_\ast$ and $F_\Delta$ are defined by (2.6), (5.2), (1.2) and (2.4)).

**Theorem 5.1** Let Condition 5.1 be fulfilled. Then for every $\varepsilon > \Delta k_\ast$ it holds that

$$k_0 \varepsilon^{1/2} \Delta^{-1/4} \leq J_z^\Delta [\varepsilon] - J_0^0 \leq 0, \quad (5.5)$$

where $k_0$ does not depend on $\varepsilon$ and $\Delta$ and can be computed explicitly.

**Proof.** The inequality

$$J_z^\Delta [\varepsilon] - J_0^0 \leq 0$$

follows from (5.4). Let us verify that

$$J_z^\Delta [\varepsilon] - J_0^0 \geq k_0 \varepsilon^{1/2} \Delta^{-1/4}.$$ 

Note that

$$\langle l, \ F_\Delta v - b_z^\Delta \rangle_{H_2} = \int_0^\phi \int_0^t \varphi_\Delta(t - \sigma) v(\sigma) \ d\sigma - b_z^\Delta(t) \ dt =$$

$$= \int_0^\phi \psi'_\phi(\sigma) v(\sigma) \ d\sigma - \int_0^\phi l'(t) b_z^\Delta(t) \ dt,$$

where

$$\psi'_\phi(\sigma) = \int_\sigma^\phi l'(t) \varphi_\Delta(t - \sigma) \ dt \equiv \int_0^\phi l'(t) \varphi_\Delta(t - \sigma) \ dt,$$

$$l(\cdot) = \{ l_1(\cdot), \ldots, l_m(\cdot) \} \in L_2(T; \mathbb{R}^m);$$
\( l_r \) is the \( r \)-th component of vector \( l \in \mathbb{R}^m \). Therefore inequality \(|F_\Delta v - b_\Delta^T|_{H_2} \leq \varepsilon \) can be rewritten into
\[
\sup_{|l|_{H_2} \leq 1} \left\{ \int_0^1 \psi'_i(\sigma)v(\sigma)\,d\sigma - \int_0^1 l'(\sigma)b_\Delta^T(\sigma)\,d\sigma \right\} \leq \varepsilon.
\]
(5.6)

Let \( \Phi \subset H_1 \) be the closure in \( H_1 \) of the linear hull of the set
\[
\{ \lambda(\sigma) \in \mathbb{R}^n, \sigma \in T : \exists t \in T \mid k \in [1 : m] \quad \lambda(\sigma) = \{ \varphi_\Delta(t - \sigma) \}_k \text{ PRI P. W. } \sigma \in [0, t] \} \subset H_1;
\]
here \( \{ \varphi_\Delta(t) \}_k \) is the \( k \)-th line of matrix \( \varphi_\Delta(t) \), \( k \in [1 : m] \). Represent element \( g \in H_1 \) in the form
\[
g = p_1 + p_2,
\]
where \( p_1 \in \Phi, \quad \langle p_2, \lambda \rangle_{H_1} = 0 \quad (\lambda \in \Phi), \text{ i.e. } p_2 \perp \Phi \). Then
\[
J(u) = \langle p_1, u_1 \rangle_{H_1} + \langle p_2, u_2 \rangle_{H_1}, \quad (u \in U_*),
\]
\[
u = u_1 + u_2, \quad u_1 \in \Phi, \quad u_2 \perp \Phi,
\]
\[
p_1(\sigma) = \sum_{j=1}^S \alpha_j \varphi^*(t_j - \sigma), \quad \sigma \in T, \quad \varphi^* = \varphi_{k_j}(t_j - \sigma) \in \Phi, \quad 0 \leq s \leq E(\Delta) \times m.
\]
Consequently
\[
\langle p_1, u_1 \rangle_{H_1} = \sum_{j=1}^S \alpha_j \{ h_\Delta(t_j) \}_{k_j} = c \quad (u_1 \in \Phi).
\]
Therefore
\[
J^0_z = c + \min \{ \langle p_2, u_2 \rangle_{H_1} : u = u_1 + u_2, \quad u_1 \in \Phi, \quad u_2 \perp \Phi, \quad |u_1|^2_{H_1} + |u_2|^2_{H_1} \leq M^2 \}.
\]

