OPTIMAL MIGRATION POLICIES

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Preface

Interest in human settlement systems and policies has been a critical part of urban-related work at IIASA since its inception. Recently this interest has given rise to a concentrated research effort focusing on migration dynamics and settlement patterns. Four sub-tasks form the core of this research effort:

I.   the study of spatial population dynamics;

II.   the definition and elaboration of a new research area called demometrics and its application to migration analysis and spatial population forecasting;

III. the analysis and design of migration and settlement policy;

IV.   a comparative study of national migration and settlement patterns and policies.

This paper, the third in the policy analysis series, develops a paradigm for a formal theory of normative demography, drawing on related work in economics and in optimal control theory. It adds a goals-means dimension to our current efforts in demographic and demometric modelling and shows how a number of apparently diverse aspects of population distribution policy may be considered within a single overall analytical framework.

Willekens' study was conducted here at IIASA this past year and forms part of a doctoral dissertation submitted to Northwestern University. This work was financially supported by the Institute by means of a research fellowship.

Related papers in the policy analysis series and other publications of the migration and settlement study are listed on the back page of this report.

A. Rogers
June 1976
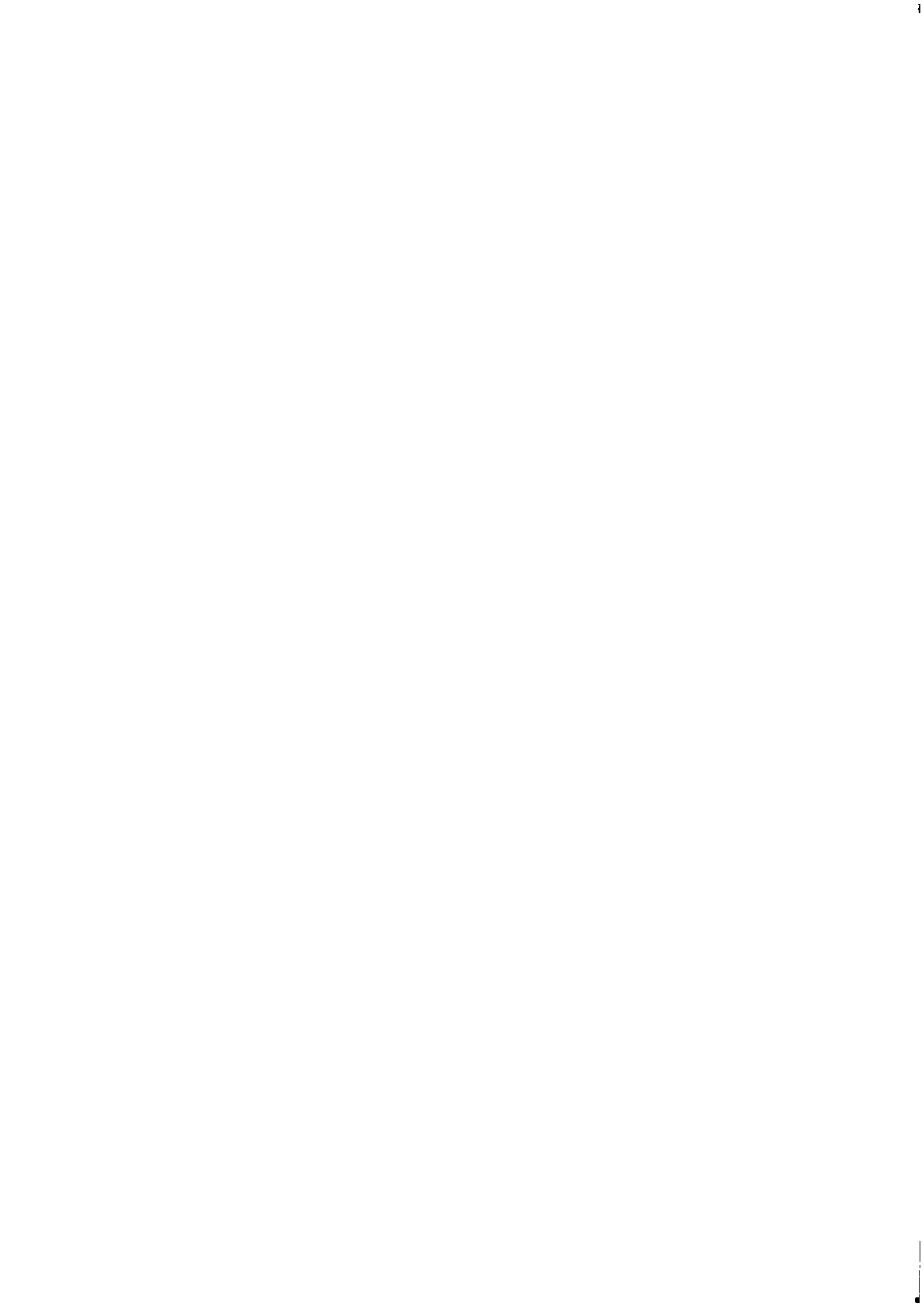
Abstract

      This paper explores the analytical features of a population distribution or human settlement policy. It examines linear static and dynamic policy models in the Tinbergen formulation and in the state-space format and shows how they may be derived from demographic and demometric models by adding a new dimension: the goals-means relationship of population distribution policy. Although our general treatment encompasses most policy models, attention is focused on models for which solutions may be expressed analytically, such as the initial period control problem and the linear-quadratic control problem.

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Foreword

In recent years there has been an increasing interest in the dynamics of spatial demographic growth. Models for multiregional population growth have been developed to describe the growth process and to analyze its impact on future population characteristics (Rogers, 1975). The various economic, social, climatological and cultural forces influencing spatial population growth have been brought together in explanatory demometric models (Greenwood, 1975a). The mathematical demographic models and the demometric models have a common feature. They are designed to describe and to explain the dynamics of the spatial population growth.

Once the dynamics of a phenomena are understood, human nature comes up with the ultimate question: can we control it and how? The models associated with this third concern are population policy models. The subject of migration policy models has been treated by Rogers (1966; 1968, Chapter 6; 1971, pp. 98-108), and more recently, MacKinnon (1975a, 1975b) devotes considerable attention to the design of optimal-seeking migration policy models.

This paper is devoted to a methodological analysis of migration policy models. We assume that a demometric or a demographic model, consisting of a system of linear simultaneous equations, has been successfully specified and estimated. Therefore, we do not devote any attention, for example, to identification and estimation procedures. The main thread of the analysis is provided by the Tinbergen paradigm, to which we will refer frequently. Chapter 1 is a conceptual survey of various possible policy models.

Each model is related back to the original Tinbergen framework. The matrix of impact multipliers, well known in economic analysis, is seen to be of crucial importance to the classification scheme. After the introductory chapter has set the scene, we devote our attention to the two central issues in the theory of policy: the concepts of existence and of design. The existence problem deals with the question whether the system is controllable, i.e., whether a set of arbitrary targets can be achieved at all, given the internal dynamics of the system and given the set of available instruments. The answer to the controllability problem provides input information for the design problem. For the design of an optimal policy, the policy maker may apply a wide range of mathematical programming techniques, assuming that he has a clear idea of his preferences. To facilitate the discussion of the controllability of dynamic systems in Chapter 2 and of the design of optimal policies in Chapter 3, we introduce in Chapter 2 the state-space representation of demometric models.

CHAPTER 1

OPTIMAL MIGRATION POLICIES:

A CONCEPTUAL FRAMEWORK

There are several analytical differences between a policy model and a conventional demographic or demometric model. The most basic classification of variables in any model consists of two categories: endogenous variables, which are determined within the model, and exogenous variables, which are predetermined. Suppose the population system is linear and may be modeled as

$$A\{y\} = E\{z\} \tag{1.1}$$

where $\{y\}$ is a M x 1 vector of endogenous variables,

$\{z\}$ is a L x 1 vector of exogenous variables,

A is a M x M matrix of coefficients,

E is a M x L matrix of coefficients.

Equation (1.1) is the reduced form of a population model. The endogenous and the exogenous variables are separated. Assuming that A is nonsingular, we obtain

$$\{y\} = A^{-1}E\{z\} = C\{z\} \tag{1.2}$$

where C is the matrix of multipliers, i.e. the reduced form matrix. The elements of C represent the impact on $\{y\}$ of a unit change in $\{z\}$.

The policy models treated here, will be discussed with reference to (1.2). Tinbergen (1963) proposed a classi-fication of the variables of (1.2) better suited for the

policy problem. His ideas are general enough to encompass the whole range of policy models. Starting from the Tinbergen paradigm, we try to present a unified treatment of various classes of models, which are relevant for population policy.

1.1. THE TINBERGEN PARADIGM

Tinbergen (1963) distinguished two categories of variables in both the endogenous and the exogenous variables. The endogenous variables consist of target variables, which are of direct interest for policy purposes, and other variables which are not. The latter are labeled by Tinbergen as irrelevant variables. However, they may be of indirect interest for policy planning, since their values may in turn influence various target variables. The exogenous variables are divided according to their controllability. Instrument variables are subject to direct control by the policy authorities. Data variables are beyond their control. The latter include exogenously predetermined and uncontrollable variables, as well as lagged endogenous variables. They define the environment in which the levels of instrument variables have to be set. Applying this approach, equation (1.2) may be partitioned to give

$$
\begin{bmatrix} \{y_1\} \\ \{y_2\} \end{bmatrix} = \begin{bmatrix} R & S \\ P & Q \end{bmatrix} \begin{bmatrix} \{z_1\} \\ \{z_2\} \end{bmatrix}
$$

where $\{y_1\}$ is the N x 1 vector of target variables,

$\{y_2\}$ is the (M - N) x 1 vector of other endogenous variables,

$\{z_1\}$ is the K x 1 vector of instrument variables,

$\{z_2\}$ is the (L - K) x 1 vector of uncontrollable

exogenous variables and lagged endogenous
variables,

R, S, P, Q are conformable partitions of the model's
reduced form matrix.

The value of the target vector is

$$\{y_1\} = R\{z_1\} + S\{z_2\} \quad . \tag{1.3}$$

The policy problem, as formulated by Tinbergen, is to
choose an appropriate value of the instrument vector $\{z_1\}$
so as to render the value of the target vector $\{y_1\}$ equal to
some previously established desired value $\{\bar{y}_1\}$. The choice
of the level of the instrument variables depends on the
levels of the uncontrollable variables, represented by
$\{z_2\}$, and on how much they affect the targets.

It is important to keep in mind that the policy model
(1.3) is derived from the explanatory model (1.2) by adding
a new dimension to (1.2). This new dimension is the goals-
means relationship of population policy. The explanatory
model may be a pure demographic model, relating population
growth and distribution to demographic factors such as fertil-
ity, mortality and migration. It may also be a demometric
model, which statistically relates spatial population growth
to socio-economic variables. Any model may be converted into
a policy model if and only if all the target variables of the
policy model are part of the set of endogenous variables of
the explanatory model and if at least one of the exogenous
variables is controllable. Most migration models found in
the literature are single-equation models with gross or net

migration as the dependent variable. They serve only a
restricted category of policy models, namely those with
targets that consist of migration levels and instruments
which are socio-economic in nature. Various regional
economic models include migration as an exogenous variable.
Therefore, they are not suited to become migration policy
models if population distribution is the goal. Simultaneous
equation models, such as the ones developed by Greenwood
(1973, 1975b) and Olvey (1972), are relevant to model popu-
lation policy problems of all types, because they include
demographic and socio-economic variables in both the set
of endogenous and the set of exogenous variables. Thus they may
be applied in situations where the goals-means relationship
consists of demographic, as well as of socio-economic measures.
Finally, the multiregional population growth models of Rogers
(1975) may be converted to policy models to study purely
demographic policy problems, i.e., both targets and instruments
are demographic in nature.

Before going into greater detail in our exposition,
we would like to stress that the analytical solution of
Tinbergen's formulation of the policy problem is restricted
to linear policy models. If the model is nonlinear, one can
only solve it numerically. The latter approach is denoted
by Naylor (1970; p. 263) as the simulation approach, and has
been applied extensively by Fromm and Taubman (1968). In
this part, we only deal with linear models and do not discuss
the simulation approach.

1.2. SURVEY OF POLICY MODELS

Conceptually, any policy model may be related to (1.3). For convenience, we drop the subscript of the target vector.

$$\{\underset{\sim}{y}\} = \underset{\sim}{R}\{\underset{\sim}{z}_1\} + \underset{\sim}{S}\{\underset{\sim}{z}_2\} \quad . \tag{1.3}$$

Throughout our discussion of policy models, it will be assumed that both the targets and the instruments are linearly independent. The matrix $\underset{\sim}{R}$ then plays a crucial role in policy analysis. The existence of an optimal policy, i.e., a solution to (1.3), depends on the rank of $\underset{\sim}{R}$. The design of an optimal policy, i.e., the assignment of values to the instrument variables, depends on the structure of $\underset{\sim}{R}$, and on the values of its entries. The matrix $\underset{\sim}{R}$ is known in the economic literature as the matrix of impact multipliers. The name refers to the fact that an element r_{ij} gives the change in the value of the target variable i when the instrument variable j is varied by one unit. The ratio $-r_{ij}/r_{ik}$ is the amount by which the j-th instrument may be cut down without changing the level of the i-th target, if the value of the k-th instrument is increased with one unit. It is, therefore, the marginal rate of substitution between the two instruments (Fromm and Taubman, 1968; p. 109).

It is the purpose of this section to classify relevant policy models without going into technical detail. Detailed treatment will be given later. The survey revolves around the matrix multiplier $\underset{\sim}{R}$ and its characteristics. A first

classification scheme is based on the rank of $\underset{\sim}{R}$, or alternatively on the relation between the number of targets and the number of instruments. A second classification scheme relates to the structure of $\underset{\sim}{R}$. The structure of $\underset{\sim}{R}$ also provides us with a link between the reduced form models and the models of optimal control.

1.2.1. Classification of Policy Models According to the Rank of the Matrix Multiplier

We may distinguish between three categories of policy models: $\underset{\sim}{R}$ is nonsingular and of rank N; $\underset{\sim}{R}$ is singular and of rank K; $\underset{\sim}{R}$ is singular and of rank N. The parameters N and K are, respectively, the number of instruments and the number of targets. An illustration is given by a typical policy model, namely the Theil (1964) model.

a. The matrix multiplier is nonsingular and of rank N.

If $\underset{\sim}{R}$ is nonsingular, i.e., there are as many instruments as there are targets, then there exists a unique combination of instruments leading to the set of desired targets. Once the targets are specified, the unique instrument vector is given by

$$\{\underset{\sim}{z}_1\} = \underset{\sim}{R}^{-1}[\{\underset{\sim}{y}\} - \underset{\sim}{S}\{\underset{\sim}{z}_2\}] \quad . \tag{1.4}$$

The solution to (1.3) is unique, and there is no need for the policy maker to provide any other information than the set of target values.

b. The matrix multiplier is singular and of

rank $K < N$.

If the number of instruments is less than the number of targets, however, the system (1.3) is inconsistent and there is no way that all the target values can be reached. This poses an additional decision problem for the policy maker. Does he give up some targets in order to reach others, or does he want to achieve all the targets as closely as possible with the limited resources? In the latter case, the policy maker may also wish to weight the targets differently. If the first alternative is chosen, some targets are deleted, and the instrument vector is given by (1.4). The second alternative often leads to the formulation of a quadratic programming model. If $\{\bar{y}\}$ is the vector of desired target values, and $\{\hat{y}\}$ is the vector of realized values, then the problem is to minimize the squared deviation between $\{\bar{y}\}$ and $\{\hat{y}\}$ subject to (1.3), which describes the behavior of the population system. That is,

$$\min \ [\{\bar{y}\} - \{\hat{y}\}]' \ A[\{\bar{y}\} - \{\hat{y}\}] \tag{1.5}$$

subject to

$$\{\hat{y}\} = R\{z_1\} + S\{z_2\} \ . \tag{1.6}$$

The weight matrix A represents the policy maker's differential preferences towards the targets. The target variables with the highest weights will be forced very close to their desired values. Those with the lowest weights will not.

c. The matrix multiplier is singular and of rank N.

If the number of instrument variables exceeds the number of targets, then there is an infinite number of solutions to (1.3) and, therefore, an infinite number of instrument vectors. To get a unique solution, the policy maker may force the number of instruments to be equal to the number of targets, by deleting some instruments. On the other hand, he may put some constraints on the instruments. There is a wide variety of possible constraints, but we consider only two categories.

c.1. Some Instruments are Linearly Dependent.

By making some instruments linearly dependent, the freedom of policy action is reduced in a way such that only one strategy is available to achieve the targets. An illustration of this constraint is the intervention model of Rogers (1971; pp. 99-101). Targets are specified only for the planning horizon, but instruments are available in each time period. In order to get a unique policy, the constraint is introduced that the values of the instruments in all the time periods are linearly related to each other.

c.2. Introduction of Acceptable Values of the Instruments.

In most cases, the policy maker has a good idea of what levels of the instrument variables are acceptable politically. Minimizing the squared deviations between the realized and the most acceptable values assures a unique instrument vector.

d. Illustration: the Theil quadratic programming

model.

We have described how policy models are related to the

rank of the matrix of impact multipliers or, equivalently,

to the number of targets and instruments. Only some alter-

native policy models have been indicated. A wider variety

is possible. For example, the targets and the instruments

may be constrained at the same time, and these constraints

need not to be linear. The objective function (1.5) may

not be quadratic, and (1.6) can be supplemented with both

equality and inequality constraints. The reader is referred

to the mathematical programming literature for such illustra-

tions. The quadratic objective function with linear constraints,

however, is common in economic policy analysis. It is

based on two assumptions. The first is that the policy

maker's preferences are quadratic in targets and controls.

The second assumption is that each of the targets depends

linearly on all the instruments, the coefficients of these

linear relations being fixed and known. The basic structure

of this linear quadratic model is due to Theil (1964;

pp. 34-35), and may be expressed as

$$\min W = \{a\}' \{z_1\} + \{b\}' \{\hat{y}\} + \frac{1}{2}\Big[\{z_1\}' A\{z_1\}$$

$$+ \{\hat{y}\}' Q\{\hat{y}\} + \{z_1\}' C\{\hat{y}\} + \{\hat{y}\} C'\{z_1\}\Big] \tag{1.7}$$

subject to

$$\{\hat{y}\} = R\{z_1\} + S\{z_2\} \tag{1.3}$$

where $\{\hat{y}\}$ is the vector of realized values of the target

variables,

$\{z_1\}$ is the vector of instrument variables,

$\{z_2\}$ is the vector of exogenous variables,

A, Q, C are weight matrices,

R, S are matrices of multipliers.

Applications of the Theil model in economic policy
literature may be found in Fox, Sengupta and Thorbecke (1972;
p. 215), and in Friedman (1975; pp. 158-160). To simplify
matters we may suppose that $\{a\} = \{b\} = \{0\}$ and $C = 0$. The
problem then reduces to

$$\min \tfrac{1}{2}\left[\{\hat{y}\}' \ Q\{\hat{y}\} + \{z_1\}' \ A\{z_1\} \right] \tag{1.8}$$

subject to

$$\{\hat{y}\} = R\{z_1\} + S\{z_2\} \quad , \quad \text{where} \tag{1.3}$$

Q and A are weights attached to the target vector and to
the instrument vector respectively.

To illustrate the application of the Theil model in
migration policy analysis, consider the following problem.
The costs of public services are held to be too high
because some regions are over-urbanized and are subject to
diseconomies of scale, while other areas have insufficient people
to reach the threshold needed for an efficient public service
system. The high costs in the public sector can, therefore,
be related to the inefficient population distribution. To
reduce the costs, a migration policy is needed. However,

there is a cost associated with the redistribution of people over space. Assume that the cost function of public services is a quadratic function of the population distribution $\{\hat{\underset{\sim}{y}}\}$, i.e.

$$C_p = \{\underset{\sim}{b}\}' \{\hat{\underset{\sim}{y}}\} + \{\hat{\underset{\sim}{y}}\}' \underset{\sim}{E}\{\hat{\underset{\sim}{y}}\} \quad . \tag{1.9}$$

Assume also that the cost associated with population distribution is quadratic in the vector of the number of people relocated by the policy program, $\{\underset{\sim}{z}_1\}$, i.e.

$$C_m = \{\underset{\sim}{z}_1\}' \underset{\sim}{F}\{\underset{\sim}{z}_1\} \quad . \tag{1.10}$$

An element z_{1i} of $\{\underset{\sim}{z}_1\}$ is positive if the program attracts people to region i. It is negative if the program has an out-migration effect. On comparing the cost functions with the preference function (1.7), we see that

$$\{\underset{\sim}{a}\} = \{\underset{\sim}{0}\} \quad , \quad \underset{\sim}{C} = \underset{\sim}{0} \quad , \quad \underset{\sim}{Q} = 2\underset{\sim}{E}$$

and

$$\underset{\sim}{A} = 2\underset{\sim}{F} \quad .$$

Since $\{\underset{\sim}{z}_1\}$ represents the additional migration, $\underset{\sim}{R} = \underset{\sim}{I}$ in the constraint. The vector of uncontrollable variables is the population distribution in the previous time period, and $\underset{\sim}{S}$ is the multiregional population growth matrix.

1.2.2. <u>Classification of Policy Models According to the</u> <u>Structure of the Matrix Multiplier</u>

We now turn to the question of how policy models may be related to the structure of the matrix R. The structure determines the nature of the dependence of $\{z_1\}$ upon $\{y\}$. Several assumptions may be adopted to simplify the form of R. They have been studied by Tinbergen (1963, Chapter 4), by Fox, Sengupta and Thorbecke (1972; pp. 24-25) and by Friedman (1975; pp. 149-153) among others. We consider four different structures of R: diagonal, triangular, block-diagonal and block-triangular. Our illustration considers the block-triangular multiperiod policy model.

a. The matrix multiplier is diagonal.

If R is diagonal, then each target variable can be associated with one and only one instrument variable and vice versa. Since R^{-1} is also diagonal, equation (1.4) implies a series of expressions

$$\bar{z}_{1i} = \frac{1}{r_{ii}} \left[\bar{y}_i - \sum_k s_{ik} z_{2k} \right] \qquad i = 1,\ldots,N \quad ,$$

each of which may be solved independently. The practical implication of this is that the policy maker can, in such an instance, pursue each target with a single specific instrument, and no coordination between the various policies is required.

b. The matrix multiplier is triangular.

Equation (1.3) is recursive. The two-way simultaneity between the vectors $\{y\}$ and $\{z_1\}$, i.e., $\{z_1\}$ affecting $\{y\}$ and $\{y\}$ affecting $\{z_1\}$, can be reduced to a unilateral dependence or a unidirectional causality. Suppose R is lower triangular, then R^{-1} is also lower triangular, and the decision making procedure is recursive:

$$\bar{z}_{11} = \frac{1}{r_{11}} \left[\bar{y}_1 - \sum_k s_{1k} z_{2k} \right]$$

$$\bar{z}_{12} = \frac{1}{r_{22}} \left[\bar{y}_2 - \sum_k s_{2k} z_{2k} - r_{21} \bar{z}_{11} \right]$$

$$\vdots$$

$$\bar{z}_{1i} = \frac{1}{r_{ii}} \left[\bar{y}_i - \sum_k s_{ik} z_{2k} - \sum_{j=1}^{i-1} r_{ij} \bar{z}_{1j} \right] \quad .$$

These expressions may be solved in sequence, and the model has a simple policy interpretation. If each equation were assigned to a different policy maker, the system of equations would specify a hierarchy. In order to make an optimal decision, each policy maker would not need to look at the instruments selected by those who were below his position in the hierarchy.

c. The matrix multiplier is block-diagonal.

In the case of a block-diagonal policy model, the overall model can be decomposed into several independent parts. This would occur if a policy can be decentralized into independent subpolicies, each having a goals-means relationship unrelated to the goals and the instruments of the other subpolicies. This would permit efficient decentralized decision making.

d. The matrix multiplier is block-triangular.

Here, as in the case of a triangular $\underset{\sim}{R}$, the set of instruments corresponding to any given block can be solved for without any knowledge of the instruments belonging to blocks which are lower in the hierarchy. The overall policy could be decomposed into a hierarchical system of policies.

e. Illustration: the multiperiod policy problem.

An important application of the block-triangular form of $\underset{\sim}{R}$ is found in dynamic policy analysis. The models presented thus far have been static, but they are general enough to handle dynamic policy problems as well. If the entries of the target vector and of the instrument vector belong to different time periods, we clearly have a dynamic or multiperiod policy model. Suppose, for example, that a target vector is given for a sequence of time periods from 1 to T, say. Then $\{\underset{\sim}{y}\}$ is itself composed of vectors, one for each time period. Suppose, moreover, that there exists an instrument vector for each time period. The reduced form model (1.3) now may be expressed as

$$\{\underset{\sim}{y}\} = \underset{\sim}{R}\{\underset{\sim}{z}_1\} + \underset{\sim}{S}\{\underset{\sim}{z}_2\} \qquad (1.11)$$

where

$$\{\underset{\sim}{y}\} = \begin{bmatrix} \{\underset{\sim}{y}^{(1)}\} \\ \{\underset{\sim}{y}^{(2)}\} \\ \vdots \\ \{\underset{\sim}{y}^{(t)}\} \\ \vdots \\ \{\underset{\sim}{y}^{(T)}\} \end{bmatrix} \qquad \{\underset{\sim}{z}_1\} = \begin{bmatrix} \{\underset{\sim}{z}_1^{(1)}\} \\ \{\underset{\sim}{z}_1^{(2)}\} \\ \vdots \\ \{\underset{\sim}{z}_1^{(t)}\} \\ \vdots \\ \{\underset{\sim}{z}_1^{(T)}\} \end{bmatrix} \qquad \{\underset{\sim}{z}_2\} = \begin{bmatrix} \{\underset{\sim}{z}_2^{(1)}\} \\ \{\underset{\sim}{z}_2^{(2)}\} \\ \vdots \\ \{\underset{\sim}{z}_2^{(t)}\} \\ \vdots \\ \{\underset{\sim}{z}_2^{(T)}\} \end{bmatrix}$$

$$\underset{\sim}{R} = \begin{bmatrix} \underset{\sim}{R}_{11} & \underset{\sim}{R}_{12} & \cdots & \underset{\sim}{R}_{1T} \\ \underset{\sim}{R}_{21} & \underset{\sim}{R}_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ \underset{\sim}{R}_{T1} & \cdots\cdots\cdots & & \underset{\sim}{R}_{TT} \end{bmatrix}$$

$$\underset{\sim}{S} = \begin{bmatrix} \underset{\sim}{S}_{11} & \underset{\sim}{S}_{12} & \cdots & \underset{\sim}{S}_{1T} \\ \underset{\sim}{S}_{21} & \underset{\sim}{S}_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ \underset{\sim}{S}_{T1} & \cdots\cdots\cdots & & \underset{\sim}{S}_{TT} \end{bmatrix}$$

Vector $\{z_1\}$ is of order KT, and $\{z_2\}$ and $\{y\}$ are of order NT. The submatrix R_{ij} is N x K and its elements are dynamic policy multipliers which express the impact on the target vector $\{y^{(t)}\}$ in time period $t = i$ of changes in the instrument vector $\{z_1^{(t)}\}$ in time period $t = j$. $\underset{\sim}{R}$ is NT x KT; $\underset{\sim}{S}$ is NT x NT and the submatrices S_{ij} are of order N x N. $\underset{\sim}{S}$ shows the dynamic effects of predetermined variables on the target variables.

Most policy models assume that policy actions do not influence events which precede them in time and, therefore, generally ignore expectational effects or advance announcement effects. This assumption of unidirectional causality yields a block-triangular $\underset{\sim}{R}$ matrix:

$$\underset{\sim}{R} = \begin{bmatrix} \underset{\sim}{R}_0 & 0 & 0 & \cdots & 0 \\ \underset{\sim}{R}_1 & \underset{\sim}{R}_0 & 0 & & \vdots \\ \underset{\sim}{R}_2 & \underset{\sim}{R}_1 & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ \underset{\sim}{R}_{T-1} & \underset{\sim}{R}_{T-2} & & & \underset{\sim}{R}_0 \end{bmatrix} \qquad (1.12)$$

where the elements of $\underset{\sim}{R}_i$ are dynamic policy multipliers. A triangular $\underset{\sim}{R}$ matrix leads to a sequential decision making procedure analogous to that of the static model. The key distinction is that here the sequence is across time, rather than across individual instrument and target variables.

By way of illustration, consider the application of the Theil model in population policy. Assume that there is a time sequence of target population distributions, and a time sequence of vectors of induced migration. Suppose that no tough policy actions are expected by the potential migrants, therefore the population distribution at time t does not depend on the migration policies beyond t. Equation (1.11) may, therefore, be written with $\underset{\sim}{R}$ being lower block-triangular.

We may reduce the form of this policy model even further. Suppose that the migration policy at time t only affects the population distribution at t + 1 directly. The impact on the population distributions at a later time is indirect in the sense that the population distribution at t + 1 affects the distribution beyond t + 1. This implies the recurrence equation

$$\{\underset{\sim}{y}(t+1)\} = \underset{\sim}{R}_0\{\underset{\sim}{z}_1(t)\} + \underset{\sim}{S}_{t+1,t}\{\underset{\sim}{y}(t)\} \quad . \tag{1.13}$$

The submatrix $\underset{\sim}{S}_{t+1,t}$ is the growth matrix of the population between t and t + 1. If we assume the growth matrix to be time-independent, i.e. $\underset{\sim}{G} = \underset{\sim}{S}_{t+1,t}$ for all t, we may write

$$\{\underset{\sim}{y}(t+1)\} = \underset{\sim}{R}_0\{\underset{\sim}{z}_1(t)\} + \underset{\sim}{G}\{\underset{\sim}{y}(t)\} \quad . \tag{1.14}$$

Therefore, (1.11) may be reduced to a set of recurrence equations

$$\{\underset{\sim}{y}(1)\} = \underset{\sim}{R}_0\{\underset{\sim}{z}_1(0)\} + \underset{\sim}{G}\{\underset{\sim}{y}(0)\}$$

$$\{\underset{\sim}{y}(2)\} = \underset{\sim}{R}_0\{\underset{\sim}{z}_1(1)\} + \underset{\sim}{G}\{\underset{\sim}{y}(1)\}$$

$$= \underset{\sim}{R}_0\{\underset{\sim}{z}_1(1)\} + \underset{\sim}{G}\underset{\sim}{R}_0\{\underset{\sim}{z}_1(0)\} + \underset{\sim}{G}^2\{\underset{\sim}{y}(0)\}$$

$$\vdots$$

$$\{\underset{\sim}{y}(t)\} = \underset{\sim}{R}_0\{\underset{\sim}{z}_1(t-1)\} + \underset{\sim}{G}\{\underset{\sim}{y}(t-1)\}$$

$$= \underset{\sim}{G}^t\{\underset{\sim}{y}(0)\} + \sum_{i=0}^{t-1}\underset{\sim}{G}^{t-1-i}\underset{\sim}{R}_0\{\underset{\sim}{z}_1(i)\} \quad .$$

In matrix form, we have that

$$
\begin{bmatrix} \{\underset{\sim}{y}(0)\} \\ \{\underset{\sim}{y}(1)\} \\ \{\underset{\sim}{y}(2)\} \\ \vdots \\ \{\underset{\sim}{y}(T)\} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{I} & & & & \\ \underset{\sim}{G} & \underset{\sim}{R}_0 & & & \\ \underset{\sim}{G}^2 & \underset{\sim}{G}\underset{\sim}{R}_0 & \underset{\sim}{R}_0 & & \\ \vdots & \vdots & \vdots & \ddots & \\ \underset{\sim}{G}^T & \underset{\sim}{G}^{T-1}\underset{\sim}{R}_0 & \underset{\sim}{G}^{T-2}\underset{\sim}{R}_0 & \cdots & \underset{\sim}{R}_0 \end{bmatrix} \begin{bmatrix} \{\underset{\sim}{y}(0)\} \\ \{\underset{\sim}{z}_1(0)\} \\ \{\underset{\sim}{z}_1(1)\} \\ \vdots \\ \{\underset{\sim}{z}_1(T-1)\} \end{bmatrix}
$$

(1.15)

Equation (1.14) has received much attention in system theory. It is called the discrete state equation and forms the central component of the discrete version of the state-space model. Stimulated by recent work in system theory and optimal control, an increasing number of authors have adopted the state-space approach to describe dynamic models in the social sciences[1]. We have shown how the

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[1]See, for example, Pindyck (1973), Kenkel (1974) and Chow (1975).

state-space model may be derived conceptually from the
reduced form model. How the transformation is done mathe-
matically will be shown later.

By introducing the assumption of unidirectional causal-
ity of the population system, we may write the Theil model
(1.8), (1.3) as

$$\min\ J = \frac{1}{2}\Big[\{\hat{y}\}'\ Q\{\hat{y}\} + \{z_1\}'\ A\{z_1\}\Big] \tag{1.8}$$

subject to

$$\{y(t + 1)\} = R_0\{z_1(t)\} + G\{y(t)\} \quad . \tag{1.16}$$

Recall that Q is a NT x NT matrix, where T is the planning
horizon, N is the number of target variables at each
period, and A is a KT x KT matrix, where K is the number
of instrument variables.

1.2.3. From the Tinbergen Model to the Optimal Control Model

In this chapter, we started out with the Tinbergen
paradigm. The original model, based on this paradigm, was
simple in nature, in the sense that the number of instruments
was equal to the number of targets and that the optimal policy
was the unique solution to a system of linear equations. When
the number of instruments and targets differs, the policy maker
is confronted with an additional decision problem. He needs to
specify his preferences in order to get a unique policy which
is optimal. This led us to the Theil model and to the broad
application of mathematical programming. When policy problems

become large, there is a need for simplification. The

simplifying assumptions, discussed up to now, are related

to the structure of the multiplier matrix $\underset{\sim}{R}$. The assumption

of unidirectional causality of the system, represented by

the block-triangularity of $\underset{\sim}{R}$ is crucial to the further

development of dynamic policy models. Now we extend the

simplification of the policy models to the objective function.

Assume that the effect of the target and control vector

at time t on the value of the objective function, is inde-

pendent of the target and control vectors at other time

periods. This implies that the matrices $\underset{\sim}{Q}$ and $\underset{\sim}{A}$ are block-

diagonal. The large multiperiod problem may then be decom-

posed into a sequence of smaller single-period problems.

The objective function becomes

$$\min \frac{1}{2} \sum_{t=0}^{T} \{\hat{\underset{\sim}{y}}(t)\}' \underset{\sim}{Q}(t) \{\hat{\underset{\sim}{y}}(t)\} + \{\underset{\sim}{z}_1(t)\}' \underset{\sim}{A}(t) \{\underset{\sim}{z}_1(t)\} \quad .$$

(1.17)

It is assumed that $\{\underset{\sim}{z}_1(T)\} = \{\underset{\sim}{0}\}$; therefore, we have

$$\min \frac{1}{2} \{\hat{\underset{\sim}{y}}(T)\}' \underset{\sim}{Q}(T) \{\hat{\underset{\sim}{y}}(T)\}$$

$$+ \frac{1}{2} \sum_{t=0}^{T-1} \left[\{\hat{\underset{\sim}{y}}(t)\}' \underset{\sim}{Q}(t) \{\hat{\underset{\sim}{y}}(t)\} + \{\underset{\sim}{z}_1(t)\}' \underset{\sim}{A}(t) \{\underset{\sim}{z}_1(t)\}\right] .$$

(1.18)

The block-diagonal structure of $\underset{\sim}{Q}$ and $\underset{\sim}{A}$ implies that the

values of the target variables at time t are independent

of their values at previous and at later time periods.

This is denoted as the assumption of inter-temporal separa-

bility of the objectives. The combination of (1.18) with

(1.16) is known as the linear-quadratic control problem,

which has received much attention in control theory. <u>This</u>

shows that the multiperiod Theil problem may be reduced to a linear-quadratic control problem by assuming inter-temporal separability of the objective and unidirectional causality of the population system. If these conditions are not met, one must apply the dynamic generalization of the Theil model (Theil, 1964, Chapter 4).

In control theory, it is common to denote the target vector $\{\underset{\sim}{y}(t)\}$ by $\{\underset{\sim}{x}(t)\}$, and the control vector $\{\underset{\sim}{z}_1(t)\}$ by $\{\underset{\sim}{u}(t)\}$. In most practical applications, it is also assumed that $\underset{\sim}{Q}(t) = \underset{\sim}{Q}$ is equal for all time periods up to $T - 1$. This assumption is only valid if the preference system and tastes do not change over time. It also implies that the contribution of a certain set of target and control values is independent of when they appear on the time path, since no discounting measure has been introduced. The matrix $\underset{\sim}{Q}(T)$ is commonly denoted by $\underset{\sim}{F}$. The weight matrices $\underset{\sim}{A}(t)$ associated with the instruments or controls are also assumed to be time independent, and are denoted by $\underset{\sim}{R}$. The multiplier matrix $\underset{\sim}{R}_0$ is denoted by $\underset{\sim}{B}$. To facilitate reference to the optimal control literature, we will adopt this notation in the remainder of this study. The linear-quadratic problem, therefore, is reformulated as

$$\min \frac{1}{2} \{\underset{\sim}{x}(T)\}' \; \underset{\sim}{F}\{\underset{\sim}{x}(T)\}$$

$$+ \frac{1}{2} \sum_{t=0}^{T-1} \left[\{\underset{\sim}{x}(t)\}' \; \underset{\sim}{Q}\{\underset{\sim}{x}(t)\} + \{\underset{\sim}{u}(t)\}' \; \underset{\sim}{R}\{\underset{\sim}{u}(t)\} \right] \tag{1.19}$$

subject to

$$\{\underset{\sim}{x}(t + 1)\} = \underset{\sim}{G}\{\underset{\sim}{x}(t)\} + \underset{\sim}{B}\{\underset{\sim}{u}(t)\} \quad .$$

To solve (1.19), one can apply the quadratic program-
ming algorithm to the original Theil problem with the
matrices A, Q, R and S of the appropriate structure. However,
if T and N are of some practical magnitude, the scale of the
problem becomes immense. In recent years, algorithms have
been sought which could solve the general linear-quadratic
problem and dynamic problems directly. The optimization of
such dynamic systems may be approached from three alter-
native perspectives:

  i.   Variational calculus, dealing with the problem
       of finding the function describing the optimal
       trajectory of the system. The solution of such
       a problem involves the determination of maxima
       and minima of functionals (Gelfand and Fomin, 1963).

  ii. Dynamic programming, based on Bellman's principle
       of optimality (Bellman, 1957).

  iii. Optimal control theory, based on the "maximum
       principle" derived by Pontryagin and his asso-
       ciates (1962).

A discussion of the three approaches is beyond the scope
of this study. The interested reader is referred to the
literature. A clear exposition of the relationship between
the calculus of functionals and the calculus of functions
is given by Connors and Teichroew (1967). How dynamic
programming and optimal control theory relate to each other,
is discussed by Noton (1972). A fine textbook on applied
optimal control is Bryson and Ho (1969).

    Optimal control has the broadest field of application.
Problems which may be solved by calculus of variations or
by dynamic programming, can also be solved by optimal

control.   Therefore, we adopt the optimal control approach
to the optimization of dynamic population systems.   This will
enable us in Chapter 3 to derive the optimal solution to
the linear-quadratic control problem.

CHAPTER 2

REPRESENTATION AND EXISTENCE THEOREMS
OF MIGRATION POLICIES

In this chapter we deal with constraints (1.6) and
(1.16); in other words, with the demometric model representation of the dynamics of a population system, and with
the policy model describing the goals-means relationship
in migration policy. Nothing will be said about goal-setting or about the selection of optimal values for the
instrument variables. These will be considered in the
next chapter.

Let us begin with a demometric model in the form of
a system of simultaneous linear equations. It is assumed
that the model has been specified and that the coefficients
have been estimated. The model relates demographic with
socio-economic variables, in a manner such as is found in
Greenwood (1973, 1975b). We assume that the model is
dynamic, i.e., that it contains lagged endogenous and
exogenous variables. It is also assumed that the goals-means relationship of migration policy is known, i.e., the
target variables and the instrument variables have been
separated from the other endogenous and exogenous variables.

We first transform the reduced form of the model into
the discrete state-space form. A general solution of
the discrete state-space equation is then derived. Next,
we consider the question whether arbitrary specified
levels of target variables can be reached by the existing
set of instruments. The existence theorems which are

derived are related to the rank of the matrix of impact
multipliers.

2.1. STATE-SPACE REPRESENTATION OF DEMOMETRIC MODELS

Stimulated by recent work in optimal control and system
theory, an increasing number of authors have adopted the
state-space approach to describe dynamic models in the
social sciences. This section describes the character-
istics of the state-space model and the procedure for its
solution. Since most demometric models are given in the
reduced form, we also consider its transformation to the
state-space form.

2.1.1. The State-Space Model

The state-space representation of a linear system is
defined by the following set of first order linear differ-
ence equations[2]:

$$\{x(t + 1)\} = G(t) \{x(t)\} + B(t) \{u(t)\} \tag{2.1}$$

$$\{y(t)\} = C(t) \{x(t)\} + E(t) \{u(t)\} \tag{2.2}$$

where $\{x(t)\}$ is an N-dimensional vector-valued function of
  time, called the state of the system,

$\{u(t)\}$ is an K-dimensional vector-valued function of
  time, called the input or control to the system,

---

[2]We shall consider only the discrete state-space model.
The continuous version is a set of differential equations.
For details see, for example, Director and Rohrer (1972)
and Wolovich (1974).

$\{y(t)\}$ is an P-dimensional vector-valued function of

time, called the output of the system,

$A(t)$, $B(t)$, $C(t)$ and $E(t)$ are real-time dependent

matrices of dimension N x N, N x K, P x N and

P x K, respectively.

If $G(t)$, $B(t)$, $C(t)$ and $E(t)$ are constant over time, the system is time-invariant. In this section, we will only consider the case where these matrices are constant. Thus,

$$\{x(t + 1)\} = G\{x(t)\} + B\{u(t)\} \tag{2.3a}$$

$$\{y(t)\} = C\{x(t)\} + E\{u(t)\} \quad . \tag{2.3b}$$

The homogenous part of (2.3a):

$$\{x(t + 1)\} = G\{x(t)\}$$

gives the growth of the system without intervention. The matrix G is the growth matrix. The discrete model of population growth, studied by Rogers (1975; p. 123), is of this form.

The interpretation of (2.3) as a migration policy model is straightforward. Suppose $\{x(t)\}$ is the interregional and/or age-specific population distribution. The matrix G is the population growth matrix, and $\{u(t)\}$ is a vector of instrument variables, which may range from pure demographic variables to socio-economic variables. It defines a policy at time t. The impact of each policy variable on the population distribution in the next period, is given by the elements of B. If $\{u(t)\}$ has no lagged instrument

variables, and if $\{u(t)\}$ has no impact on $\{x(t)\}$, then B is the matrix of impact multipliers. If the policy is a direct population influencing policy, then $\{u(t)\}$ is expressed in numbers of people, exactly as $\{x(t)\}$, and therefore B is the identity matrix.

In demographic policy problems with socio-economic goals, the target vector is not expressed in terms of popu- lation distribution, but in terms of socio-economic variables. The matrix C transforms the population distribution $\{x(t)\}$ into the vector $\{y(t)\}$ of socio-economic target variables, whereas E gives the direct impact of the policy variables on the new target variables. In fact, C can be any transformation matrix. For example, suppose $\{x(t)\}$ is the regional distri- bution of the population by age. If the policy maker is interested only in the spatial distribution of the total population, then C will be a consolidation matrix.