It is easily seen that the above minimum is reached at the element
\[
u^0 = u_1^0 + u_2^0,
\]
\[
u_1^0 = p_1 c|p_1|^{-2}_{H_1}, \quad u_2^0 = -p_2 (M^2 - c^2 |p_1|^{-2}_{H_1})^{1/2} |p_2|^{-1}.
\]
Then,
\[
J^0_z = c - (M^2 - c^2 |p_1|^{-2}_{H_1})^{1/2}.
\]
(5.7)

Consider \( J^z_{\varepsilon} \). We have
\[
J^z_{\varepsilon} = \min \{ \langle p_1, u_1 \rangle_{H_1} + \langle p_2, u_2 \rangle_{H_1} : u = u_1 + u_2 \in U_*, \quad u_1 \in \Phi, \quad u_2 \perp \Phi
\]
and inequality (5.6) is true }.

Let \( d \in (0, \Delta] \) and vector functions \( l_j(\cdot) \in H_2, \quad j \in [1 : s] \), be such that
\[
l_j(t)_{k_j} = \begin{cases} d^{-1}, & t \in \Delta_j \equiv [t_j, t_j + d], \\ 0, & t \in T \setminus \Delta_j, \end{cases}
\]
\[
l_j(t)_r = 0 \quad \forall t \in T, \quad r \neq k_j, \quad r \in [1 : E(\Delta)].
\]
\[
(5.8)
\]
Since \( d \in (0, \Delta] \), the following inequalities hold:

\[
\int_0^\varphi l_j'(t) \varphi'(t - \sigma)_{k_j} \, dt = \varphi'(t_j - \sigma)_{k_j},
\]

where

\[
\int_0^\varphi l_j'(t) b_j^\Delta(t) \, dt = b_j^\Delta(t_j)_{k_j}.
\]

Therefore, using (5.6), (5.8) and (5.9), we obtain

\[
|\langle p_1, u \rangle_{H_1} - \sum_{j=1}^S \alpha_j \{ b_j(t_j) \}_{k_j}| = |\langle p_1, u \rangle_{H_1} - c| \leq \mu, \quad \mu = k_1 \varepsilon d^{-1/2}.
\]

Hence

\[
J^\Delta_\varepsilon[\varepsilon] \geq c - \mu + J_\varepsilon[\varepsilon],
\]

where

\[
J_\varepsilon[\varepsilon] = \min \{ \langle p_2, u_2 \rangle_{H_1} : u = u_1 + u_2, \ u_1 \in \Phi, \ u_2 \perp \Phi, \ |u|_{H_1} \leq M, \ \langle p_1, u \rangle_{H_1} - c \leq \mu \}.
\]

Fix a \( u_1 = u_1^* \in \Phi \). We have

\[
\arg \min \{ \langle p_2, u_2 \rangle_{H_1} : u = u_1^* + u_2, \ u_2 \perp \Phi, \ |u_1^* + u_2|_{H_1} \leq M \} = -p_2(M^2 - |u_1^*|_{H_1}^2)^{1/2}/|p_2|_{H_1}^{-1},
\]

and

\[
\min \{ \langle p_2, u_2 \rangle_{H_1} : u = u_1^* + u_2, \ u_2 \perp \Phi, \ |u_1^* + u_2|_{H_1} \leq M \} = -(M^2 - |u_1^*|_{H_1}^2)^{1/2}/|p_2|_{H_1}.
\]

Besides,

\[
\arg \min \{ |u|_{H_1} : c - \mu \leq \langle p_1, u \rangle_{H_1} \leq c + \mu \} = \min \{ |c - \mu|, |c + \mu| \} p_1 \cdot |p_1|_{H_1}^{-2},
\]

\[
\mu = \mu(\varepsilon, d).
\]

From (5.11)–(5.14) follows that

\[
J_\varepsilon[\varepsilon] = -(M^2 - \min \{ |c - \mu|^2, |c + \mu|^2 \}|p_1|_{H_1}^{-2})^{1/2}.
\]

Hence by (5.7), (5.10) we deduce

\[
J^\Delta_\varepsilon[\varepsilon] - J^0_\varepsilon \geq \mu + (M^2 - \varepsilon^2|p_1|_{H_1}^{-2})^{1/2} - \frac{1}{2}
\]

\[
-((M^2 - \varepsilon^2|p_1|_{H_1}^{-2}) + (2|c|\mu - \mu^2)|p_1|_{H_1}^{-2})^{1/2} \geq k_2\varepsilon^{1/2} d^{-1/4}.
\]

Now inequality (5.5) follows with \( d = \Delta \). The theorem is proved.