2.1.2. Solution of the State-Space Model

In order to derive the solution to (2.3a), we write (2.3a) for various t:

$$\{x(1)\} = G\{x(0)\} + B\{u(0)\}$$

$$\{x(2)\} = G\{x(1)\} + B\{u(1)\} = G^2\{x(0)\} + GB\{u(0)\} + B\{u(1)\}$$

$$\cdot$$
$$\cdot$$
$$\cdot$$

$$\{x(T)\} = G^T\{x(0)\} + \sum_{i=0}^{T-1} G^{T-1-i} B\{u(i)\} \quad .$$

Therefore the general solution to (2.3) is

$$\{\underset{\sim}{x}(t)\} = \underset{\sim}{G}^t\{\underset{\sim}{x}(0)\} + \sum_{i=0}^{t-1} \underset{\sim}{G}^{t-1-i} \underset{\sim}{B}\{\underset{\sim}{u}(i)\}$$

$$(2.4)$$

$$\{\underset{\sim}{y}(t)\} = \underset{\sim}{C}\underset{\sim}{G}^t\{\underset{\sim}{x}(0)\} + \sum_{i=0}^{t-1} \underset{\sim}{C}\underset{\sim}{G}^{t-1-i} \underset{\sim}{B}\{\underset{\sim}{u}(i)\} + \underset{\sim}{E}\{\underset{\sim}{u}(t)\} \quad .$$

The solution to the homogenous part of (2.3a) is

$$\{\underset{\sim}{x}(t)\} = \underset{\sim}{G}^t\{\underset{\sim}{x}(0)\} \qquad (2.5)$$

where $\underset{\sim}{G}^t = \underset{\sim}{\Phi}(t,0)$ is known as the discrete <u>state-transition</u> <u>matrix</u>. The solution in terms of the state-transition matrix is:

$$\{\underset{\sim}{x}(t)\} = \underset{\sim}{\Phi}(t)\{\underset{\sim}{x}(0)\} + \sum_{i=0}^{t-1} \underset{\sim}{\Phi}(t-1-i) \underset{\sim}{B}\{\underset{\sim}{u}(i)\} \qquad (2.6)$$

and

$$\{\underset{\sim}{y}(t)\} = \underset{\sim}{C}\underset{\sim}{\Phi}(t)\{\underset{\sim}{x}(0)\} + \underset{\sim}{C} \sum_{i=0}^{t-1} \underset{\sim}{\Phi}(t-1-i) \underset{\sim}{B}\{\underset{\sim}{u}(i)\} + \underset{\sim}{E}\{\underset{\sim}{u}(t)\}$$

$$(2.7)$$

where $\underset{\sim}{\Phi}(t) = \underset{\sim}{G}^t \quad .$

Consider the system where $\underset{\sim}{E} = \underset{\sim}{0}$. Then

$$\{\underset{\sim}{y}(t)\} = \underset{\sim}{C}\underset{\sim}{G}^t\{\underset{\sim}{x}(0)\} + \sum_{i=0}^{t-1} \underset{\sim}{C}\underset{\sim}{G}^{t-1-i} \underset{\sim}{B}\{\underset{\sim}{u}(i)\} \quad .$$

Let $\underset{\sim}{H}(t) = \underset{\sim}{C}\underset{\sim}{G}^{t-1} \underset{\sim}{B}$, then

$$\{\underset{\sim}{y}(t)\} = \underset{\sim}{C}\underset{\sim}{G}^t\{\underset{\sim}{x}(0)\} + \sum_{j=1}^{t-1} \underset{\sim}{H}(t-i)\{\underset{\sim}{u}(i)\}$$

and, if $j = t - i$,

$$\{\underset{\sim}{y}(t)\} = \underset{\sim}{C}\underset{\sim}{G}^t\{\underset{\sim}{x}(0)\} + \sum_{j=1}^{t} \underset{\sim}{H}(j)\{\underset{\sim}{u}(t - j)\} \quad . \quad (2.8)$$

If $\{\underset{\sim}{x}(0)\}$ is the initial population distribution, if $\{\underset{\sim}{u}(t)\}$ is vector of control or policy actions at time t, and if $\{\underset{\sim}{y}(t)\}$ describes the population distribution at time t (in this case, $\underset{\sim}{C} = \underset{\sim}{I}$), then $\underset{\sim}{H}(j)$ is the matrix of dynamic impact multipliers. The element $h_{rs}(j)$ represents the change of the population in group or region r at time t due to a unit change in the s-th instrument at time $t - j$. $\underset{\sim}{H}(j)$ can also be thought of as the contribution of the policy action at time $(t - j)$ to the population distribution at time t. Each matrix $\underset{\sim}{H}(j)$ corresponds to the various submatrices of (2.15), which are not in the first column.

2.1.3. State-Space Representation of the Reduced Form Model

The reduced form of a demometric model is

$$\underset{\sim}{A}\{\underset{\sim}{y}\} = \underset{\sim}{E}\{\underset{\sim}{z}\} \quad , \quad (2.1)$$

where $\{\underset{\sim}{y}\}$ is the vector of endogenous variables, and

$\{\underset{\sim}{z}\}$ is the vector of predetermined variables consist-
ing of exogenous and lagged endogenous variables.

The general reduced form is

$$A\{\underset{\sim}{y}(t)\} = \underset{\sim}{E}_1\{\underset{\sim}{y}(t - 1)\} + \underset{\sim}{E}_2\{\underset{\sim}{y}(t - 2)\}...+ \underset{\sim}{E}_r\{\underset{\sim}{y}(t - r)\}$$

$$+ \underset{\sim}{D}_0\{\underset{\sim}{z}_1(t)\} + \underset{\sim}{D}_1\{\underset{\sim}{z}_1(t - 1)\}...+ \underset{\sim}{D}_s\{\underset{\sim}{z}_1(t - s)\}$$

$$(2.9)$$

where $(t - i)$ indicates a time lag of i periods. In order to put (2.9) into state-space form, we must define new variables and corresponding equations to replace the reduced form variables that have second order or higher order lags. The procedure is then one of the replacement of an r-th order difference equation by r first-order difference equations.

First, let

$$\{\underset{\sim}{z}(t)\} = \begin{bmatrix} \{\underset{\sim}{z}_1(t)\} \\ \{\underset{\sim}{z}_1(t - 1)\} \\ \cdot \\ \cdot \\ \cdot \\ \{\underset{\sim}{z}_1(t - s)\} \end{bmatrix}$$

and

$$\underset{\sim}{D} = [\underset{\sim}{D}_0, \underset{\sim}{D}_1, ..., \underset{\sim}{D}_s] \quad .$$

Equation (2.9) then may be simplified to yield

$$\underset{\sim}{A}\{\underset{\sim}{y}(t)\} = \underset{\sim}{E}_1\{\underset{\sim}{y}(t - 1)\} + \underset{\sim}{E}_2\{\underset{\sim}{y}(t - 2)\}...+ \underset{\sim}{E}_r\{\underset{\sim}{y}(t - r)\}$$

$$(2.10)$$

$$+ \underset{\sim}{D}\{\underset{\sim}{z}(t)\} \quad .$$

Following Kenkel (1974; pp. 295-299), we define a set of new vectors:

$$\{\underset{\sim}{y}_1(t)\} = \{\underset{\sim}{y}(t)\}$$

$$\{\underset{\sim}{y}_2(t)\} = \{\underset{\sim}{y}(t - 1)\}$$

$$\vdots$$

$$\{\underset{\sim}{y}_r(t)\} = \{\underset{\sim}{y}(t - r + 1)\}$$

$$\{\underset{\sim}{y}_r(t - 1)\} = \{\underset{\sim}{y}(t - r)\} \quad .$$

Therefore (2.10) becomes

$$\{\underset{\sim}{y}_1(t)\} = \underset{\sim}{A}^{-1}\underset{\sim}{E}_1\{\underset{\sim}{y}_1(t - 1)\} + \underset{\sim}{A}^{-1}\underset{\sim}{E}_2\{\underset{\sim}{y}_2(t - 1)\}\ldots$$

$$+ \underset{\sim}{A}^{-1}\underset{\sim}{E}_r\{\underset{\sim}{y}_r(t - 1)\}$$

which may now be rewritten as a recurrence equation of the form

$$\{\underset{\sim}{x}(t)\} = \underset{\sim}{G}\{\underset{\sim}{x}(t - 1)\} + \underset{\sim}{B}\{\underset{\sim}{u}(t)\} \tag{2.11}$$

where

$$\{\underset{\sim}{x}(t)\} = \begin{bmatrix} \{\underset{\sim}{y}_1(t)\} \\ \{\underset{\sim}{y}_2(t)\} \\ \vdots \\ \{\underset{\sim}{y}_r(t)\} \end{bmatrix} \qquad \{\underset{\sim}{u}(t)\} = \{\underset{\sim}{z}(t)\}$$

$$
\underset{\sim}{G} = \begin{bmatrix}
\underset{\sim}{A}^{-1}\underset{\sim}{E}_1 & \underset{\sim}{A}^{-1}\underset{\sim}{E}_2 & \cdots\cdots & \underset{\sim}{A}^{-1}\underset{\sim}{E}_{r-1} & \underset{\sim}{A}^{-1}\underset{\sim}{E}_r \\
\underset{\sim}{I} & \underset{\sim}{0} & \cdots\cdots\cdots & \underset{\sim}{0} & \underset{\sim}{0} \\
\vdots & & & & \vdots \\
\vdots & & \underset{\sim}{0} & & \vdots \\
\underset{\sim}{0} & & & \underset{\sim}{I} & \underset{\sim}{0}
\end{bmatrix} \qquad (2.12)
$$

$$
\underset{\sim}{B} = \begin{bmatrix}
\underset{\sim}{A}^{-1}\underset{\sim}{D} \\
\underset{\sim}{0} \\
\vdots \\
\underset{\sim}{0}
\end{bmatrix} \qquad .
$$

Equation (2.11) is the state-space representation. The submatrices in the first row denote the impact on $\{\underset{\sim}{y}(t)\}$ of the vectors of lagged endogenous variables. The submatrix $\underset{\sim}{A}^{-1}\underset{\sim}{D}$ denotes the direct effect on $\{\underset{\sim}{y}(t)\}$ of the exogenous variables. $\underset{\sim}{A}^{-1}\underset{\sim}{D}_0$ is the matrix of impact multipliers. The matrix $\underset{\sim}{A}^{-1}\underset{\sim}{D}_i$ gives the direct effect on $\{\underset{\sim}{y}(t)\}$ of the vector of exogenous variables, lagged by i periods. These are not total delay multipliers, since $\{\underset{\sim}{z}(t - i)\}$ also affects $\{\underset{\sim}{y}(t)\}$ through its impact on $\{\underset{\sim}{y}(t - k)\}$, $k = 1,\ldots,i$.

There is another transformation of the reduced form to the state-space form. This transformation has no direct motivation for demometric models, but it facilitates the study of the state-space model. Equation (2.10) may be written as

$$\underset{\sim}{E}_r\{\underset{\sim}{y}(t - r)\} = - \underset{\sim}{E}_{r-1}\{\underset{\sim}{y}(t - r + 1)\} - \underset{\sim}{E}_{r-2}\{\underset{\sim}{y}(t - r + 2)\}\ldots$$

$$\hspace{8cm} (2.13)$$

$$- \underset{\sim}{E}_1\{\underset{\sim}{y}(t - 1)\} - \underset{\sim}{A}\{\underset{\sim}{y}(t)\} + \underset{\sim}{D}\{\underset{\sim}{z}(t)\} \quad .$$

Suppose $\underset{\sim}{E}_r$ is nonsingular, and define the new vectors

$$\{\underset{\sim}{y}_1(t - 1)\} = \{\underset{\sim}{y}(t)\}$$

$$\{\underset{\sim}{y}_1(t - 1)\} = \{\underset{\sim}{y}(t - 1)\}$$

$$\vdots$$

$$\{\underset{\sim}{y}_i(t - 1)\} = \{\underset{\sim}{y}(t - (i - 1))\}$$

$$\vdots$$

$$\{\underset{\sim}{y}_r(t - 1)\} = \{\underset{\sim}{y}(t - (r - 1))\} \quad .$$

The extended version of (2.13) then is

$$\{\underset{\sim}{x}(t)\} = \underset{\sim}{\hat{G}}\{\underset{\sim}{x}(t - 1)\} + \underset{\sim}{\hat{B}}\{\underset{\sim}{u}(t)\}$$

where $\{\underset{\sim}{x}(t)\}$ and $\{\underset{\sim}{u}(t)\}$ are as defined in (2.12),

$$\underset{\sim}{\hat{G}} = \begin{bmatrix} \underset{\sim}{0} & \underset{\sim}{I} & \underset{\sim}{0} & \cdots\cdots & \underset{\sim}{0} \\ \vdots & & \ddots & & \vdots \\ \underset{\sim}{0} & & & & \underset{\sim}{I} \\ -\underset{\sim}{E}_r^{-1}\underset{\sim}{A} & -\underset{\sim}{E}_r^{-1}\underset{\sim}{E}_1 & & & -\underset{\sim}{E}_r\underset{\sim}{E}_{r-1} \end{bmatrix} \hspace{1.5cm} (2.14)$$

$$
\hat{\underset{\sim}{B}} = \begin{bmatrix} \underset{\sim}{0} \\ \underset{\sim}{0} \\ \vdots \\ \underset{\sim r}{E}^{-1}\underset{\sim}{D} \end{bmatrix} .
$$

The matrices $\underset{\sim}{G}$ and $\hat{\underset{\sim}{G}}$ are generalized companion matrices. In a previous paper, we have already introduced the companion matrix in the demographic analysis (Willekens, 1976b, p. 49). It has been indicated that this matrix can play an important role in the reconciliation of discrete and continuous models of demographic growth. Here we have shown that the companion matrix provides the natural link between the reduced form model and the state-space model. A similar link may be formulated between the structural form and the state-space model. A detailed description of the technique is given by Pindyck (1973; pp. 89-94).

2.2. EXISTENCE THEOREMS OF MIGRATION POLICIES

It is argued that there are two central issues in the theory of policy. These are the concepts of existence and of design. Existence of policy refers to the controllability of the system or the ability to design any policy at all; design refers to the techniques for designing optimal policies once existence is assured. Although both issues have been recognized for a long time in system theory, policy analysis in the social sciences, led by the theory of economic policy, has focused almost entirely on the design problem. Only Tinbergen (1963) has given considerable attention to

both issues. His policy model is formulated in the reduced form. An alternative representation is the state-space format.

This section is divided into two parts. The first deals with the existence of optimal policies in the Tinbergen framework. The other derives existence theorems for the state-space model. Until very recently, the existence of optimal policies in the state-space framework has not been investigated in the theory of economic policy. Based on findings of system theory, Aoki (1973, 1974, 1975) and Preston (1974) have supplemented Tinbergen's existence theorem with theorems related to state-space economic models.

2.2.1. Existence Theorem in the Tinbergen Model

Recall the Tinbergen model

$$\{y\} = R\{z_1\} + S\{z_2\} \ . \tag{1.3}$$

In the original formulation, (1.3) represented a static policy problem, i.e. the targets and the instruments belonged to the same time period. The model, however, may include lagged variables in the vector of uncontrollable variables $\{z_2\}$. Contrary to Preston's (1974; p. 65) claim, the Tinbergen model also fits dynamic situations, where the targets and instruments belong to different time periods. This is shown in (1.11). The cornerstone of Tinbergen's theory of policy is the condition for which there exists for any $\{\bar{y}\}$ a corresponding unique policy vector $\{\bar{z}_1\}$ such that (1.3) is satisfied. In other words, under what conditions has (1.3) a unique solution for $\{z_1\}$? The necessary

and sufficient condition is that R is of full rank, and that the number of targets is equal to the number of instruments. This statement is general enough to encounter dynamic policy problems where time series of targets are given and where time series of instruments are sought. The uniqueness of the instrument vector is an unnecessarily restrictive condition. An infinite number of policy vectors may exist which lead to the same target vector. The controllability theorem for the Tinbergen model is, therefore, stated as follows:

THEOREM 1:   Tinbergen Controllability Theorem

The policy model

$$\{y\} = R\{z_1\} + S\{z_2\} \tag{1.3}$$

is controllable for all $\{y\} = \{\bar{y}\}$ if and only if the matrix multiplier R satisfies the condition

$$\text{rank } (R) = N$$

where N is the number of targets. The control vector $\{z_1\}$ is unique if R is N x N. This condition is a reformulation of Tinbergen's proposition that there exist as many instruments as there are targets.

To prove Theorem 1, recall that R is a N x K matrix, where N is the number of targets and K is the number of instruments. In the previous chapter, we made the assumption that both the targets and the instruments are linearly

independent. This implies that the equations of (1.3) are
independent. The system (1.3) is consistent, i.e. has a
solution if and only if the number of unknowns K is greater
than or equal to the number of equations N. But this
implies that the rank of R is N. If K is less than N, the
rank of R is $K < N$, and the system is inconsistent. The
general solution to a consistent system is (Rogers, 1971;
p. 258):

$$\{\bar{z}_1\} = R^{(1)}\left[\{\bar{y}\} - S\{z_2\}\right] + \left[I - R^{(1)}R\right]\{c\} \qquad (2.15)$$

where $R^{(1)}$ is a generalized inverse of R, satisfying

$$R\,R^{(1)}\,R = R \quad,$$

and $\{c\}$ is an arbitrary vector.

If $K > N$, there exists an infinite number of instrument
vectors associated with $\{\bar{y}\}$. However, in most cases, there
is only one instrument vector which is most suited to the
policy maker's preferences. The design of such a policy
vector will be discussed in the next chapter. If on the
other hand $K = N$, then R is nonsingular and (1.3) has a
unique solution:

$$\{\bar{z}_1\} = R^{-1}\left[\{\bar{y}\} - S\{z_2\}\right] \quad . \qquad (2.16)$$

2.2.2. Existence Theorems in the State-Space Model

In the previous chapter, the state-space model was derived by introducing the assumptions of unidirectional causality and time independence into the Tinbergen model. Recall

$$\{x(t + 1)\} = G\{x(t)\} + B\{u(t)\} \qquad (2.3a)$$

$$\{y(t)\} = C\{x(t)\} + E\{u(t)\} \ . \qquad (2.3b)$$

Two existence problems may be distinguished. The first deals with the question of whether there is a sequence of control vectors $\{u(t)\}$, $t = 0,\ldots,T-1$, such that a desired target vector can be achieved at a predefined planning horizon T. The second deals with the question whether there exists a sequence of control vectors $\{u(t)\}$, $t = 0,\ldots,T-1$, such that any sequence of target vectors $\{y(t)\}$, $t = 1,\ldots,T$ can be realized. The first existence problem is known in system theory as state and output controllability; the latter is sometimes referred to as output function controllability. The state and output controllability has received most attention in the literature. Both existence problems will be dealt with below. Two applications will be discussed: the separation of the controllable and the noncontrollable parts of the system, and the achievement of the targets with a minimum number of instruments.

a. State and Output Controllability

The system

$$\{\underset{\sim}{x}(t + 1)\} = G\{\underset{\sim}{x}(t)\} + B\{\underset{\sim}{u}(t)\} \qquad (2.3a)$$

is said to be controllable (state controllable) if and only if there exists a control $\{\underset{\sim}{u}(t)\}$ which transfers any initial state $\{\underset{\sim}{x}(t_0)\}$ at any time t_0 to any arbitrary final state $\{\underset{\sim}{x}(t_1)\}$ at any time $t_1 > t_0 \geq 0$, (Wolovich, 1974; p. 65). Otherwise, the system is uncontrollable or only "controllable in part," i.e. it may be possible to transfer certain states to any desired final states or to transfer all the states to a position close to the desired states.

The controllability concept assumes that there is no constraint on $\{\underset{\sim}{u}(t)\}$. The only requirement is that there exists a trajectory between the initial and the terminal state.

In order to determine conditions for controllability of (2.3a), consider its solution

$$\{\underset{\sim}{x}(t)\} = G^t\{\underset{\sim}{x}(0)\} + \sum_{i=0}^{t-1} G^{t-1-i} B\{\underset{\sim}{u}(i)\} \qquad (2.4)$$

which may be rewritten as

$$\{\underset{\sim}{x}(t)\} - G^t\{\underset{\sim}{x}(0)\} = [B \vdots GB \vdots G^2B \vdots \cdots \vdots G^{t-1}B] \begin{bmatrix} \{\underset{\sim}{u}(t-1)\} \\ \{\underset{\sim}{u}(t-2)\} \\ \vdots \\ \{\underset{\sim}{u}(0)\} \end{bmatrix}$$

$$(2.17)$$

where $\{\underset{\sim}{x}(t)\}$ and $\{\underset{\sim}{x}(0)\}$ are given and $\underset{\sim}{G}$ is time-invariant. The dimension of the target vector $\{\underset{\sim}{x}(t)\}$ is N (t = 1,...,T) and of the control vector $\{\underset{\sim}{u}(t)\}$ is K (t = 0,...,T-1). The matrix

$$\underset{\sim}{D} = [\underset{\sim}{B} \vdots \underset{\sim}{GB} \vdots \underset{\sim}{G}^2 \underset{\sim}{B} \vdots \cdots \vdots \underset{\sim}{G}^{t-1} \underset{\sim}{B}]$$

is therefore of dimension N x Kt. Equation (2.3a) is controllable, or there exists a solution to (2.17) if the rank of $\underset{\sim}{D}$ is N. If t > N, i.e. if the number of control intervals is greater than the number of targets, then we don't need to consider the whole matrix $\underset{\sim}{D}$ to evaluate the controllability of (2.3a).

According to the Cayley-Hamilton theorem, each matrix satisfies its own characteristic equation. If $\underset{\sim}{G}$ has the characteristic equation

$$\lambda^N + c_1 \lambda^{N-1} + c_2 \lambda^{N-2} + \ldots + c_N = 0$$

then

$$\underset{\sim}{G}^N + c_1 \underset{\sim}{G}^{N-1} + c_2 \underset{\sim}{G}^{N-2} + \ldots + c_N \underset{\sim}{I} = \underset{\sim}{0} \quad . \qquad (2.18)$$

Therefore $\underset{\sim}{G}^N$ and any $\underset{\sim}{G}^{N+i}$ (i $\geq$ 0) is linearly dependent on $[\underset{\sim}{I} \vdots \underset{\sim}{G} \vdots \underset{\sim}{G}^2 \vdots \cdots \vdots \underset{\sim}{G}^{N-1}]$. It follows that no extra independent column vectors would be added to $\underset{\sim}{D}$ if there are more than N control intervals.

This result is formulated as follows:

THEOREM 2: State Controllability Theorem

The dynamic system

$$\{\underset{\sim}{x}(t + 1)\} = \underset{\sim}{G}\{\underset{\sim}{x}(t)\} + \underset{\sim}{B}\{\underset{\sim}{u}(t)\} \qquad (2.3a)$$

is completely controllable for all $\{\underset{\sim}{x}(t)\} = \{\overline{\underset{\sim}{x}}(t)\}$ if and only if the N x KN matrix

$$\underset{\sim}{D} = [\underset{\sim}{B} \vdots \underset{\sim}{GB} \vdots \cdots \vdots \underset{\sim}{G}^{N-1}\underset{\sim}{B}] \qquad (2.19)$$

is of rank N. This theorem has been considered by Preston (1974, p. 68) as the dynamic generalization of Tinbergen's theory of policy. Several observations may be made at this point.

a) It is a corollary to the theorem that if a target vector $\{\overline{\underset{\sim}{x}}\}$ cannot be reached in N control intervals, it will never be reached. This is important for policy purposes, since it answers the question of how fast a target population distribution, for example, can be achieved[3].

b) Tinbergen-controllability implies state controllability. If the rank of $\underset{\sim}{B}$ is N, as required for Tinbergen controllability, then the rank of $\underset{\sim}{D}$ is N also. If the rank of $\underset{\sim}{B}$ is N, then the targets can be reached in only one control interval ($t_1 = t_0 + 1$).

c) An argument similar to the one leading to Theorem 2, may be used to derive the conditions for output

---

[3]Under the assumption that no constraints are imposed on the trajectory of control and state variables.

controllability. The system

$$\{x(t + 1)\} = G\{x(t)\} + B\{u(t)\} \tag{2.3a}$$

$$\{y(t)\} = C\{x(t)\} + E\{u(t)\} \tag{2.3b}$$

is output controllable if and only if there exists a control vector $\{u(t)\}$ which transfers any initial output $\{y(t_0)\}$ at time t_0 to any arbitrary final output $\{y(t_1)\}$ at any $t_1 > t_0$. Wolovich (1974; p. 71) states the condition for output controllability to be

$$\text{rank } [CB \vdots CGB \vdots \cdots \vdots CG^{N-1}B \vdots E] = P \tag{2.20}$$

if P x 1 is the dimension of the output vector. Output controllability is sometimes referred to as <u>reproducibility</u> (Brockett and Mesarović, 1965; p. 549).

   d)   The "dual" notion of controllability is <u>observability</u>. The system (2.3) is said to be observable if and only if the entire state $\{x(t)\}$ can be determined over any finite interval $[t_0, t_1]$ from complete knowledge of $\{u(t)\}$ and $\{y(t)\}$ over the interval $[t_0, t_1]$ with $t_1 > t_0 \geq 0$ (Wolovich, 1974; p. 73). The condition for observability is that the MN x N matrix

$$Q = \begin{bmatrix} C \\ \cdots \\ CG \\ \vdots \\ CG^{N-1} \end{bmatrix} \tag{2.21}$$

be of rank N. Equation (2.3b), written out for the time

periods t = 0,...,N-1, while noting that

$$\{\underset{\sim}{x}(t)\} = \underset{\sim}{G}^t\{\underset{\sim}{x}(0)\} + \underset{\sim}{B}\{\underset{\sim}{u}(t - 1)\} \quad ,$$

gives

$$
\begin{bmatrix}
\{\underset{\sim}{y}(0)\} \\
\{\underset{\sim}{y}(1)\} \\
\vdots \\
\{\underset{\sim}{y}(N - 1)\}
\end{bmatrix}
=
\begin{bmatrix}
\underset{\sim}{C} \\
\underset{\sim}{C}\underset{\sim}{G} \\
\vdots \\
\underset{\sim}{C}\underset{\sim}{G}^{N-1}
\end{bmatrix}
\{\underset{\sim}{x}(0)\} +
\begin{bmatrix}
\{\underset{\sim}{0}\} \\
\underset{\sim}{C}\underset{\sim}{B}\{\underset{\sim}{u}(0)\} \\
\vdots \\
\underset{\sim}{C}\underset{\sim}{B}\{\underset{\sim}{u}(N - 2)\}
\end{bmatrix}
+
\begin{bmatrix}
\underset{\sim}{E}\{\underset{\sim}{u}(0)\} \\
\underset{\sim}{E}\{\underset{\sim}{u}(1)\} \\
\vdots \\
\underset{\sim}{E}\{\underset{\sim}{u}(N - 1)\}
\end{bmatrix}
$$

$$(2.22)$$

where $\{\underset{\sim}{u}(t)\}$ and $\{\underset{\sim}{y}(t)\}$ are known for t = 0,...,N-1, and

$\{\underset{\sim}{x}(0)\}$ is unknown. System (2.22) consists of MN equations in

N unknowns. $\{\underset{\sim}{x}(0)\}$ can be calculated if $\underset{\sim}{Q}$ has rank N. If

$\{\underset{\sim}{x}(0)\}$ is known, the whole sequence of state vectors is known

by (2.3a).

The notion of observability might be useful in the

study of populations with incomplete data. For example,

let $\{\underset{\sim}{x}(0)\}$ be the spatial distribution of a population by

age group, at time t = 0. Let $\{\underset{\sim}{y}(t)\}$ be the observed spatial

distribution of the total population at time t and let

$\{\underset{\sim}{u}(t)\} = \{\underset{\sim}{0}\}$ for all t. The matrix $\underset{\sim}{C}$ is then a consolidation

matrix. Assuming that the condition for observability is met,

and that $\underset{\sim}{G}$ is known and remains constant in time, $\{\underset{\sim}{x}(0)\}$ can

be computed from $\{\underset{\sim}{y}(t)\}$ [t = 0,...,N-1]. If $\underset{\sim}{G}$ is unknown, it

may be approximated by some underlying model mortality,

fertility and migration schedules.

The problem of controllability and observability has

been studied by Vajda (1975) in manpower planning, although

the author does not refer to the concepts and theorems just described and instead focuses on totally different techniques. He uses the simplex algorithm to determine the population distribution from which a given distribution can be obtained, and to find out if a target distribution can be reached from the present distribution in one, two or more steps.

b. Output Function Controllability

The controllability concept discussed in the previous section dealt with the existence of a control vector, such that a desired target vector can be achieved at a predefined planning horizon. In practice, policy makers would be interested in not only achieving desired target values, but also keeping them on some desired time trajectory once achieved, or achieving the targets along a desired path. It is not uncommon in politics that short term objectives conflict with long term goals. In designing a policy to achieve the short term objectives, the policy maker includes elements which make the long term goals unattainable. The careful policy-maker, therefore, will design a policy that enables him not only to achieve, for example, a desired population distribution at a certain point in time, but also to control the growth path of the multiregional population system once the target distribution is achieved. A system whose trajectory is controllable is called output function controllable or, equivalently, functionally reproducible (Brockett and Mesarović, 1965; p. 556).

Recall the dynamic system described by (2.3)

$$\{\underset{\sim}{x}(t + 1)\} = \underset{\sim}{G}\{\underset{\sim}{x}(t)\} + \underset{\sim}{B}\{\underset{\sim}{u}(t)\} \qquad (2.3a)$$

$$\{\underset{\sim}{y}(t)\} = \underset{\sim}{C}\{\underset{\sim}{x}(t)\} \qquad (2.3b)$$

where $\{\underset{\sim}{x}(t)\}$ is the N x 1 state vector,

$\{\underset{\sim}{u}(t)\}$ is the K x 1 control or input vector, and

$\{\underset{\sim}{y}(t)\}$ is the P x 1 output vector.

If the target is related to the state of the system, (2.3b) may be deleted, or $\underset{\sim}{C}$ may be set identical to the identity matrix. In order to derive the condition for output function controllability, we take z-transforms in equation (2.3) (Director and Rohrer, 1972; p. 317):

$$z\{\underset{\sim}{x}(z)\} - z\{\underset{\sim}{x}(0)\} = \underset{\sim}{G}\{\underset{\sim}{x}(z)\} + \underset{\sim}{B}\{\underset{\sim}{u}(z)\}$$

$$\{\underset{\sim}{y}(z)\} = \underset{\sim}{C}\{\underset{\sim}{x}(z)\} \qquad .$$

Thus

$$[z\underset{\sim}{I} - \underset{\sim}{G}]\{\underset{\sim}{x}(z)\} = z\{\underset{\sim}{x}(0)\} + \underset{\sim}{B}\{\underset{\sim}{u}(z)\}$$

$$\{\underset{\sim}{x}(z)\} = [z\underset{\sim}{I} - \underset{\sim}{G}]^{-1} z\{\underset{\sim}{x}(0)\} + [z\underset{\sim}{I} - \underset{\sim}{G}]^{-1} \underset{\sim}{B}\{\underset{\sim}{u}(z)\}$$

$$\{\underset{\sim}{y}(z)\} = \underset{\sim}{C}[z\underset{\sim}{I} - \underset{\sim}{G}]^{-1} z\{\underset{\sim}{x}(0)\} + \underset{\sim}{C}[z\underset{\sim}{I} - \underset{\sim}{G}]^{-1} \underset{\sim}{B}\{\underset{\sim}{u}(z)\} \qquad .$$

$$(2.23)$$

The P x K matrix

$$\underset{\sim}{C}[z\underset{\sim}{I} - \underset{\sim}{G}]^{-1} \underset{\sim}{B} = \underset{\sim}{H}(z) \qquad\qquad (2.24)$$

is called the discrete <u>transfer matrix</u> (Director and Rohrer, 1972; p. 317)[4]. The transfer matrix describes the relationship between the output $\{\underset{\sim}{y}(t)\}$ and the input $\{\underset{\sim}{u}(t)\}$ of the system. It is independent of any particular choice of $\{\underset{\sim}{x}(0)\}$. Equation (2.23) may be written as

$$\underset{\sim}{H}(z) \{\underset{\sim}{u}(z)\} = \{\underset{\sim}{y}(z)\} - \underset{\sim}{C}[z\underset{\sim}{I} - \underset{\sim}{G}]^{-1} z\{\underset{\sim}{x}(0)\} \quad . \qquad (2.25)$$

This allows us to formulate precisely the question of output function controllability and to answer it.

The question of output function controllability is: given any desired P-dimensional output vector $\{\bar{\underset{\sim}{y}}(t)\}$, defined for all $t \geq t_0$, and the initial state $\{\underset{\sim}{x}(0)\}$, can the sequence $\{\bar{\underset{\sim}{y}}(t)\}$, $t \geq t_0$ be obtained by choosing the appropriate sequence $\{\underset{\sim}{u}(t)\}$, $t \geq t_0$? The answer to this question is formulated in the following theorem.

THEOREM 3: <u>Output Function Controllability Theorem</u>

The system

$$\{\underset{\sim}{x}(t + 1)\} = \underset{\sim}{G}\{\underset{\sim}{x}(t)\} + \underset{\sim}{B}\{\underset{\sim}{u}(t)\}$$

---

[4]The discrete transfer matrix is the analogue of the transfer matrix of continuous models, derived using Laplace transforms:

$$\underset{\sim}{T}(s) = \underset{\sim}{C}[s\underset{\sim}{I} - \underset{\sim}{G}]^{-1} \underset{\sim}{B} \qquad\qquad (2.26)$$

See Director and Rohrer (1972; p. 303) and Wolovich (1974; p. 101).

is output function controllable if and only if the rank of
the transfer matrix

$$H(z) = C[zI - G]^{-1} B \qquad (2.24)$$

is equal to P. The control $\{u(t)\}$, $t \geq t_0$, is unioue if P
is equal to K. The existence theorem, formulated by Wolovich
(1974; p. 164) states that the transfer matrix must have an
inverse, i.e. must be nonsingular. The control sequence he
derives, is, therefore, unique. However, the uniqueness of
$\{u(t)\}$ is not a necessary condition for output function
controllability. If P < K, an infinite number of control
sequences leads to the desired output sequence.

The condition for output function controllability may
also be expressed in terms of the matrices G, B and C of
the original system (2.3) (Brockett and Mesarović, 1965;
p. 556).

THEOREM 3':

The system

$$\{x(t + 1)\} = G\{x(t)\} + B\{u(t)\} \qquad (2.3a)$$

$$\{y(t)\} = C\{x(t)\} \qquad (2.3b)$$

is output function controllable if and only if the
PN x (2N - 1) K matrix

$$
\underset{\sim}{F} = \begin{bmatrix}
\underset{\sim\sim}{CB} & \underset{\sim\sim\sim}{CGB} & \underset{\sim\sim}{CG^2B} & \cdots & \underset{\sim\sim}{CG^{N-1}B} & \underset{\sim\sim}{CG^NB} & \cdots & \underset{\sim\sim}{CG^{2N-1}B} \\
0 & \underset{\sim\sim}{CB} & \underset{\sim\sim}{CGB} & \cdots & \underset{\sim\sim}{CG^{N-2}B} & \underset{\sim\sim}{CG^{N-1}B} & \cdots & \underset{\sim\sim}{CG^{2N-2}B} \\
0 & 0 & \underset{\sim\sim}{CB} & \cdots & \underset{\sim\sim}{CG^{N-3}B} & \underset{\sim\sim}{CG^{N-2}B} & \cdots & \underset{\sim\sim}{CG^{2N-3}B} \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\
0 & 0 & 0 & & \underset{\sim\sim}{CB} & \underset{\sim\sim\sim}{CGB} & & \underset{\sim\sim}{CG^{N-1}B}
\end{bmatrix} \quad (2.27)
$$

is of rank PN. (See R. Brockett and M. Mesarović (1965; pp. 556-559) for the formal proof.) Two observations, which are corollaries to theorem 3', may be made at this point.

a) Output function controllability implies output controllability. In a corollary to Theorem 2, the system (2.3) was said to be output controllable if

$$
\text{rank } \underset{\sim\sim}{CQ} = [\underset{\sim\sim}{CB} \vdots \underset{\sim\sim\sim}{CGB} \vdots \cdots \vdots \underset{\sim\sim}{CG^{N-1}B}] = P \quad .
$$

The matrix $\underset{\sim\sim}{CQ}$ is the last row of $\underset{\sim}{F}$. Now, if the rank of $\underset{\sim}{F}$ is PN, then the rank of $\underset{\sim\sim}{CQ}$ must be P and the system is output controllable.

b) A sufficient condition for $\underset{\sim}{F}$ to be of rank PN is that

$$
PN \leq 2(N - 1) K + K
$$

or

$$
P \leq \frac{2N - 1}{N} K \quad (2.28)
$$

for any N. This means that the number of target variables must be less than or equal to the number of instrument variables (Aoki, 1975; p. 295). This leads Aoki to conclude

that the condition for output function controllability is a
more proper dynamic generalization of Tinbergen's theory of
policy than is the condition for output controllability pro-
posed by Preston (1974; p. 68), because the former contains the
original Tinbergen condition that the number of targets
cannot exceed the number of instruments.

c. Separation of Controllable and Non-Controllable
   Parts of a System

If a system is not completely state controllable, i.e.
the rank of $\underset{\sim}{D}$ is less than N, it is important for policy
purposes to determine the controllable part of the system.
Two relevant methods are given below. The first is based
on the diagonalization of the matrix $\underset{\sim}{G}$. The other method
starts directly from the controllability condition.

Assume that the growth matrix $\underset{\sim}{G}$ is primitive, a common
assumption in the mathematical demography literature. Then
$\underset{\sim}{G}$ has N distinct eigenvalues, and N linearly independent
eigenvectors. Now, any square matrix of order N that has N
linearly independent eigenvectors may be diagonalized[5].
Let $\underset{\sim}{P}$ be the modal matrix, formed by stacking the N eigen-
vectors side by side. Because the eigenvectors are linearly
independent, $\underset{\sim}{P}$ is nonsingular. Equation (2.3a) can be
written in its canonical form

$$\{\hat{\underset{\sim}{x}}(t + 1)\} = \underset{\sim}{\Lambda}\{\hat{\underset{\sim}{x}}(t)\} + \hat{\underset{\sim}{B}}\{\underset{\sim}{u}(t)\} \qquad (2.29)$$

---

[5]The condition of distinct eigenvalues is sufficient
but not necessary. (Rogers, 1971; p. 412.)

where

$$\{\hat{\underset{\sim}{x}}(t)\} = \underset{\sim}{P}^{-1}\{\underset{\sim}{x}(t)\}$$

$$\underset{\sim}{\Lambda} = \underset{\sim}{P}^{-1}\underset{\sim}{G}\underset{\sim}{P}$$

$$\underset{\sim}{B} = \underset{\sim}{P}^{-1}\underset{\sim}{B} \quad , \qquad \text{rank } (\hat{\underset{\sim}{B}}) = K \leq N \quad .$$

$\underset{\sim}{\Lambda}$ is the diagonal matrix of eigenvalues of $\underset{\sim}{G}$.

We now use the result that system controllability is unaffected by any equivalence transformation of the state (Wolovich, 1974; p. 76). The system (2.3a) is controllable if and only if (2.29) is controllable. With $\underset{\sim}{\Lambda}$ diagonal, an element $\hat{x}_i(t + 1)$ is only affected by $\hat{x}_i(t)$ and is uncoupled from $\hat{x}_j(t)$, $j \neq i$. Therefore, a control of $\hat{x}_i(t + 1)$ requires that

$$\{\hat{\underset{\sim}{b}}_i\}' \{\underset{\sim}{u}(t)\} \neq 0$$

where $\{\hat{\underset{\sim}{b}}_i\}'$ is the i-th row of $\hat{\underset{\sim}{B}}$. The vector $\{\hat{\underset{\sim}{b}}_i\}'$ must have at least one nonzero element. Preston (1974; p. 69) labels the condition that there exists at least one nonzero element in each row of the transformed instrument coefficient matrix $\hat{\underset{\sim}{B}} = \underset{\sim}{P}^{-1}\underset{\sim}{B}$ as the coupling criterion. The coupling criterion is an alternative condition for the controllability of the dynamic system (2.3a).

In order to separate the controllable part of a system, it is not necessary to compute all the eigenvalues and eigenvectors. An alternative transformation is given by MacFarlane (1970; pp. 466-469). It starts out from the matrix:

$$\underset{\sim}{D} = [\underset{\sim}{B} \vdots \underset{\sim}{G}\underset{\sim}{B} \vdots \cdots \vdots \underset{\sim}{G}^{N-1}\underset{\sim}{B}] \quad . \tag{2.19}$$

Define $\underset{\sim}{S}$ as the $N \times N_k$ matrix obtained by selecting from left to right as many linearly independent columns of $\underset{\sim}{D}$ as possible. The column vectors of $\underset{\sim}{S}$ span the controllable subspace of the target space, and any vector in this subspace can be expressed as a linear combination of these basic vectors. If the system is controllable, $\underset{\sim}{S}$ is of full rank, i.e. $N = N_k$. If the system is only controllable in part, $N_k < N$. Define any $N \times (N - N_k)$ matrix $\underset{\sim}{X}$ such that

$$\underset{\sim}{T} = [\underset{\sim}{S} \ \underset{\sim}{X}]$$

is nonsingular. Then

$$\underset{\sim}{T}^{-1} = \begin{bmatrix} \underset{\sim}{Y} \\ \underset{\sim}{W} \end{bmatrix}$$

$$\underset{\sim}{T}^{-1}\underset{\sim}{T} = \begin{bmatrix} \underset{\sim}{Y} \\ \underset{\sim}{W} \end{bmatrix} [\underset{\sim}{S} \ \underset{\sim}{X}] = \begin{bmatrix} \underset{\sim}{Y}\underset{\sim}{S} & \underset{\sim}{Y}\underset{\sim}{X} \\ \underset{\sim}{W}\underset{\sim}{S} & \underset{\sim}{W}\underset{\sim}{X} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{I}_{(N_k)} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{I}_{(N - N_k)} \end{bmatrix} \quad .$$

Hence, $\underset{\sim}{Y}$ and $\underset{\sim}{W}$ satisfy the conditions

$$\underset{\sim}{Y}\underset{\sim}{S} = \underset{\sim}{I} \qquad\qquad \underset{\sim}{Y}\underset{\sim}{X} = \underset{\sim}{0}$$

$$\underset{\sim}{W}\underset{\sim}{S} = \underset{\sim}{0} \qquad\qquad \underset{\sim}{W}\underset{\sim}{X} = \underset{\sim}{I} \quad .$$

And the dynamic system

$$\{\underset{\sim}{x}(t + 1)\} = \underset{\sim}{G}\{\underset{\sim}{x}(t)\} + \underset{\sim}{B}\{\underset{\sim}{u}(t)\} \tag{2.3a}$$

is transformed to

$$\underset{\sim}{T}^{-1}\{\underset{\sim}{x}(t + 1)\} = \underset{\sim}{T}^{-1}\underset{\sim}{G}\underset{\sim}{T}\underset{\sim}{T}^{-1}\{\underset{\sim}{x}(t)\} + \underset{\sim}{T}^{-1}\underset{\sim}{B}\{\underset{\sim}{u}(t)\}$$

$$\{\hat{\underset{\sim}{x}}(t + 1)\} = \underset{\sim}{T}^{-1}\underset{\sim}{G}\underset{\sim}{T}\{\hat{\underset{\sim}{x}}(t)\} + \underset{\sim}{T}^{-1}\underset{\sim}{B}\{\underset{\sim}{u}(t)\}$$

$$\{\hat{\underset{\sim}{x}}(t + 1)\} = \begin{bmatrix} \underset{\sim\sim}{YGS} & \underset{\sim\sim}{YGX} \\ \underset{\sim\sim}{WGS} & \underset{\sim\sim}{WGX} \end{bmatrix} \{\hat{\underset{\sim}{x}}(t)\} + \begin{bmatrix} \underset{\sim}{YB} \\ \underset{\sim}{WB} \end{bmatrix} \{\underset{\sim}{u}(t)\} \quad .(2.30)$$

It already has been stated that the controllability of (2.3a) is not affected by an equivalence transformation. It is also true that the controllable subspace is invariant under the operator $\underset{\sim}{G}$. Therefore, for any vector $\{\underset{\sim}{s}_i\}$ in the controllable subspace, the vector $\underset{\sim}{G}\{\underset{\sim}{s}_i\}$ must lie in the same subspace. However, since $\underset{\sim}{W}\underset{\sim}{S} = \underset{\sim}{0}$, the rows of $\underset{\sim}{W}$ are orthogonal to the columns of $\underset{\sim}{S}$, and, therefore, to any vector lying in the subspace spanned by the columns of $\underset{\sim}{S}$. This implies

$$\underset{\sim\sim}{WGS} = \underset{\sim}{0} \quad .$$

The column vectors of $\underset{\sim}{B}$ also belong to the controllable subspace spanned by $\underset{\sim}{S}$, so that

$$\underset{\sim}{WS} = \underset{\sim}{0} \quad .$$

We may write

$$\begin{bmatrix} \{\hat{\underset{\sim}{x}}_1(t + 1)\} \\ \{\hat{\underset{\sim}{x}}_2(t + 1)\} \end{bmatrix} = \begin{bmatrix} \underset{\sim\sim}{YGS} & \underset{\sim\sim}{YGX} \\ \underset{\sim}{0} & \underset{\sim\sim}{WGX} \end{bmatrix} \begin{bmatrix} \{\hat{\underset{\sim}{x}}_1(t)\} \\ \{\hat{\underset{\sim}{x}}_2(t)\} \end{bmatrix} + \begin{bmatrix} \underset{\sim}{YB} \\ \underset{\sim}{0} \end{bmatrix} \{\underset{\sim}{u}(t)\} \quad .$$

$$(2.31)$$

It follows that the controllable part of the system is given
by

$$\{\hat{\underset{\sim}{x}}_1(t + 1)\} = \underset{\sim\sim\sim}{YGS}\{\hat{\underset{\sim}{x}}_1(t)\} + \underset{\sim\sim}{YB}\{\underset{\sim}{u}(t)\} \tag{2.32}$$

where $\{\hat{\underset{\sim}{x}}_1(t)\}$ has dimension $N_k \times 1$. The vector $\underset{\sim\sim\sim}{YGX}\{\hat{\underset{\sim}{x}}_2(t)\}$ can
be treated as a known disturbance. The $(N - N_k)$ dimensional
subsystem defined by the remaining rows of (2.31), namely

$$\{\hat{\underset{\sim}{x}}_2(t + 1)\} = \underset{\sim\sim\sim}{WGX}\{\hat{\underset{\sim}{x}}_2(t)\} \tag{2.33}$$

is completely independent of $\{\underset{\sim}{u}(t)\}$, and therefore is
uncontrollable.

d. Achieving the Targets with a Minimal Number of Instruments

Applying the above transformations to uncouple the
controllable part of the system from the uncontrollable
part, an important question in policy-making may be answered:
what is the minimal number of dynamic instruments, needed
to steer the system towards a set of targets. Consider
(2.3a). Assume that only one instrument $u_i(t)$ is used in
policy implementation. Whether this instrument can transfer
the system from $\{\underset{\sim}{x}(0)\}$ to $\{\underset{\sim}{x}(T)\}$ depends on the control-
lability of the subsystem

$$\{\underset{\sim}{x}(t + 1)\} = \underset{\sim}{G}\{\underset{\sim}{x}(t)\} + \{\underset{\sim}{b}_i\} u_i(t) \tag{2.34}$$

where $\{\underset{\sim}{b}_i\}$ is the i-th column of the matrix B. The system
(2.34) will be controllable with the i-th instrument if

the matrix

$$Q = [\{b_i\} \ G\{b_i\} \ G^2\{b_i\} \ \ldots \ G^{N-1}\{b_i\}]$$

is of rank N.