Theorem 3.1, 5.1 yield the following.
Theorem 5.2 Let Condition 5.1 be fulfilled and \( \delta \)-trajectory \( y_{\delta,h}^{0}(i)/ (i\delta) \) be defined by \((3.4), (3.5).\) Then
\[
k_{0}\{ (2K_{J}\alpha/i\delta) + g_{i}(\delta,h)/(i\delta)) \}^{1/2} \Delta^{-1/2} + k_{*}\Delta^{1/2} \leq \]
\[
 J(y_{\delta,h}^{0}(i)/(i\delta) - J_{z}^{0} \leq g_{i}(\delta,h)/\alpha. 
\]

Proof. By virtue of \((3.7),\) and taking into account the definition of set \( U_{z}[\gamma, \beta], \) we conclude that
\[
|F(y_{\delta,h}^{0}(i)/(i\delta)) - b_{z}|_{H_{2}} \leq \{2K_{J}\alpha/(i\delta) + g_{i}(\delta,h)/(i\delta)) \}^{1/2},
\]
\[
 J(y_{\delta,h}^{0}(i)/(i\delta)) \leq J_{z}^{0} + g_{i}(\delta,h)/\alpha.
\]

Hence and from \((5.3)\) we deduce
\[
|F_{\Delta}(y_{\delta,h}^{0}(i)/(i\delta)) - b_{z}^{\Delta}|_{H_{2}} \leq \{2K_{J}\alpha/(i\delta) + g_{i}(\delta,h)/(i\delta)) \}^{1/2} + k_{*}\Delta.
\]

Consequently (see the rule of definition \( J_{z}^{\Delta}[\varepsilon] \) \((5.2))
\[
 J(y_{\delta,h}^{0}(i)/(i\delta)) \leq J_{z}^{\Delta} \{[2K_{J}\alpha/(i\delta) + g_{i}(\delta,h)/(i\delta)) \}^{1/2} + k_{*}\Delta\].
\]

The inequality \((5.15)\) follows from \((5.16)\) and Theorem 5.1. The theorem is proved.

The following theorem is stated similarly.

Theorem 5.3 Let Condition 5.1 be fulfilled and \( \delta \)-trajectory \( y_{\delta,h}(i) \) be generated by mappings \((4.4), (4.5).\) Then the inequality
\[
k_{0}\{ (g_{i}(\delta,h)/(i\delta))^{1/2} \Delta^{-1/2} + k_{*}\Delta^{1/2} \}^{1/2} \leq J(y_{\delta,h}(i)/(i\delta) - J_{z}^{0} \leq 0 \]
is true.

The research was supported partially by International Science and Technology Center (Project 008-94) and the Fund for Fundamental Research of the Russian Academy of Sciences (Grant 96-01-00846).

References


Lasiecka, I. 1978, Boundary control of parabolic systems: regularity of optimal solutions. Applied Mathem. and Optim., V. 4, No 4, p. 301–328


Lions, J. L. 1971, Optimal control of systems governed by partial differential equations. Springer, NY


Maksimov, V. I. 1993, Numerical solution for any inverse problem of heat equation, Avtomatika i telemekhanika, No 2, p. 83–92 (Russian)

Marchuk, G. I. 1982, Mathematical modelling in environmental problems, Moscow, Nauka, (Russian)


Tikhonov, A. N. and Arsenin, V. Ya. 1979, Solution methods for ill-posed problems, Moscow, Nauka, (Russian)

Vasilyev, F. P. 1981, Solution methods for extremal problems, Moscow, Nauka, (Russian)