In terms of the coupling criterion, the existence condition is that

$$\{\hat{b}_i\} = P^{-1}\{b_i\}$$

contains N nonzero elements, since the zero elements indicate the noncontrollable part of the system. A zero element occurs in $\{\hat{b}_i\}$ whenever a row of P^{-1} is orthogonal to the vector $\{b_i\}$. The rows of P^{-1} are the normalized left eigenvectors of G.[6] Zero elements in $\{\hat{b}_i\}$ are precluded if and only if $\{b_i\}$ is linearly dependent on all the N eigenvectors of G.[7] Preston (1970; p. 70) refers to this condition as the eigenvector condition. If there exists one instrument that does not violate the eigenvector condition, the system can be controlled by just one instrument. If no instrument satisfies the eigenvector condition, a combination of instruments may still satisfy the coupling criterion, if their nonzero elements mutually offset the zero elements that disqualify them individually. Therefore, the minimal set

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[6] The first row of P^{-1} has special meaning in demography. It shows the reproductive values of the population. (Keyfitz, 1968; p. 53.)

[7] This may be compared with the possibility of writing the observed population distribution as a linear combination of the right eigenvectors of G. (Keyfitz, 1968; p. 56.)

of instruments necessary and sufficient for dynamic control-lability is equal to the number of columns, $\hat{k}$, of the smallest $\hat{B}_{\tilde{k}}$ matrix possessing N nonzero rows.

The result that under certain circumstances defined by $\hat{\tilde{B}}$, all the targets can be reached by using only one instru-ment, is rather intriguing and is totally contrary to the thinking engendered by the Tinbergen framework. It means, for example, that a desired population distribution over N regions can be realized by having a population policy in only one region. The achievement of the target distribution, however, needs time. From looking at the $\tilde{Q}$-matrix, it is clear that if there is only one instrument, the objective can only be reached after N periods of time[8]. Therefore, there exists a trade-off between the minimal length of the planning horizon and the minimal number of instruments.

If the targets must be reached immediately (T = 1), the minimal number of instruments is N, since

$$\tilde{Q} = [\tilde{G}^0\tilde{B}] = \tilde{B} \qquad (2.35)$$

must be of rank N. Equation (2.35) is the static control-lability condition, discussed earlier.

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[8] It should be remembered that the controllability condition is based on the assumption that no constraints are imposed on the instrument. Constraints would reduce the degrees of freedom associated with dynamic controllability.

CHAPTER 3

DESIGN OF OPTIMAL MIGRATION POLICIES

Any design of optimal policies should begin with a
statement of objectives. Thus far we have focused our
attention on the description of system dynamics by means of
a demometric model. We have answered the question under what
conditions it is possible to specify certain objectives or
targets and to achieve them by the instruments at hand.
Under very specific conditions, there is a unique instrument
vector assuring the achievement of the targets. The optimal
levels of the instrument variables then follow directly. Under
other conditions, however, there is an infinite number of
combinations of the instruments that lead to the desired
targets. In this case, the policy maker is confronted with
an additional decision problem: which alternative set of
instruments to choose. This requires the set-up of a cost
function or welfare loss function which aggregates the
relative costs incurred in the implementation of each instru-
ment. Or the feasible set of instruments may be limited by
imposing constraints on them. A further possibility is that
the objectives are overstated, i.e. that no combination of
instruments can be found that realizes all the targets.
The system is uncontrollable and again the policy maker
has an additional decision to make: where should he modify
his preference system? Is he willing to give up some
targets completely in order to achieve the others, or is
he satisfied with approximating all the targets without
reaching them exactly? This amounts to specifying a welfare
function of the target variables of interest. The

coefficients of the welfare function are the trade-offs
between the target variables. The specification of the
cost and the welfare function is the most difficult and
the most socially sensitive task in the policy design
process. In this paper, we make the assumption that these
functions are given by the policy maker.

This chapter is divided into three sections. The
first discusses the design of optimal policies in the
Tinbergen framework. It will be shown that in some instances
implicit objective functions may be used to derive the
optimal policy. The unifying feature of this section is
the notion of the generalized inverse. The importance of
the minimizing properties of generalized inverses for policy
analysis will be illustrated. The other two sections are
related to the state-space model and consider time series
of controls. The policy problem in which all targets relate
to the planning horizon is discussed in the second section.
The last section treats the policy design in the case that
a target trajectory is given. It applies the theory of
optimal control to migration policy problems.

3.1. DESIGN IN THE TINBERGEN FRAMEWORK

From the previous chapter, we know that an optimal
policy exists if the rank of the impact multiplier matrix
R is equal to the number of targets. The targets may
belong to one time period or to different periods. Follow-
ing Tinbergen, we consider three cases according to the
relationship between the number of targets (N) and the
number of instruments (K) or, equivalently, to the rank of
the multiplier matrix and its singularity property.

3.1.1. The Matrix Multiplier is Nonsingular and of Rank N

Recall equation (1.3):

$$\{y\} = R\{z_1\} + S\{z_2\} \quad . \tag{1.3}$$

If R is nonsingular, then the optimal policy is unique and given by (1.4)

$$\{\bar{z}_1\} = R^{-1}\left[\{\bar{y}\} - S\{z_2\}\right] \tag{1.4}$$

It is clear from (1.4) that the policy depends not only on the target vector, but also on the uncontrollable variables. If $\{z_2\}$ has some lagged endogenous variables, then the effects of past policies will be felt in the current policy.

The nature of the dependence of $\{\bar{z}_1\}$ upon $\{\bar{y}\}$ is associated with different types of structures of the matrix R. They were discussed in Chapter 1. Since there is only one possible set of instruments leading to the target vector $\{\bar{y}\}$, no cost or welfare function is needed to distinguish between alternatives.

3.1.2. The Matrix Multiplier is Singular and of Rank N

If $N < K$, there exists an infinite number of instrument vectors which lead to the achievement of a preassigned value of the target vector. The solution set to (1.3) may be represented by

$$R\{\bar{z}_1\} = \{\bar{y}\} - S\{z_2\}$$

$$\tag{3.1}$$

$$\{\bar{z}_1\} = R^{(1)}\left[\{\bar{y}\} - S\{z_2\}\right] + \left[I - R^{(1)} R\right]\{c\}$$

where $R^{(1)}$ is a generalized inverse of R, satisfying

$$RR^{(1)} R = R$$

and $\{c\}$ is an arbitrary vector.

In order to get a unique instrument vector, one must impose additional conditions on $\{\bar{z}_1\}$. Two illustrations are given of how this may be done. Both minimize a function of $\{z_1\}$ over a constrained set. The first illustration is the formulation of a general mathematical programming problem. The second makes use of the minimizing properties of some types of generalized inverses.

Illustration a: Suppose a cost or welfare loss function $f(\{z_1\})$ has been defined. One wants to minimize this function subject to the dynamic behavior of the system and to some other constraints imposed upon the instrument vector and represented by the vector-valued inequality $g(\{z_1\}) \geq 0$. The problem then may be formulated as a mathematical programming problem,

$$\min f(\{z_1\})$$

subject to

$$\{\bar{y}\} = R\{z_1\} + S\{z_2\} \tag{3.2}$$

$$g(\{z_1\}) \geq 0 \quad .$$

If $g(\{z_2\})$ and $f(\{z_1\})$ are both linear, the problem is a linear programming problem and can be solved by the simplex technique.

Illustration b: This illustration is a special case
of the problem (3.2). We delete the constraint $g(\{z_1\}) \geq 0$,
and we let $f(\{z_1\})$ be the Euclidean norm defined on $\{z_1\}$,
i.e.,

$$f(\{z_1\}) = \left[\{z_1\}' \{z_1\} \right]^{\frac{1}{2}} . \tag{3.3}$$

Ben-Israel and Greville (1974; p. 114) prove that the unique
solution to this problem is given by

$$\{z_1\} = R^{(1,4)} \left[\{\bar{y}\} - S\{z_2\} \right] \tag{3.4}$$

where $R^{(1,4)}$ is a generalized inverse satisfying

$$RR^{(1,4)} R = R$$

and

$$\left[R^{(1,4)} R \right]' = \left[R^{(1,4)} R \right] .$$

Because $R^{(1,4)}$ defines a minimum norm solution to (1.3),
it is often called the "minimum-norm inverse."

There may be other norms defined on the instrument
vector. Suppose the policy maker lists some most acceptable
values of the instrument variables $\{\bar{z}_1\}$, and wants to
minimize the squared deviation between the optimal values
and these preassigned values. The policy model is then

$$\min f(\{z_1\}) = ||\{z_1\} - \{\bar{z}_1\}|| = \left[\left[\{z_1\} - \{\bar{z}_1\}\right]' \left[\{z_1\} - \{\bar{z}_1\}\right]\right]^{\frac{1}{2}}$$

(3.5)

$$\text{s.t. } \{\bar{y}\} = R\{z_1\} + S\{z_2\} \quad .$$

The optimal solution is given by

$$\{z_1\} = R^{(1,4)}\left[\{\bar{y}\} - S\{z_2\}\right] + \left[I - R^{(1,4)} R\right] \{\bar{z}_1\} \quad .$$

(3.6)

The matrix $R^{(1,4)}$ has a special meaning for policy analysis. An element $r_{ij}^{(1,4)}$ indicates the change in the i-th instrument variable required for a unit change in the j-th target variable, assuming that $\{z_2\}$, and, in the second case, also $\{\bar{z}_1\}$ remain unchanged. It is, therefore, a multiplier in the economic sense, measuring the relative effectiveness of the i-th instrument.

3.1.3. The Matrix Multiplier is Singular and of Rank K

If $N > K$, the system (1.3) is inconsistent and no solution exists, i.e. the residual vector $\{r\}$ is nonzero, where

$$\{r\} = \left[\{\bar{y}\} - S\{z_2\}\right] - R\{z_1\} = \{\bar{y}\} - \{\hat{y}\}$$

where $\{\hat{y}\}$ is the realized value of the target vector.

In this case, it is common to search for an approximate solution of (1.3), which makes $\{r\}$ closest to zero in some sense. Again two illustrations will be given. As before, the first is a mathematical programming model, namely, a quadratic programming model, and the second applies the minimizing properties of some generalized inverses.

Illustration a: Theil (1964; p. 159) was the first to assume that a policy-maker, confronted with an over-statement of his goals set, i.e. N > K, formulates his preferences as a quadratic function of the target and control variables. The Theil model has been given in Chapter 1 without proposing a solution to it. Recall the model

$$\min W(\{z_1\}) = \{a\}'\{z_1\} + \{b\}'\{\hat{y}\} + \frac{1}{2}\left[\{z_1\}'A\{z_1\}\right.$$

(1.7)

$$\left. + \{\hat{y}\}'B\{\hat{y}\} + \{z_1\}'C\{\hat{y}\} + \{\hat{y}\}'C'\{z_1\}\right]$$

$$\text{s.t. } \{\hat{y}\} = R\{z_1\} + S\{z_2\}$$

(1.3)

where A, B, C are symmetric positive definite weight matrices. This optimization problem may be solved by means of the Lagrangean technique. An alternative method of deriving the optimum consists of using the constraints to eliminate the target vector in the objective function and then minimizing this function unconditionally with respect to the instruments (Theil, 1964; pp. 40-41). This solution procedure is also followed by Friedman (1975; pp. 159). Substituting the constraint in the objective function gives

$$W(\{z_1\}) = K_0 + \{k\}'\{z_1\} + \frac{1}{2}\{z_1\}'K\{z_1\}$$

(3.7)

where

$$K_0 = \{b\}'S\{z_2\} + \frac{1}{2}\left[[S\{z_2\}]'B[S\{z_2\}]\right]$$

$$\{\underset{\sim}{k}\} = \{\underset{\sim}{a}\} + \underset{\sim}{R}'\{\underset{\sim}{b}\} + [\underset{\sim}{C} + \underset{\sim}{R}'\underset{\sim}{B}][\underset{\sim}{S}\{\underset{\sim}{z}_2\}]$$

$$\underset{\sim}{K} = \underset{\sim}{A} + \underset{\sim}{R}'\underset{\sim}{B}\underset{\sim}{R} + \underset{\sim}{C}\underset{\sim}{R} + \underset{\sim}{R}'\underset{\sim}{C}' \quad .$$

The first order condition for minimizing $W(\{\underset{\sim}{z}_1\})$ with respect to the instrument vector $\{\underset{\sim}{z}_1\}$ is

$$\frac{dW(\{\underset{\sim}{z}_1\})}{d\{\underset{\sim}{z}_1\}} = \{\underset{\sim}{0}\} = \{\underset{\sim}{k}\} + \underset{\sim}{K}\{\underset{\sim}{z}_1\} \quad .$$

The optimal solution follows immediately

$$\{\underset{\sim}{z}_1\} = -\underset{\sim}{K}^{-1}\{\underset{\sim}{k}\} \tag{3.8}$$

where $\underset{\sim}{K}$ and $\{\underset{\sim}{k}\}$ are as defined in (3.7). The second order condition for the minimization of $W(\{\underset{\sim}{z}_1\})$ with respect to $\{\underset{\sim}{z}_1\}$ is that $\underset{\sim}{K}$ is positive definite. The corresponding value of the target vector is

$$\{\underset{\sim}{\hat{y}}\} = \underset{\sim}{R}\underset{\sim}{K}^{-1}\{\underset{\sim}{k}\} + \underset{\sim}{S}\{\underset{\sim}{z}_2\} \quad . \tag{3.9}$$

It should be noted that a nontrivial solution to (3.9) exists only if $\{\underset{\sim}{k}\}$ is nonzero.

Illustration b: Suppose the policy maker only wants to minimize $\{\underset{\sim}{r}\}$. The model may be considered as a variant of the Theil model.

$$\min \left[[\{\underset{\sim}{\bar{y}}\} - \{\underset{\sim}{\hat{y}}\}]'[\{\underset{\sim}{\bar{y}}\} - \{\underset{\sim}{\hat{y}}\}] \right]^{\frac{1}{2}} \tag{2.43}$$

$$\text{s.t.} \quad \{\underset{\sim}{\hat{y}}\} = \underset{\sim}{R}\{\underset{\sim}{z}_1\} + \underset{\sim}{S}\{\underset{\sim}{z}_2\} \quad .$$

The objective function defines the Euclidean norm of $\{\underset{\sim}{r}\}$. Ben-Israel and Greville (1974; p. 104) show that the optimal solution to this problem is given by:

$$\{\underset{\sim}{z}_1\} = \underset{\sim}{R}^{(1,3)} [\{\underset{\sim}{\bar{y}}\} - \underset{\sim}{S}\{\underset{\sim}{z}_2\}] \tag{3.10}$$

where $\underset{\sim}{R}^{(1,3)}$ is the generalized inverse of $\underset{\sim}{R}$ satisfying

$$\underset{\sim}{R}\underset{\sim}{R}^{(1,3)} \underset{\sim}{R} = \underset{\sim}{R}$$

$$\left[\underset{\sim}{R}\underset{\sim}{R}^{(1,3)}\right]' = \underset{\sim}{R}\underset{\sim}{R}^{(1,3)} \quad .$$

Because of the property that $\underset{\sim}{R}^{(1,3)}$ minimizes the Euclidean norm of the residual vector, i.e., the sum of squares of the residuals, it is called the "least-squares inverse." An element $r_{ij}^{(1,3)}$ indicates how much the i-th instrument has to change for a unit change in the j-th target variable, in order to maintain the smallest sum of squared deviations between the realized and the preassigned values of the target variables. The general least-squares solution is

$$\{\underset{\sim}{z}_1\} = \underset{\sim}{R}^{(1,3)}\left[\{\underset{\sim}{\bar{y}}\} - \underset{\sim}{S}\{\underset{\sim}{z}_2\}\right] + \left[\underset{\sim}{I} - \underset{\sim}{R}^{(1,3)} \underset{\sim}{R}\right] \{\underset{\sim}{c}\} \tag{3.11}$$

where $\{\underset{\sim}{c}\}$ is an arbitrary K x 1 vector.

Ben-Israel and Greville note that the least-squares solution is unique only when $\underset{\sim}{R}$ is of full column rank. This condition is always satisfied in policy models discussed here, since we have assumed initially that the instruments are linearly independent.

This illustration shows that the least-squares generalized inverse is the solution to a special variant of the Theil model. A similar observation has recently been made by Russell and Smith (1975; p. 143).

3.2 DESIGN IN THE STATE-SPACE FRAMEWORK: FIXED TARGETS AT THE PLANNING HORIZON

Consider the discrete system

$$\{x(t + 1)\} = G\{x(t)\} + B\{u(t)\} \tag{2.3a}$$

$$\{y(t)\} = C\{x(t)\} \tag{2.3b}$$

where $\{x(t)\}$ is the population distribution at time t. It can be the age distribution, the regional distribution, or both.

$\{y(t)\}$ is any policy relevant measure dependent on the population distribution.

$\{u(t)\}$ is the intervention vector, control or instrument vector at time t.

G is the N x N growth matrix without intervention.

B is the N x K dynamic impact multiplier matrix.

C is the P x N conversion matrix.

In the following, we make the simplifying assumption that C is the identity matrix. The solution to (2.3a) for $t_0 = 0$ is

$$\{\underset{\sim}{x}(t)\} = \underset{\sim}{G}^{t}\{\underset{\sim}{x}(0)\} + \sum_{i=0}^{t-1} \underset{\sim}{G}^{(t-1-i)} \underset{\sim}{B}\{\underset{\sim}{u}(i)\} \quad . \tag{2.4}$$

The policy design problem starts out from (2.4) and seeks to answer the question: what is the sequence of control vectors $\{\underset{\sim}{u}(i)\}$, such that, given the initial condition $\{\underset{\sim}{x}(0)\}$ and the assumption of time-invariance of the coefficient matrices, a target vector at the horizon $\{\underset{\sim}{x}(T)\}$ will be reached in an optimal manner. The intermediate states are of no importance in this formulation.

Equation (2.17) may be written as

$$\{\underset{\sim}{x}(T)\} - \underset{\sim}{G}^{T}\{\underset{\sim}{x}(0)\} = [\underset{\sim}{B} \vdots \underset{\sim}{GB} \vdots \cdots \vdots \underset{\sim}{G}^{T-1}\underset{\sim}{B}] \begin{bmatrix} \{\underset{\sim}{u}(T-1)\} \\ \vdots \\ \{\underset{\sim}{u}(1)\} \\ \hline \{\underset{\sim}{u}(0)\} \end{bmatrix} \tag{3.12}$$

$$= \underset{\sim}{D}\{\underset{\sim}{\hat{u}}\} \quad , \quad \text{say} \quad . \tag{3.13}$$

The system is state controllable if the N x KT matrix $\underset{\sim}{D}$ is of rank N, where N is the dimension of the target vector $\{\underset{\sim}{x}(T)\}$. The controllability condition implies that $N \leq KT$. We distinguish two cases: $N = KT$ and $N < KT$.

CASE 1: $N = KT$

In the dynamic policy model, it is the combined magnitude of the number of instruments and the planning horizon that determines the state controllability. In the previous chapter, we saw that a trade-off exists between the minimal length of the planning horizon and the minimal number of instruments. Any target vector may be reached

by only one instrument, provided that the planning horizon
is not less than N. Also, any target vector can be achieved
in only one time period, if the policy maker may handle at
least N instruments[9]. If $N = KT$, and if the instruments
of the different time periods are independent, then $\underset{\sim}{D}$ is
nonsingular, and the unique control sequence is

$$\{\bar{\underset{\sim}{u}}\} = \underset{\sim}{D}^{-1} [\{\bar{\underset{\sim}{x}}(T)\} - \underset{\sim}{G}^{T}\{\underset{\sim}{x}(0)\}] \quad , \tag{3.14}$$

where $\{\bar{\underset{\sim}{x}}(T)\}$ is the target vector at the planning horizon.

CASE 2: $N < KT$

If $\underset{\sim}{D}$ has rank N and is rectangular, then an infinite
number of combinations of the controls leads to the pre-
defined target population. The solution of (3.13) is

$$\{\bar{\underset{\sim}{u}}\} = \underset{\sim}{D}^{(1)} [\{\bar{\underset{\sim}{x}}(T)\} - \underset{\sim}{G}^{T}\{\underset{\sim}{x}(0)\}] + [\underset{\sim}{D}^{(1)}\underset{\sim}{D} - \underset{\sim}{I}] \{\underset{\sim}{c}\} \tag{3.15}$$

where $\{\underset{\sim}{c}\}$ is arbitrary,
and $\underset{\sim}{D}^{(1)}$ is a generalized inverse of $\underset{\sim}{D}$.

In order to find a unique policy, the policy maker may
minimize a cost function of the instruments, he may put
constraints on the instruments, or he may do both. The
introduction of a cost function will be discussed at the
end of this section. First, we deal with the imposition

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[9]This is exactly the controllability condition
derived by Tinbergen for a static policy model.

of constraints on the instruments to ensure uniqueness of
the instrument vector. The idea is to reduce the degrees
of freedom of the policy measures by making the instrument
vector at a certain time period depend on the controls
exercised at previous time periods. Two reduction methods
are distinguished. The first formulates the control vector
at time t as a linear combination of the control vector
at t - 1. This implies that the control at t may be direct-
ly related to the control vector at the initial time period.
Therefore, we call this the <u>initial period control</u>. This
method has been developed by Rogers (1966; 1968, Chapter 6;
1971; pp. 98-108) for migration policy purposes. The
second reduction method, known as <u>feedback control</u>, makes
the control vector at time t a linear function of the state
vector at the same time t.

3.2.1. Initial Period Control

Suppose that the control vector at time period t is

$$\{\underset{\sim}{u}(t)\} = \underset{\sim}{W}\{\underset{\sim}{u}(t - 1)\} \qquad (3.16)$$

where $\underset{\sim}{W}$ is nonsingular, fixed, and known.

Recall the state-space model of (2.3a):

$$\{\underset{\sim}{x}(t + 1)\} = \underset{\sim}{G}\{\underset{\sim}{x}(t)\} + \underset{\sim}{B}\{\underset{\sim}{u}(t)\} \quad . \qquad (2.3a)$$

Its solution is given by (2.4). Let t = T be the planning
horizon, then

$$\{\underset{\sim}{x}(T)\} = \underset{\sim}{G}^T\{\underset{\sim}{x}(0)\} + \sum_{i=0}^{T-1} \underset{\sim}{G}^{T-1-i} \underset{\sim}{B}\{\underset{\sim}{u}(i)\} \quad . \qquad (3.17)$$

But

$$\{\underset{\sim}{u}(i)\} = \underset{\sim}{W}\{\underset{\sim}{u}(i - 1)\} = \underset{\sim}{W}^i\{\underset{\sim}{u}(0)\} \quad .$$

Therefore (3.17) becomes

$$\{\underset{\sim}{x}(T)\} = \underset{\sim}{G}^T\{\underset{\sim}{x}(0)\} + \left[\sum_{i=0}^{T-1} \underset{\sim}{G}^{T-1-i} \underset{\sim}{B}\underset{\sim}{W}^i \right] \{\underset{\sim}{u}(0)\}$$

and

$$\{\underset{\sim}{x}(T)\} - \underset{\sim}{G}^T\{\underset{\sim}{x}(0)\} = \left[\sum_{i=0}^{T-1} \underset{\sim}{G}^{T-1-i} \underset{\sim}{B}\underset{\sim}{W}^i \right] \{\underset{\sim}{u}(0)\} \quad . \qquad (3.18)$$

Let

$$\underset{\sim}{A}(T) = \sum_{i=0}^{T-1} \underset{\sim}{G}^{T-1-i} \underset{\sim}{B}\underset{\sim}{W}^i \quad , \qquad \text{then}$$

equation (3.18) may be written as

$$\{\underset{\sim}{x}(T)\} - \underset{\sim}{G}^T\{\underset{\sim}{x}(0)\} = \underset{\sim}{A}(T)\{\underset{\sim}{u}(0)\} \qquad (3.19)$$

which is in fact the formulation of the Tinbergen model, with $\{\underset{\sim}{x}(T)\}$ the target vector, $\{\underset{\sim}{x}(0)\}$ the vector of uncontrollable variables and $\{\underset{\sim}{u}(0)\}$ the control vector. The multiperiod problem (3.17) with the target vector given for the planning horizon, and with the control vector at each

time period being a linear combination of the control
vector at the initial time period, is in fact a single-
period problem. Only the control at the initial time
period must be specified. The existence and the uniqueness
of $\{\underset{\sim}{u}(0)\}$ depends only on the rank of $\underset{\sim}{B}$ and is independent
of the choice of T. If $\underset{\sim}{B}$ is nonsingular, then the unique
and optimal value of $\{\underset{\sim}{u}(0)\}$ is given by

$$\{\underset{\sim}{u}(0)\} = \underset{\sim}{A}^{-1}(\underset{\sim}{T}) \left[\{\underset{\sim}{x}(T)\} - \underset{\sim}{G}^T\{\underset{\sim}{x}(0)\}\right] \quad . \tag{3.20}$$

A special case of the initial period control is discussed
by Rogers (1971; pp. 99-100). Suppose that $\underset{\sim}{W}$ is a scalar
matrix, i.e.

$$\underset{\sim}{W} = w\underset{\sim}{I}$$

with w being a scalar. It means that the controls change
in time at a constant rate. Equation (3.18) may be
rewritten as

$$\{\underset{\sim}{x}(T)\} - \underset{\sim}{G}^T\{\underset{\sim}{x}(0)\} = \left[\sum_{i=0}^{T-1} w^{T-1-i}\underset{\sim}{G}^i\underset{\sim}{B}\right]\{\underset{\sim}{u}(0)\} \quad .$$

Premultiplying both sides with $(w\underset{\sim}{I} - \underset{\sim}{G})$ gives

$$(w\underset{\sim}{I} - \underset{\sim}{G}) \left[\{\underset{\sim}{x}(T)\} - \underset{\sim}{G}^T\{\underset{\sim}{x}(0)\}\right] = (w\underset{\sim}{I} - \underset{\sim}{G}) \left[\sum_{i=0}^{T-1} w^{T-1-i}\underset{\sim}{G}^i\underset{\sim}{B}\right]\{\underset{\sim}{u}(0)\}$$

$$= \left[\sum_{i=0}^{T-1} w^{T-i}\underset{\sim}{G}^i - \sum_{i=0}^{T-1} w^{T-1-i}\underset{\sim}{G}^{i+1}\right] \underset{\sim}{B}\{\underset{\sim}{u}(0)\}$$

$$= \left[w^T\underset{\sim}{I} + w^{T-1}\underset{\sim}{G} + w^{T-1}\underset{\sim}{G}^2 + \ldots - w^{T-1}\underset{\sim}{G} - w^{T-1}\underset{\sim}{G}^2 \ldots\right.$$

$$\left. - w^0\underset{\sim}{G}^T\right] \underset{\sim}{B}\{\underset{\sim}{u}(0)\}$$

$$= \left[w^T \underset{\sim}{I} - \underset{\sim}{G}^T \right] \underset{\sim}{B} \{ \underset{\sim}{u}(0) \} \quad .$$

Therefore

$$(w\underset{\sim}{I} - \underset{\sim}{G}) \left[\{ \underset{\sim}{x}(T) \} - \underset{\sim}{G}^T \{ \underset{\sim}{x}(0) \} \right] = \left[w^T \underset{\sim}{I} - \underset{\sim}{G}^T \right] \underset{\sim}{B} \{ \underset{\sim}{u}(0) \}$$

$$\tag{3.21}$$

$$\{ \underset{\sim}{x}(T) \} = \underset{\sim}{G}^T \{ \underset{\sim}{x}(0) \} + (w\underset{\sim}{I} - \underset{\sim}{G})^{-1} (w^T \underset{\sim}{I} - \underset{\sim}{G}^T) \underset{\sim}{B} \{ \underset{\sim}{u}(0) \}$$

and, given that $\underset{\sim}{B}$ is nonsingular,

$$\{ \underset{\sim}{u}(0) \} = \underset{\sim}{B}^{-1} (w^T \underset{\sim}{I} - \underset{\sim}{G}^T)^{-1} (w\underset{\sim}{I} - \underset{\sim}{G}) \left[\{ \underset{\sim}{x}(T) \} - \underset{\sim}{G}^T \{ \underset{\sim}{x}(0) \} \right]$$

$$\tag{3.22}$$

which is in fact also a single-period problem:

$$\{ \bar{\underset{\sim}{u}} \} = \hat{\underset{\sim}{A}}(T) \{ \underset{\sim}{x}(T) \} - \{ \underset{\sim}{a}(T) \} \tag{3.23}$$

where

$$\hat{\underset{\sim}{A}}(T) = \underset{\sim}{B}^{-1} (w^T \underset{\sim}{I} - \underset{\sim}{G}^T)^{-1} (w\underset{\sim}{I} - \underset{\sim}{G})$$

$$\{ \underset{\sim}{a}(T) \} = \underset{\sim}{B}^{-1} (w^T \underset{\sim}{I} - \underset{\sim}{G}^T)^{-1} (w\underset{\sim}{I} - \underset{\sim}{G}) \underset{\sim}{G}^T \{ \underset{\sim}{x}(0) \} \quad .$$

The special case, $w = 1$, is the intervention model of Rogers (1971; pp.99-100) with constant policy.

We now consider two illustrations of the initial period control model. We will assume that $\underset{\sim}{W}$ is equal to the identity matrix. The constant instrument vector is given by

$$\underset{\sim}{B} \{ \bar{\underset{\sim}{u}} \} = (\underset{\sim}{I} - \underset{\sim}{G}^T)^{-1} (\underset{\sim}{I} - \underset{\sim}{G}) \left[\{ \underset{\sim}{x}(T) \} - \underset{\sim}{G}^T \{ \underset{\sim}{x}(0) \} \right] \tag{3.24}$$

and the target vector is

$$\{x(T)\} = G^T\{x(0)\} + (I - G)^{-1} (I - G^T) B\{\bar{u}\} \quad . \quad (3.25)$$

The first illustration is the stationary population model and the second is the pure migration model.

Illustration a: Stationary Population Model.

In the literature on zero population growth, it is emphasized that an immediate reduction of fertility to replacement level will result in a further increase of the population for at least 50 years in most countries. A policy that would keep the population constant at the current level, would result in an unrealistic fluctuation of fertility and mortality rates over the next decades (Coale, 1972; p. 595).

In the multiregional case, keeping total population as well as the population distribution at the current level implies that $\{x(T)\} = \{x(0)\}$, hence we have that

$$\{x(0)\} = G^T\{x(0)\} + (I - G)^{-1} (I - G^T) B\{\bar{u}\}$$

$$(I - G^T) \{x(0)\} = (I - G)^{-1} (I - G^T) B\{\bar{u}\}$$

$$B\{\bar{u}\} = (I - G^T)^{-1} (I - G)(I - G^T) \{x(0)\}$$

$$= \{x(0)\} - (I - G^T)^{-1} G(I - G^T) \{x(0)\}$$

$$= \{x(0)\} - (I - G^T)^{-1} [G - G^{T+1}] \{x(0)\}$$

$$= \{x(0)\} - (I - G^T)^{-1} (I - G^T) G\{x(0)\}$$

$$\underset{\sim}{B}\{\bar{u}\} = (\underset{\sim}{I} - \underset{\sim}{G}) \{\underset{\sim}{x}(0)\} \quad .$$

If $\underset{\sim}{B}$ is nonsingular then

$$\{\bar{\underset{\sim}{u}}\} = \underset{\sim}{B}^{-1}(\underset{\sim}{I} - \underset{\sim}{G}) \{\underset{\sim}{x}(0)\} \quad . \tag{3.26}$$

If $\underset{\sim}{B}$ is singular, we have

$$\{\bar{\underset{\sim}{u}}\} = \underset{\sim}{B}^{(1)}(\underset{\sim}{I} - \underset{\sim}{G}) \{\underset{\sim}{x}(0)\} + [\underset{\sim}{B}^{(1)}\underset{\sim}{B} - \underset{\sim}{I}] \{\underset{\sim}{c}\} \quad . \tag{3.27}$$

The result then may be given as follows: If equation (3.24) has a solution for an arbitrary T, then there exists a constant policy vector $\{\bar{\underset{\sim}{u}}\}$ which keeps the total population as well as its distribution constant at the current level. The vector $\{\bar{\underset{\sim}{u}}\}$ does not depend on the planning horizon, but only on the current population level and distribution.

Illustration b: Pure Migration Model.

The procedure to compute the intervention vector is described by Rogers (1971; p. 106) as follows. The migration rates are taken out of the growth matrix and the migration flows are introduced via the control vector $\{\bar{\underset{\sim}{u}}\}$. The new matrix is $\underset{\sim}{S}$. However, an in-migrant with respect to one region is an out-migrant with respect to another region, and therefore net internal migration must be equal to zero. The instruments are not independent. After computing $\{\bar{\underset{\sim}{u}}\}$ by (3.24) with the revised growth matrix $\underset{\sim}{S}$, and a target vector $\{\underset{\sim}{x}(T)\}$ by (3.25), some elements of $\{\bar{\underset{\sim}{u}}\}$ are adjusted such that

$$\{\underset{\sim}{1}\}' \{\bar{\underset{\sim}{u}}\} = 0$$

where $\{\underset{\sim}{1}\}$ is a vector of ones. A change of an element $\bar{u}_i$ implies that the target population of region i will not be reached. $x_i(T)$ becomes uncontrollable. After the adjustment procedure, the revised target population is computed using (3.25).

For an illustration of another approach that draws on the controllability concept, consider (3.25) once again:

$$\{\underset{\sim}{x}(T)\} = \underset{\sim}{G}^T\{\underset{\sim}{x}(0)\} + (\underset{\sim}{I} - \underset{\sim}{G})^{-1} (\underset{\sim}{I} - \underset{\sim}{G}^T) \underset{\sim}{B}\{\underset{\sim}{\bar{u}}\} \qquad (3.25)$$

where $\underset{\sim}{G}$ is the unreduced growth matrix. Any linear constraints on $\{\underset{\sim}{\bar{u}}\}$ may be introduced in (3.25) via $\underset{\sim}{B}$. The idea is similar to the introduction of linear restrictions in the general linear regression model (Johnston, 1972; p. 157). In the general case where $\{\underset{\sim}{\bar{u}}\}$ is unrestricted, $\underset{\sim}{B}$ is the identity matrix.

Suppose the policy problem is to find $\{\underset{\sim}{\bar{u}}\}$ such that $\{\underset{\sim}{x}(T)\} = \{\underset{\sim}{\bar{x}}(T)\}$ is the target vector, and such that the level of the fourth control variable is equal to the sum of the first and the third variable, i.e.

$$u_4 = u_1 + u_3 \quad . \qquad (3.28)$$

Equation (3.25) may then be written as:

$$\{\underset{\sim}{x}(T)\} - \underset{\sim}{G}^T\{\underset{\sim}{x}(0)\} = (\underset{\sim}{I} - \underset{\sim}{G})^{-1} (\underset{\sim}{I} - \underset{\sim}{G}^T) \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & 0 & \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 1 & \\ & & 0 & & & 1 \end{bmatrix} \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \bar{u}_3 \\ \bar{u}_4 \\ \vdots \\ \bar{u}_N \end{bmatrix}$$

$$(3.29)$$

Because of the linear restriction (3.28), $\underset{\sim}{B}$ is no longer of full rank. Since the instrument vector must remain constant in time, (3.25) may be written as

$$\{\underset{\sim}{y}(T)\} = \underset{\sim}{F}\underset{\sim}{B}\{\bar{\underset{\sim}{u}}\} \qquad (3.30)$$

where

$$\{\underset{\sim}{y}(T)\} = \{\bar{\underset{\sim}{x}}(T)\} - \underset{\sim}{G}^T\{\underset{\sim}{x}(0)\}$$

$$\underset{\sim}{F} = (\underset{\sim}{I} - \underset{\sim}{G})^{-1} (\underset{\sim}{I} - \underset{\sim}{G})^T \quad .$$

Equation (3.29) is equivalent to a static policy problem. By Theorem 1, it has a solution if

$$\text{rank} (\underset{\sim}{F}\underset{\sim}{B}) = N \quad .$$

Since $\underset{\sim}{I}$, $\underset{\sim}{G}$ and $\underset{\sim}{G}^T$ are nonsingular, $\underset{\sim}{F}$ is nonsingular. Therefore (Lancaster, 1969; p. 45):

$$\text{rank} (\underset{\sim}{F}\underset{\sim}{B}) = \text{rank} (\underset{\sim}{B}) = N - 1 \quad ,$$

and the system is not controllable[10]. Because the fourth column of $\underset{\sim}{F}\underset{\sim}{B}$ is $\{0\}$, $\bar{u}_4$ may be deleted, and (3.30) becomes

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[10]The fourth column of $\underset{\sim}{F}\underset{\sim}{B}$ is $\{\underset{\sim}{0}\}$.

$$\{\bar{\underset{\sim}{y}}(T)\} = \underset{\sim}{F} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ & & & & \ddots \\ & & & & & 1 \end{bmatrix} \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \bar{u}_3 \\ \bar{u}_5 \\ \vdots \\ \bar{u}_N \end{bmatrix} . \qquad (3.31)$$

The non-controllable part of the system may be determined by the methods described in Chapter 2. If $\{\bar{\underset{\sim}{u}}\}$ is a vector of in-migrations, it is immediately clear that $x_4(T)$, or equivalently $y_4(T)$, cannot be controlled. One may delete $\bar{y}_4(T)$ and the fourth row of $\underset{\sim}{FB}$, giving a new vector $\{\bar{\underset{\sim}{y}}_1(T)\}$ and a new matrix $(\underset{\sim}{FB})_1$ respectively. The instrument vector $\{\bar{\underset{\sim}{u}}\}$ is then found as

$$\{\bar{\underset{\sim}{u}}\} = \left[(\underset{\sim}{FB})_1\right]^{-1} \{\underset{\sim}{y}_1(T)\} . \qquad (3.32)$$

Entering $\{\bar{\underset{\sim}{u}}\}$ in (3.31) gives the value of $\bar{y}_4(T)$, which will not coincide with the target value.

In the pure migration model the net internal migration must add up to zero. The restriction on $\{\bar{\underset{\sim}{u}}\}$ is

$$\{\underset{\sim}{1}\}' \{\bar{\underset{\sim}{u}}\} = 0 . \qquad (3.33)$$

In a two region case, the people leaving one region must enter the other. The incorporation of this constraint in (3.25) or (3.30) yields

$$\{\bar{\underset{\sim}{y}}(T)\} = \underset{\sim}{F} \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{bmatrix}$$

hence $\bar{u}_2$ may be deleted. The system is not controllable.
If the number of regions is greater than two, and the policy
maker is interested in setting a target for only one region,
then various combinations of $\bar{u}_i$'s satisfy the constraint
(3.33). At the planning horizon, the population distribution
over the other regions depends on the combination chosen
initially, i.e. the entries of $\underset{\sim}{B}$.

It has been assumed throughout this section that the
policy maker is willing and able to give up an element of
his target vector for each linear constraint on the instru-
ment variables. By doing so, he makes it possible to
achieve the other target variables exactly. In some
situations, it may be more realistic to assume that he
wants to approximate the target vector as closely as possible
with the restricted instrument vector. The vector $\{\bar{\underset{\sim}{u}}\}$
which minimizes the deviation between the realized $\{\underset{\sim}{y}(T)\}$
and the target $\{\bar{\underset{\sim}{y}}(T)\}$ is

$$\{\bar{\underset{\sim}{u}}\} = \underset{\sim}{B}^{(1,3)} \underset{\sim}{F}^{-1}\{\bar{\underset{\sim}{y}}(T)\} \tag{3.34}$$

where $\underset{\sim}{B}^{(1,3)}$ is the least-squares generalized inverse of
$\underset{\sim}{B}$, defined in (3.10).

3.2.2. Linear Feedback Control

Suppose that the intervention vector at time t is a
linear function of the population distribution:

$$\{\underset{\sim}{u}(t)\} = \underset{\sim}{Z}\{\underset{\sim}{x}(t)\} \qquad . \tag{3.35}$$

Equation (2.3a) may then be written as

$$\{x(t + 1)\} = G\{x(t)\} + BZ\{x(t)\}$$

$$\{x(t + 1)\} = [G + BZ] \{x(t)\} \qquad (3.36)$$

$$\{x(t)\} = [G + BZ]^t \{x(0)\} \quad . \qquad (3.37)$$

Suppose $\{\bar{x}(T)\}$ is the desired population distribution at time T. The problem is to find Z, such that $\{\bar{x}(T)\}$ is a solution of the equation system

$$\{\bar{x}(T)\} = [G + BZ]^T \{x(0)\} \quad . \qquad (3.38)$$

Feedback control changes the growth matrix of the system and, therefore, also its properties at stability. The impact of the linear output feedback control on the stable population characteristics depends on the eigen-values and eigenvectors of $[G + BZ]$. If $[G + BZ]$ is nonnegative and primitive, then we can apply the Perron-Frobenius theorem to this controlled growth matrix. It has a dominant eigenvalue, and a corresponding eigenvector associated with it. The former is the stable growth ratio and the latter represents the stable distribution of the population controlled by an output feedback law. In this regard, a direct application of the feedback control is the determination of the feedback matrix Z, such that the population will converge to a desired stable distribution. Useful algorithms are given by Schulze (1974), Kreisselmeier (1975), and Mahesh and Kumar (1975). We will not elaborate on this aspect of the problem in this study.

Equation (3.35) is known as the <u>linear state variable</u> <u>feedback control law</u> (Wolovich, 1974; p. 195). If the policy authorities choose the value of the policy instruments according to equation (3.35), then their actions cease to represent an external influence, but instead become part of the population system. The feedback control law defines a closed-loop solution to the optimal control problem.

Equation (3.35) is the simplest case of linear state feedback control. It is unrealistic in the sense that it takes all freedom of action out of the hands of the policy-makers. A linear state feedback control of the form

$$\{u(t)\} = Z\{x(t)\} + H\{v(t)\} \quad . \tag{3.39}$$

is certainly more realistic.

Here $\{v(t)\}$ is an external input or a vector of real exogenous variables. The state space representation of the compensated system is obtained by substituting for $\{u(t)\}$ in (2.3)

$$\{x(t + 1)\} = (G + BZ) \{x(t)\} + BH\{v(t)\} \tag{3.40a}$$

$$\{y(t)\} = (C + EZ) \{x(t)\} + EH\{v(t)\} \quad . \tag{3.40b}$$

Instead of a state feedback, one can also imagine a linear output feedback

$$\{u(t)\} = F\{y(t)\} + H\{v(t)\} \quad . \tag{3.41}$$

Substituting (3.41) in (2.3), we obtain

$$\{x(t + 1)\} = G\{x(t)\} + BF\{y(t)\} + BH\{v(t)\}$$

$$\{y(t)\} = C\{x(t)\} + EF\{y(t)\} + EH\{v(t)\}$$

$$(I - EF) \{y(t)\} = C\{x(t)\} + EH\{v(t)\} \quad .$$

If $(I - EF)$ is nonsingular, then

$$\{y(t)\} = [I - EF]^{-1} [C\{x(t)\} + EH\{v(t)\}] \qquad (3.42)$$

$$\{x(t + 1)\} = \left[G + BF[I - EF]^{-1} C \right] \{x(t)\}$$

$$(3.43)$$

$$+ \left[BH + BF[I - EF]^{-1} EH \right] \{v(t)\} \quad .$$

If $\{v(t)\} = \{0\}$, i.e., the dynamics of the system is governed by a pure output feedback, then we have the closed loop system

$$\{x(t + 1)\} = \left[G + BF[I - EF]^{-1} C \right] \{x(t)\} \quad . \qquad (3.44)$$

There are two noteworthy special cases of (3.44):

   i) $E = 0$, i.e., the output $\{y(t)\}$ depends only on
        the state vector $\{x(t)\}$. Equation (3.44) becomes:

$$\{x(t + 1)\} = [G + BFC] \{x(t)\} \quad .$$

ii) $\underset{\sim}{E} = \underset{\sim}{0}$ and $\underset{\sim}{C} = \underset{\sim}{I}$, i.e., the output vector is equal to the state vector. Equation (3.44) then reduces to the expression for a linear state feedback control law

$$\{\underset{\sim}{x}(t + 1)\} = [\underset{\sim}{G} + \underset{\sim\sim}{BF}] \{\underset{\sim}{x}(t)\} \quad . \qquad (3.45)$$

To illustrate the usefulness of the feedback control model for migration policy, we take a policy problem described by Hansen (1974; p. 17). In the last twenty years, central governments of several Western countries have been trying to decrease regional differences in living conditions. A popular strategy to achieve this objective, which is based on equity considerations, was to allocate development funds to lagged regions. The funds allocated by the central government to the regions are a function of its "backwardness." A major indicator of it is the level of out-migration. In order to model this policy, assume that the development funds each region gets at time t is a linear combination of its level of out-migration and the level of out-migration of all the other regions.

Let $\{\underset{\sim}{x}(t)\}$ be the regional population distribution at time t,

$\{\underset{\sim}{u}(t)\}$ be the regional distribution of the development funds,

$\{\underset{\sim}{y}(t)\}$ be the level of out-migration of the regions.

The dynamics of the multiregional system is described by the state-space model

$$\{x(t + 1)\} = G\{x(t)\} + B\{u(t)\} \qquad (2.3a)$$

$$\{y(t)\} = C\{x(t)\} \qquad (2.3b)$$

where G is the population growth matrix,

B is the matrix of impact multipliers. The element b_{ij} is the effect of a dollar allocated to region j at time t on the population of region i at time $t + 1$,

C is a diagonal matrix of out-migration rates.

The policy may be written as a linear output feedback control law

$$\{u(t)\} = F\{y(t)\}$$

where the i-th row of F gives the coefficients of the linear combination between $u_i(t)$ and the regional levels of out-migration. The dynamics of the controlled population system is then given by:

$$\{x(t + 1)\} = [G + BFC] \{x(t)\} \qquad (3.45)$$

where G is a nonnegative matrix,

B has supposedly nonnegative diagonal elements and nonpositive off-diagonal elements,

$\underset{\sim}{F}$ describes the trade-offs set by the policy maker.
It is realistic to assume that the diagonal
elements are positive and most off-diagonal
elements are nonpositive. A positive off-diagonal
element f_{ij} would mean that the funds region i
gets increase with the out-migration of region j.
This is not unrealistic if the out-migrants of j,
who go to i, cause a congestion problem in region
i necessitating additional investments (population
responsive policy).

3.2.3. Horizon Constrained Optimal Control

If the number of target variables at the planning
horizon is less than the product of the number of instruments
and the length of the planning horizon, then there is an
infinite number of combinations of controls leading to the
desired target variables. Suppose, as before, that the
target is the regional population distribution at the
horizon T. All the feasible control vectors are given by
(3.15), which is the general solution to (3.12).

To arrive at a unique instrument vector, the policy
maker may apply the design techniques described under the
Tinbergen framework to this multiperiod situation. The
first technique is based on the minimizing properties of
the generalized inverse. If (3.15) is the general solution
to (3.12), then there is a unique solution which minimizes
the Euclidean norm of the instrument vector $\{\underset{\sim}{u}\}$. This
solution is given by

$$\{\bar{\underset{\sim}{u}}\} = \underset{\sim}{D}^{(1,4)} \left[\{\bar{\underset{\sim}{x}}(T)\} - \underset{\sim}{G}^T \{x(0)\} \right] \qquad (3.46)$$

where $D^{(1,4)}$ is the "minimum norm inverse" of D.

The other approach is to formulate a mathematical programming model, similar to (3.2). However, we have seen in Chapter 1 that by assuming inter-temporal separability of the objectives, and by neglecting the inequality constraint of (3.2), we may write it as an optimal control problem. Assuming, in addition, a quadratic objective functional, the problem becomes identical to (1.19), except for the addition of the horizon constraint,

$$\{x(T)\} = \{\bar{x}(T)\} \; . \tag{3.47}$$

In the literature, this problem is known as the linear-quadratic control problem with zero terminal error or with a right-hand-side constraint. The solution will be discussed in the next section.

3.3 DESIGN IN THE STATE-SPACE FRAMEWORK: TRAJECTORY OPTIMIZATION

In the models discussed in the previous section, the migration policy objectives were formulated only for the planning horizon. It was assumed that the policy-maker did not care about how the target variables converged to their desired values. In order to identify a unique combination of instruments, we have imposed severe restrictions on the path of the control vector. Now, we broaden the perspective by allowing the policy-maker to define a dynamic preference system, i.e., the targets are defined for each time period instead of only one. The instruments may vary more freely

in the sense that no fixed pattern is imposed. The range
of admissible instruments and their variation, however,
may be constrained for economic, political or stability
reasons. The latter means that the inclusion of the
instruments in the policy maker's preference function is
an appropriate way to avoid an excessive fluctuation of
the values of the instruments over time (Holbrook, 1972;
p. 57).

It has been argued in Chapter 1 that, if the policy-
maker seeks to define a time path of the control vector
out of all the feasible trajectories, such that his dynamic
preference system, expressed in the form of a functional,
is optimized, the policy problem becomes very similar to
the optimal control problem. The models presented earlier
may also be encompassed in this framework. In what follows,
we specify a dynamic policy model using the optimal
control technique. This enables us to list the set of
necessary conditions for optimizing the preference func-
tional. These conditions are known as the Pontryagin
minimum (or maximum) principle.

The optimal control problem specified here covers a
wide variety of dynamic policy problems. Its solution
however is at least quite difficult and its interpretation
is not always easy. A frequently used policy model in the
economic literature is the linear-quadratic model[11]. It
is characterized by a quadratic objective function and a

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[11]See, for example, Sengupta (1970), Turnovsky (1971),
Pindyck (1971, 1973a, 1973b), Vishwakarma (1974), Garbade
1975), and Chow (1970, 1972, and 1975, Chapter 9).

linear constraint. This problem formulation is attractive because it allows one to express a direct relation between the control vector and the target vector at each time period, thereby leading to a simple analytic solution of the optimal control problem. It is also interesting because it is a direct extension of the Theil model to dynamic situations.

3.3.1. Specification of the Optimal Control Model

Policy problems of dynamic systems may be solved by the theory of optimal control. The basic ingredients of a discrete optimal control model are:

1) A set of difference equations that represent the system to be controlled. The system is described by a demometric model in state-space notation

$$\{x(t + 1)\} = F(\{x(t)\}, \{u(t)\}, t) \quad , \qquad t = 0,\ldots,T-1 \quad .$$

$$(3.48)$$

In control theory, $\{x(t)\}$ is called the state vector and describes the state of the system. The vector $\{u(t)\}$ is the control vector, and $\{F(\cdot)\}$ is a vector-valued function of dimension T x 1. The equation is known as the state equation or transition equation. Throughout this study, we have dealt with a linear time invariant system, i.e.

$$\{x(t + 1)\} = G\{x(t)\} + B\{u(t)\} \quad . \qquad (2.3a)$$

2) A set of constraints on the state and control variables,

$$g(\{\underset{\sim}{x}(t)\}, \{\underset{\sim}{u}(t)\}, t) \geq \{\underset{\sim}{0}\} \quad . \tag{3.49}$$

where $\{g(\cdot)\}$ is a vector-valued function of dimension M. This function defines the admissible set of state and control variables.

3) A set of boundary conditions. The initial state is given

$$\{\underset{\sim}{x}(0)\} = \{\underset{\sim}{x}_0\} \quad . \tag{3.50}$$

We may also require that at the terminal time, or planning horizon, the state vector satisfies the vector-valued function $\left\{\underset{\sim}{m}(\{\underset{\sim}{x}(T)\})\right\} = \{\underset{\sim}{0}\}$. $\tag{3.51}$

4) A preference functional, welfare functional, cost functional or a performance index which is to be minimized. The functional may be written

$$J = K(\{\underset{\sim}{x}(T)\}) + \sum_{i=0}^{T-1} L(\{\underset{\sim}{x}(t)\}, \{\underset{\sim}{u}(t)\}, t) \tag{3.52}$$

The functional reduces all the utilities and disutilities of the controlled dynamic system to a single scalar. All the functions of the cost functional and of the constraints are assumed to be known and to be continuously differentiable with respect to $\{\underset{\sim}{x}(t)\}$ and $\{\underset{\sim}{u}(t)\}$. Note that the control $\{\underset{\sim}{u}(t)\}$ affects the objective functional both directly and indirectly through the value imparted to the states $\{\underset{\sim}{x}(t + \ell)\}$, $\ell > 0$.

The optimal control problem is formulated now as the
determination of the control sequence {u*(t)} for
t = 0,...,T-1, and the corresponding trajectory of the
state vector {x*(t)} for t = 0,...,T, such that the con-
straints (3.48) and (3.49), and the boundary conditions
(3.50) and (3.51) are satisfied and such that the cost
functional (3.52) is minimized. The sequence {u*(t)} is
then called the optimal control, and {x*(t)} the optimal
trajectory. In other words, the optimal control problem
is to steer a dynamic system, so as to optimize a perfor-
mance index, subject to constraints. This formulation is
very general and explains why the theory pertaining to
its solution has found such a wide range of applications,
and why it is also has relevance for population policy
problems[12].

3.3.2. The Discrete Minimum Principle

We now turn to the necessary conditions for optimality.
Originally, these conditions were derived by Pontryagin
and his associates (1962) for continuous-time systems,
described by differential equations. For a thorough
statement of the Pontryagin minimum principle, the reader
is referred to Athans and Falb (1966). To remain consistent
with the other parts of this study, we will state the
discrete version of the minimum principle. Several
derivations of the discrete time minimum principle have

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[12]For a survey of applications of optimal control
in economic policy planning and of possible extensions,
see Athans and Kendrick (1974) and the two special issues
of the Annals of Economic and Social Measurement (1972,
1974).

appeared in the literature[13]. We will state the principle

without proof, since it may be found in the literature.

The discrete minimum principle: Suppose the sequence

$\{u^*(t)\}$, $t = 0,\ldots,T-1$ constitutes an optimal control and

$\{x^*(t)\}$, $t = 0,\ldots,T$ is an optimal trajectory of the system

described by (3.48), and constrainted by (3.49), (3.50) and

(3.51). In order for $\{u^*(t)\}$, $t = 0,\ldots,T-1$ to minimize

the cost functional (3.52), it is necessary that there

exist a sequence of $N \times 1$ vectors $\{\lambda^*(t)\}$, $t = 1,\ldots,T$,

and a sequence of $M \times 1$ vectors $\{\mu^*(t)\}$, $t = 1,\ldots,T$,

such that the following conditions hold:

1) The scalar function

$$H\left(\{x^*(t)\},\{u(t)\},\{\lambda^*(t + 1)\},\{\mu^*(t + 1)\}\right)$$

$$= L\left(\{x^*(t)\},\{u(t)\},t\right) + \{\lambda^*(t + 1)\}'\{f(\{x^*(t)\},\{u(t)\},t)\}$$

$$- \{\mu^*(t + 1)\}'\{g(\{x^*(t)\},\{u(t)\},t)\} \tag{3.53}$$

is minimized as a function of $\{u(t)\}$ at $\{u(t)\} = \{u^*(t)\}$

for all $t = 0,\ldots,T-1$. This implies that

$$\frac{\delta H}{\delta\{u(t)\}}\Big|_* = \{0\} . \tag{3.54}$$

The vector $\{\lambda(t)\}$ is the co-state vector, and $\{\mu(t)\}$ is

the co-constraint vector. With each difference equation

(3.48) is associated a co-state vector, and with each

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[13]See, for example, Halkin (1964), Holtzman (1966)
and Pindyck (1973a, 1973b).

constraint (3.49) a co-constraint vector[14]. The function $H(\cdot)$ is called the <u>Hamiltonian</u>.

2) The dynamics of $\{\underset{\sim}{x}^*(t)\}$, $\{\underset{\sim}{\lambda}^*(t)\}$ and $\{\underset{\sim}{\mu}^*(t)\}$ are governed by the equations:

$$\{\underset{\sim}{x}^*(t+1)\} = \frac{\delta H}{\delta\{\underset{\sim}{\lambda}(t+1)\}} = \{\underset{\sim}{f}(\{\underset{\sim}{x}^*(t)\},\{\underset{\sim}{u}^*(t)\},t)\} \tag{3.55}$$

$$\{\underset{\sim}{x}^*(0)\} = \{\underset{\sim}{x}_0\} \tag{3.56}$$

$$\{\underset{\sim}{\lambda}^*(t)\} = \frac{\delta H}{\delta\{\underset{\sim}{x}(t)\}} \bigg|_* \tag{3.57}$$

$$\{\underset{\sim}{\lambda}^*(T)\} = \frac{\delta K(\{\underset{\sim}{x}(T)\})}{\delta\{\underset{\sim}{x}(T)\}} \bigg|_* \tag{3.58}$$

$$\{\underset{\sim}{g}(\{\underset{\sim}{x}^*(t)\},\{\underset{\sim}{u}^*(t)\},t)\} = - \frac{\delta H}{\delta\{\underset{\sim}{\mu}(t+1)\}} \bigg|_* \geq 0 \tag{3.59}$$

$$\{\underset{\sim}{\mu}^*(t)\} \geq 0 \tag{3.60}$$

$$\{\underset{\sim}{g}(\{\underset{\sim}{x}^*(t)\},\{\underset{\sim}{u}^*(t)\},t)\}'\{\underset{\sim}{\mu}^*(t+1)\} = 0 \quad . \tag{3.61}$$

Condition (3.55) repeats the difference equation (3.48), and (3.59) is the constraint (3.49). The necessary conditions are essentially equivalent to the Kuhn-Tucker conditions of nonlinear programming. Equations (3.55) and (3.57) are referred to as the canonical difference equations (Athans, 1971; p. 458). Conditions (3.56) and (3.58) are the boundary conditions.

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[14] Co-state and co-constraint variables in optimal control are similar to Lagrange multipliers in function optimization. They may be interpreted as shadow prices associated with the constraints (Pindyck, 1973; pp. 35-38).

Note that the minimum principle yields only necessary conditions for optimality, which are valid locally. Global optimality also reauires sufficiency conditions. These involve the convexity of the functions.

If (3.54) is solved for $\{u(t)\}$ in terms of $\{x(t)\}$, $\{\lambda(t)\}$ and $\{\mu(t)\}$, and if the resulting expression for $\{u(t)\}$ is then substituted into equation (3.48) and (3.57), a two-point boundary value problem results. A number of numerical methods are available for solving these problems. Methods such as steepest descent, conjugate directions, conjugate gradient, quasi-linearization, and the Newton-Raphson method are the best known. A description of these algorithms falls beyond the scope of this study. The interested reader should consult Bryson and Ho (1969, Chapter 7), Sage (1968), McReynolds (1970) or Noton (1972). Noton illustrates his exposition with simple numerical examples. Special algorithms, which fit some specific population policy models, have been developed by Evtushenko and MacKinnon (1975) and by Mehra (1975).

3.3.3. The Linear-Ouadratic Control Problem

The linear-quadratic (LQ) control problem is one of many possible optimal control problems. It deserves special attention because it is the only optimal control problem for which the solution may be expressed analytically, and because it generalized Theil's idea of quadratic objective function with linear constraints. The LQ control problem fits two types of policy problems. In the first,

the policy maker desires to transform an initial state, say
the actual population distribution, to a desired state at
the planning horizon, while exhibiting an acceptable
behavior of the control and state variables on the way.
In the second situation, he tries to keep a system within
an acceptable deviation from a reference condition using
acceptable amounts of control. In both situations, the
optimal control is described by feedback equations known
as <u>terminal controllers</u> and as <u>regulators</u>, respectively
(Bryson and Ho, 1969, Chapter 5).

The basic ingredients of the LQ problem are[15]:

1) A linear state equation,

$$\{\underset{\sim}{x}(t + 1)\} = \underset{\sim}{G}\{\underset{\sim}{x}(t)\} + \underset{\sim}{B}\{\underset{\sim}{u}(t)\} \ . \tag{2.3a}$$

2) The boundary condition,

$$\{\underset{\sim}{x}(0)\} = \{\underset{\sim}{x}_0\} \ . \tag{3.50}$$

The planning horizon T is fixed.

3) A quadratic performance index,

$$J = \frac{1}{2} \{\underset{\sim}{x}(T)\}' \ \underset{\sim}{F}\{\underset{\sim}{x}(T)\} + \frac{1}{2} \sum_{i=0}^{T-1} \left[\{\underset{\sim}{x}(t)\}' \ \underset{\sim}{Q}\{\underset{\sim}{x}(t)\} \right. \tag{3.62}$$

$$\left. + \{\underset{\sim}{u}(t)\}' \ \underset{\sim}{R}\{\underset{\sim}{u}(t)\} \right] \ .$$

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[15]The LQ problem has received much attention in the
literature. See, for example, Bryson and Ho (1969,
Chapter 5), Pindyck (1973; pp. 27-35), Noton (1972;
pp. 158-165) and Bar-Ness (1975; pp. 49-56).

The rationale for the quadratic performance index is identical to the one on which the Theil model is based. To assure the convexity of the objective functional, the matrices $\underset{\sim}{F}$ and $\underset{\sim}{Q}$ are assumed to be positive semi-definite, while $\underset{\sim}{R}$ is positive definite. They may be functions of time. However, the t-index is deleted for convenience. The matrices $\underset{\sim}{F}$, $\underset{\sim}{Q}$ and $\underset{\sim}{R}$ give the weights attached to the state and the control variables. They will normally be diagonal. The matrix $\underset{\sim}{G}$ is N x N and $\underset{\sim}{B}$ is N x K with N the number of targets and K the number of instruments at each period of time.

The optimal control problem is to minimize (3.62) subject to (2.3a) and (3.50). How the LQ model relates to the Theil model and to other policy models has been discussed in Chapter 1. The optimal controls $\{\underset{\sim}{u}*(t)\}$, t = 0,...,T-1 are found by applying the discrete minimum principle. Not all of the necessary conditions listed in the previous paragraph must be met, since there are no inequality constraints. The Hamiltonian is

$$H = \frac{1}{2} \{\underset{\sim}{x}(T)\}' \, \underset{\sim}{F}\{\underset{\sim}{x}(T)\} + \frac{1}{2} \sum_{t=0}^{T-1} \Bigg[\{\underset{\sim}{x}(t)\}'\underset{\sim}{Q}\{\underset{\sim}{x}(t)\} + \{\underset{\sim}{u}(t)\}'\underset{\sim}{R}\{\underset{\sim}{u}(t)\}$$

$$+ \{\underset{\sim}{\lambda}(t+1)\}'[\underset{\sim}{G}\{\underset{\sim}{x}(t)\} + \underset{\sim}{B}\{\underset{\sim}{u}(t)\}] \Bigg] \quad .$$

$$(3.63)$$

where $\{\underset{\sim}{\lambda}(t+1)\}$ is the co-state vector evaluated at period t + 1. From (3.57), we see that $\{\underset{\sim}{\lambda}(t+1)\}$ is the solution of:

$$\{\underset{\sim}{\lambda}(t)\} = \frac{\delta H}{\delta \{\underset{\sim}{x}(t)\}} = \underset{\sim}{Q}\{\underset{\sim}{x}(t)\} + \underset{\sim}{G}'\{\underset{\sim}{\lambda}(t+1)\} \qquad (3.64)$$

or

$$\{\lambda(t + 1)\} - \{\lambda(t)\} = - \underset{\sim}{Q}\{\underset{\sim}{x}(t)\} - [\underset{\sim}{G} - \underset{\sim}{I}]'\{\lambda(t + 1)\}$$

with the final value fixed by (3.58):

$$\{\lambda(T)\} = \frac{\delta}{\delta\{\underset{\sim}{x}(T)\}} \left[\{\underset{\sim}{x}(T)\}'\underset{\sim}{F}\{\underset{\sim}{x}(T)\}\right] = \underset{\sim}{F}\{\underset{\sim}{x}(T)\} \quad . \quad (3.65)$$

Along the optimal trajectory, J and H are minimized with respect to $\{u(t)\}$. The necessary conditions yielding the extremum are:

a) $\quad \dfrac{\delta H}{\delta\{\underset{\sim}{u}(t)\}} = \{\underset{\sim}{0}\} = \underset{\sim}{R}\{\underset{\sim}{u}(t)\} + \underset{\sim}{B}'\{\lambda(t + 1)\} \quad . \quad (3.66)$

b) The constraint (3.3a). This condition is formulated as

$$\{\underset{\sim}{x}(t + 1)\} = \frac{\delta H}{\delta\{\lambda(t + 1)\}} = \{\underset{\sim}{0}\} = \underset{\sim}{G}\{\underset{\sim}{x}(t)\} + \underset{\sim}{B}\{\underset{\sim}{u}(t)\}$$

$$(3.67)$$

with the initial condition

$$\{\underset{\sim}{x}(t_0)\} = \{\underset{\sim}{x}_0\} \quad . \tag{3.68}$$

Since $\underset{\sim}{R}$ is positive definite, we derive from (3.66) the optimal trajectory of the control vector

$$\{\underset{\sim}{u}^*(t)\} = - \underset{\sim}{R}^{-1}\underset{\sim}{B}'\{\lambda(t + 1)\} \quad . \tag{3.69}$$

In order for $\{\underset{\sim}{u}^*(t)\}$ to minimize H, $\underset{\sim}{R}$ must have an inverse

and

$$\frac{\delta^2 H}{\delta\{\underset{\sim}{u}(t)\}^2} = \underset{\sim}{R}$$

must be positive definite. After substituting (3.69) into (3.67), we have a system of 2N first-order difference equations to solve, together with 2N boundary conditions

$$\{\underset{\sim}{\lambda}(t)\} = \underset{\sim}{Q}\{\underset{\sim}{x}(t)\} + \underset{\sim}{G}'\{\underset{\sim}{\lambda}(t + 1)\} \tag{3.64}$$

$$\{\underset{\sim}{x}(t + 1)\} = \underset{\sim}{G}\{\underset{\sim}{x}(t)\} + \underset{\sim}{B}\underset{\sim}{R}^{-1}\underset{\sim}{B}'\{\underset{\sim}{\lambda}(t + 1)\} \tag{3.70}$$

$$\{\underset{\sim}{x}(0)\} = \{\underset{\sim}{x}_0\} \tag{3.68}$$

$$\{\underset{\sim}{\lambda}(T)\} = \underset{\sim}{F}\{\underset{\sim}{x}(T)\} \quad . \tag{3.65}$$

The solution to this two-point boundary-value problem is derived in the Appendix to this part. It starts out from the assumption that there exists a linear relation between $\{\underset{\sim}{\lambda}(t)\}$ and $\{\underset{\sim}{x}(t)\}$ at the optimum:

$$\{\underset{\sim}{\lambda}*(t)\} = \underset{\sim}{K}(t) \{\underset{\sim}{x}*(t)\} \quad . \tag{3.71}$$

The feedback matrix $\underset{\sim}{K}(t)$ is the solution of the Riccati equation. Once $\underset{\sim}{K}(t)$ is known for all t, the trajectory of the state vector is given by

$$\{\underset{\sim}{x}*(t + 1)\} = \left[\underset{\sim}{I}_N - \underset{\sim}{B}[\underset{\sim}{R} + \underset{\sim}{B}'\underset{\sim}{K}(t + 1) \underset{\sim}{B}]^{-1} \underset{\sim}{B}'\underset{\sim}{K}(t + 1)\right] \{\underset{\sim}{x}*(t)\}$$

$$\tag{3.72}$$

and the optimal control, or control law, is ·

$$\{u*(t)\} = - \underset{\sim}{R}^{-1}\underset{\sim}{B}'\underset{\sim}{K}(t + 1) \{\underset{\sim}{x}*(t + 1)\} \qquad (3.73)$$

which gives the control vector in linear state feedback form.
The trajectory of the co-state variables is

$$\{\underset{\sim}{\lambda}*(t)\} = \underset{\sim}{K}(t) \{\underset{\sim}{x}*(t)\} \quad . \qquad (3.74)$$

The optimal value of the cost functional

$$J* = \frac{1}{2} \{\underset{\sim}{x}(0)\}' \underset{\sim}{K}(0) \{\underset{\sim}{x}(0)\} \qquad (3.75)$$

depends only on the initial condition $\{\underset{\sim}{x}(0)\}$ and on $\underset{\sim}{K}(0)$.
The matrix $\underset{\sim}{K}(0)$, however, depends on $\underset{\sim}{G}$, $\underset{\sim}{F}$, $\underset{\sim}{Q}$, $\underset{\sim}{R}$ and $\underset{\sim}{B}$ and
on the feedback matrices $\underset{\sim}{K}(t)$, $t = 1,\ldots,T$.

Some useful extensions of the LQ model have been made.
We present them here as illustrations of how the LQ model
may fit policy problems. The first is the dual tracking
problem where the policy-maker is looking for a regulator
to keep the target and control variables as close as possible
to predefined, most acceptable values. By way of a second
illustration, we take up the horizon constrained optimal
control problem again. Finally, it is shown how the LQ
model may handle additional constraints. The idea is to
assign penalties for the constraints which are not met.

Illustration a: The dual tracking problem.

In most policy applications of the linear quadratic
problem, the objective is to minimize the deviations from
desired values of the target vector and eventually also of

the control vector. Rather than having the objective to minimize a function with the arguments expressed as deviations from zero, we have

$$\min J = \frac{1}{2} \{\hat{x}(T)\}' \, F\{\hat{x}(T)\} +$$

$$+ \frac{1}{2} \sum_{t=0}^{T-1} \left[\{\hat{x}(t)\}' \, Q\{\hat{x}(t)\} + \{\hat{u}(t)\}' \, R\{\hat{u}(t)\} \right] \tag{3.76}$$

where

$$\{\hat{x}(t)\} = \{x(t)\} - \{\bar{x}(t)\}$$

$$\{\hat{u}(t)\} = \{u(t)\} - \{\bar{u}(t)\}$$

with $\{\bar{x}(t)\}$ and $\{\bar{u}(t)\}$ the desired or most acceptable values for the trajectory of the target vector and the control vector, respectively.

The optimum may be found in the same manner as in the original problem.

Illustration b: Zero terminal error problem.

The dual tracking problem may be supplemented by the additional requirement that at the planning horizon some, say $\bar{N}$, of the desired levels of the state or target variables must be met exactly, rather than approximately. This means that the following constraint must hold

$$\{\hat{x}_1(T)\} = \{0\} \tag{3.77}$$

where $\{\hat{x}_1(T)\}$ is a $\bar{N} \times 1$ vector with $\bar{N} \leq N$.

The control problem is now

$$\min \; J = \frac{1}{2} \{\hat{\underset{\sim}{x}}(T)\}' \; \underset{\sim}{F}\{\hat{\underset{\sim}{x}}(T)\} +$$

$$+ \; \sum_{t=0}^{T-1} \left[\{\hat{\underset{\sim}{x}}(t)\}' \; \underset{\sim}{Q}\{\hat{\underset{\sim}{x}}(t)\} + \{\hat{\underset{\sim}{u}}(t)\}' \; \underset{\sim}{R}\{\hat{\underset{\sim}{u}}(t)\} \right] \qquad (3.76)$$

subject to

$$\{\underset{\sim}{x}(t + 1)\} = \underset{\sim}{G}\{\underset{\sim}{x}(t)\} + \underset{\sim}{B}\{\underset{\sim}{u}(t)\} \qquad (2.3a)$$

$$\{\hat{\underset{\sim}{x}}_1(T)\} = \{\underset{\sim}{0}\} \qquad (3.77)$$

and with $\{\underset{\sim}{x}(0)\} = \{\underset{\sim}{x}_0\}$ being given.

This is the exact formulation of the horizon constrained optimal control problem of the previous paragraph. Therefore, policy problems where the target vector is given for the planning horizon, and where the restrictions on the state and control trajectory are not so stringent as those discussed previously, may be formulated as dual tracking problems with zero terminal error.

To form the Hamiltonian, we adjoin equation (2.3a) to J with a multiplier sequence $\{\underset{\sim}{\lambda}(t)\}$, $t = 1,\ldots,T$, and, in addition, we adjoin (3.77) with a set of $\bar{N}$ multipliers $(\nu_1, \nu_2, \ldots, \nu_{\bar{N}}) = \{\underset{\sim}{\nu}\}'$. Thus

$$H = \frac{1}{2} \{\underset{\sim}{x}(T)\}' \; \underset{\sim}{F}\{\underset{\sim}{x}(T)\} + \sum_{t=0}^{T-1} \left[\{\underset{\sim}{x}(t)\}' \underset{\sim}{Q}\{\underset{\sim}{x}(t)\} + \{\underset{\sim}{u}(t)\}' \underset{\sim}{R}\{\underset{\sim}{u}(t)\} \right.$$

$$\left. + \{\underset{\sim}{\lambda}(t + 1)\}' [\underset{\sim}{G}\{\underset{\sim}{x}(t)\} + \underset{\sim}{B}\{\underset{\sim}{u}(t)\}] \right] + \{\underset{\sim}{\nu}\}' \{\underset{\sim}{x}_1(T)\} \qquad .$$

$$(3.78)$$

Application of the minimum principle yields a two-point boundary-value problem. Solution algorithms have been discussed by Bryson and Ho (1969; pp. 158-164) and by Mehra (1975; pp. 12-16).

Illustration c: Sign restriction.

In policy making it is often desirable to restrict a target or a control variable in sign. For example, let $\{u(t)\}$ be the net migrants of each region, and suppose that the policy maker, in addition to his quadratic objective function, would like to make sure that some regions have no net out-migration or only an "allowable" net out-migration for some or all the periods between 1 and T. It implies that the value of the control variable for these regions and time periods must be positive. He also might want to impose the restriction that the total population of some regions may not fall below a predetermined level. Such constraints may be handled by the formulation of penalty functions. The procedure has been described by Mueller and Wang (1975; p. 610). Although their exposition relates to the continuous model, the application to the discrete version is straight-forward. To each state and control variable is attached a number, which plays the role of a penalty or cost if the sign restriction is violated. The extended objective functional becomes:

$$\min J = \frac{1}{2} \{x(T)\}' \; F\{x(T)\} + \frac{1}{2} \sum_{t=0}^{T-1} \Big[\{x(t)\}' \; Q\{x(t)\}$$

$$+ \{u(t)\}' \; R\{u(t)\} + 2\{x(t)\}' \; \{g(t)\} \qquad (3.79)$$

$$+ 2\{u(t)\}' \; \{r(t)\} \Big]$$

where the elements of $\{g(t)\}$ and $\{r(t)\}$ are penalties. If an element $g_i(t)$ is positive, then $x_i(t)$ will be penalized when it is positive. A similar idea holds for $\{r(t)\}$. The magnitudes of the elements of penalty-vectors reflect the weight that the policy-maker puts on the nonnegativity restrictions of the elements of $\{x(t)\}$ and $\{u(t)\}$. The objective (3.79) may also be formulated in terms of $\{\hat{x}(t)\}$ and $\{\hat{u}(t)\}$. The optimal control is found by applying the necessary conditions to the Hamiltonian. No special difficulties are introduced by the sign restrictions.

The use of penalty functions may be extended to include other equality and inequality constraints as well. The reader may refer to Evtushenko and MacKinnon (1975).

CHAPTER 4

CONCLUSION

The purpose of this paper has been the discussion of some of the analytical problems of population distribution policy. It extends the work of Rogers (1975) on spatial population dynamics to the policy domain.

The growth of a multiregional population may be represented by a system of linear, first-order, homogenous difference equations with constant coefficients:

$$\{x(t + 1)\} = G\{x(t)\} \qquad (4.1)$$

with $\{x(t)\}$ the state vector representing the distribution of the population over space and/or age, and G the growth matrix. To transform (4.1) into a policy model, we add a control vector $\{u(t)\}$:

$$\{x(t + 1)\} = G\{x(t)\} + B\{u(t)\} \quad . \qquad (2.3a)$$

The vector $\{u(t)\}$ contains the instruments of population distribution policy. It has been argued that <u>a fundamental feature of population distribution policy is that it does not occur in a vacuum. It is subordinate to social and economic policies</u>. The ultimate goals are non-demographic in nature, and the instruments are socio-economic. The policy models must, therefore, reflect this connection. The elements of $\{u(t)\}$ are socio-economic variables representing the instruments. The relationship between $\{u(t)\}$ and the population distribution $\{x(t)\}$ is assumed to be

linear and constant in time. The matrix multiplier $\underset{\sim}{B}$ plays
a pivotal role in our discussion of policy models. The
relation between the population distribution $\{\underset{\sim}{x}(t)\}$ and
the vector of socio-economic policy objectives $\{\underset{\sim}{y}(t)\}$ is
assumed to be linear too:

$$\{\underset{\sim}{y}(t)\} = \underset{\sim}{C}\{\underset{\sim}{x}(t)\} \ . \tag{2.3b}$$

Equations (2.3a) and (2.3b) constitute the policy model
we have devoted our attention to. It takes the form of a
state-space model. Without loss of generality, we have
assumed in several instances that $\underset{\sim}{C} = \underset{\sim}{I}$, which means that
the objectives of the population distribution policy are
expressed in terms of the multiregional distribution of
people. The policy model becomes then (2.3a).

The state-space model is a powerful tool for policy
analysis, once the behavior of the system is known and the
policy objectives and the range of instruments are
identified. In most of the literature on quantitative
policy, it has been assumed that these conditions are
satisfied. We made similar assumptions in this study.
The validity of those assumptions have been questioned
in Willekens (1976a, Chapter 1). The usefulness of the
state-space model for the analytical treatment of policy
problems is maximal if it is time-invariant. Time invari-
ance of the coefficients of the policy models has therefore
been assumed. For an analytical treatment of the impact of
changes in coefficients on the outcome of the modeling effort,
see Willekens (1976b).

4.1. MIGRATION POLICY MODELS AND DEMOMETRICS

The derivation of policy models from descriptive or explanatory models is based on the assumptions that the behavior of the system to be controlled has been described by a system of linear equations, denoted as a demometric model, and that the objectives and instruments of population distribution policy have been formulated in precise terms by the policy maker. The policy dimension is introduced into the demometric model, following the Tinbergen paradigm: the policy-relevant part of the system is isolated. It has been shown that any linear descriptive or explanatory model may be converted to a policy model if and only if all the target variables of the policy model belong to the set of endogenous variables of the descriptive or explanatory model, and if at least one of the exogenous variables is controllable.

The general formulation of a policy model is (Tinbergen, 1963):

$$\{y\} = R\{z_1\} + S\{z_2\} \qquad (1.3)$$

with $\{y\}$ the vector of target variables, $\{z_1\}$ the vector of instrument variables and $\{z_2\}$ the vector of uncontrollable exogenous and lagged endogenous variables. An important role in policy analysis is played by the matrix multiplier R. Our discussion of policy models centers around this multiplier. This is consistent with the economic literature on policy models. However, we go beyond the traditional approach in economics and draw from recent findings of mathematical

system theory and the theory of optimal control. To present

an overview of policy models, a classification scheme has

been set up that is based on the rank and the structure of

$\underset{\sim}{R}$. This scheme enables us to relate seemingly unrelated

models to each other. For example, it has been shown that

the linear-quadratic control problem may be derived from

the Tinbergen and Theil model by assuming inter-temporal

separability of the objectives and unidirectional causality

of the population system. The state-space model of (2.3)

also may be derived from the Tinbergen model, and from the

reduced form model in general.

The fundamental questions of quantitative migration

policy may be expressed in terms of existence and design.

In Chapters 2 and 3, we have dealt with these two topics.

The discussion revolves around the matrix multiplier.

Whether arbitrarily specified levels of target variables

can be reached by the existing set of instruments, depends

on the rank of $\underset{\sim}{R}$. The conditions that must be satisfied

for a population system to be controllable are formulated

in a number of existence theorems. These theorems enable

us to uncouple the controllable parts of a not-completely

controllable system, and to compute the minimal number

of instruments that assure the achievement of the targets.

It has been shown, for example, that under well-defined

circumstances represented by a specific transformation of

the matrix multiplier, all the desired target-values can

be reached with a single instrument. This result is

intriguing and totally contrary to the thinking engendered

by Tinbergen's Theory of Policy.

The design procedure of optimal policies is dictated
by the structure and the rank of the matrix multiplier $\underset{\sim}{R}$.
If $\underset{\sim}{R}$ is nonsingular, then the unique solution to (1.3) for
$\{\underset{\sim}{z}_1\}$ is found by simply inverting $\underset{\sim}{R}$. When $\underset{\sim}{R}$ is singular,
there may be no instrument vector leading to the desired
target values, or there may be an infinite number of them.
To find a unique optimal solution, an objective function
reflecting the policy maker's preferences is introduced,
and mathematical programming techniques may be applied.
There is a wide variety of algorithms available in the
literature. The common characteristic of most of them
is that they determine the optimal solution numerically.
In this study, we have directed our attention to cases
where solutions to policy problems can be found analytically.

In this regard, there is the applicability of the
notion of generalized inverse. We have shown how the
minimizing properties of generalized inverses may be relevant
in solutions of policy models with a singular multiplier
matrix. For example, no matter what the rank of the N x K
matrix $\underset{\sim}{R}$ is, a unique solution to (1.3) is given by

$$\{\underset{\sim}{z}_1\} = \underset{\sim}{R}^p [\{\bar{\underset{\sim}{y}}\} - \underset{\sim}{S}\{\underset{\sim}{z}_2\}]$$

where $\underset{\sim}{R}^p$ is the Moore-Penrose inverse (Ben-Israel and
Greville, 1974; p. 7). If $\underset{\sim}{R}$ is nonsingular, then $\underset{\sim}{R}^p$
is the ordinary inverse; if $\underset{\sim}{R}$ is singular and of rank N,
i.e., the number of instruments exceeds the number of
targets, then $\underset{\sim}{R}^p$ defines a minimum norm solution to (1.3);
and if $\underset{\sim}{R}$ is singular and of rank K, i.e., the targets

exceed the instruments in number, then R^p defines a solution to (1.3) that minimizes the squared deviations between the desired and the realized values of the target variables. No explicit objective function has been specified, but it is implicit in the minimizing properties of the generalized inverses. The interesting feature of generalized inverses is that they provide an analytical solution to policy models.

Another case for which the optimal policy may be found analytically is the initial period control problem, namely: the case where the target vector is given for the planning horizon and the control vector at each time period is a linear combination of the elements of the control vector of the previous time period. It then can be shown that the initial period control problem reduces to a single-period problem and the control only needs to be specified at the initial period.

A final policy problem for which a solution may be expressed analytically, is the linear-quadratic control problem. In this trajectory-optimization problem, the policy maker wants to minimize a quadratic function of target variables and instrument variables, subject to linear constraints imposed by the behavior of the system and by the initial condition.

4.2. RECOMMENDATIONS FOR FUTURE RESEARCH

We have based our treatment of population distribution policy models on three fundamental assumptions:

i) The dynamic behavior of the population system and its interaction with socio-economic conditions can be modeled adequately.

ii) This model takes the form of a system of simultaneous linear equations with constant coefficients.

iii) There exists a policy maker who expresses the goals-means relationship of population distribution policy in specific terms of a target vector or in terms of a social welfare function,

who sets up a range of instruments, and who

is willing and able to implement the policy.

The validity of those assumptions may be questioned.
More research is needed in this regard. The prerequisite for good population distribution models is a well developed migration theory. There is no consensus yet on the determinants of migration and on the way the population system interacts with the socio-economic system. As long as the dynamics of the population system are not fully understood, government intervention cannot have a sound basis.

Apart from the problem of identifying the determinants of population growth and distribution, there is the problem of modeling the population system once the determinants are known. Specification and estimation of population models is the subject of demometrics. This new science,

initiated by Rogers, ultimately should provide the necessary
input information for policy analysis.

The third assumption on which our discussion of
policy models has been based concerns the goals-means
relationship of population policy. Not much research has
been done to provide a theoretical underpinning for this
relationship. The approach has instead been pragmatic.
The emerging theories of externalities and of government
intervention may be important building blocks for a theory
of population distribution policy (Willekens, 1976a,
Chapter 1). We are convinced that this theory is a
limiting factor for a sound analysis of population distri-
bution policy.

APPENDIX

THE LINEAR-QUADRATIC CONTROL MODEL:
SOLUTION OF THE TWO-POINT
BOUNDARY-VALUE PROBLEM

The application of the discrete minimum principle to
the LQ control problem yields a system of first-order
difference equations, together with a system of equations
representing the boundary conditions at the initial and
at the terminal time periods, respectively. The optimal
control of the LQ model is given by the solution of this
two-point boundary-value problem. The system has been
derived in Chapter 3, and is given by (3.64), (3.70), (3.68)
and (3.65):

$$\{\lambda(t) \quad = Q\{x(t)\} + G'\{\lambda(t + 1)\} \qquad (A.1)$$

$$\{x(t + 1)\} = G\{x(t)\} + BR^{-1}B'\{\lambda(t + 1)\} \qquad (A.2)$$

$$\{x(0)\} = \{x_0\} \qquad (A.3)$$

$$\{\lambda(T)\} = F\{x(T)\} \qquad (A.4)$$

where $\{x(t)\}$ and $\{\lambda(t)\}$ are the state vector and the co-
state vector, respectively.

The solution to the two-point boundary-value problem
starts out with the assumption that there exists a linear
relation between the co-state vector and the state-vector
at the optimum:

$$\{\lambda *(t)\} = \underset{\sim}{K}(t) \ \{x *(t)\} \tag{A.5}$$

where the feedback matrix $\underset{\sim}{K}(t)$ is the solution of the discrete Riccati equation. Since by (3.69)

$$\{u *(t)\} = - \underset{\sim}{R}^{-1} \underset{\sim}{B}' \{\underset{\sim}{\lambda}(t + 1)\} \quad , \tag{A.6}$$

we may write the feedback control law as

$$\{u *(t)\} = - \underset{\sim}{R}^{-1} \underset{\sim}{B}' \underset{\sim}{K}(t + 1) \ \{\underset{\sim}{x} *(t + 1)\} \quad . \tag{A.7}$$

The closed-loop system then is

$$\{\underset{\sim}{x} *(t + 1)\} = \underset{\sim}{G}\{\underset{\sim}{x} *(t)\} - \underset{\sim}{B}\underset{\sim}{R}^{-1} \underset{\sim}{B}' \underset{\sim}{K}(t + 1) \ \{\underset{\sim}{x} *(t + 1)\}$$

$$[\underset{\sim}{I} + \underset{\sim}{B}\underset{\sim}{R}^{-1} \underset{\sim}{B}' \underset{\sim}{K}(t + 1)] \ \{\underset{\sim}{x} *(t + 1)\} = \underset{\sim}{G}\{\underset{\sim}{x} *(t)\} \quad . \tag{A.8}$$

The matrix

$$[\underset{\sim}{I} + \underset{\sim}{B}\underset{\sim}{R}^{-1} \underset{\sim}{B}' \underset{\sim}{K}(t + 1)]$$

is nonsingular, as will be shown later. Therefore

$$\{\underset{\sim}{x} *(t + 1)\} = [\underset{\sim}{I} + \underset{\sim}{B}\underset{\sim}{R}^{-1} \underset{\sim}{B}' \underset{\sim}{K}(t + 1)]^{-1} \ \underset{\sim}{G}\{\underset{\sim}{x} *(t)\} \quad .$$

$$\tag{A.9}$$

The solution to this system of homogenous difference equations is

$$\{\underset{\sim}{x} *(t)\} = \underset{\sim}{\phi}(t,0) \ \{\underset{\sim}{x}(0)\} \tag{A.10}$$

where $\underset{\sim}{\phi}(t,0)$ is the discrete state transition matrix. The matrix

$$\underset{\sim}{V}(t) = [\underset{\sim}{I} + \underset{\sim}{B}\underset{\sim}{R}^{-1}\underset{\sim}{B}'\underset{\sim}{K}(t + 1)] \tag{A.11}$$

is function of time. Therefore, the solution of time invariant systems

$$\{\underset{\sim}{x}^*(t)\} = [\underset{\sim}{V}^{-1}\underset{\sim}{G}]^t \{\underset{\sim}{x}(0)\}$$

is incorrect.

We must find a sequence $\underset{\sim}{K}(t)$ such that (A.9) and (A.10) hold for any value of $\{\underset{\sim}{x}(0)\}$. Substituting (A.5) into (A.1) gives

$$\underset{\sim}{K}(t) \{\underset{\sim}{x}^*(t)\} = \underset{\sim}{Q}\{\underset{\sim}{x}^*(t)\} + \underset{\sim}{G}'\underset{\sim}{K}(t + 1) \{\underset{\sim}{x}^*(t + 1)\}$$

where $\{\underset{\sim}{x}^*(t + 1)\}$ is given by (A.9). We find then

$$\underset{\sim}{K}(t) \{\underset{\sim}{x}^*(t)\} = \underset{\sim}{Q}\{\underset{\sim}{x}^*(t)\} + \underset{\sim}{G}'\underset{\sim}{K}(t + 1) [\underset{\sim}{V}^{-1}(t) \underset{\sim}{G}] \{\underset{\sim}{x}^*(t)\}$$

where $\underset{\sim}{V}(t)$ is given by (A.11), whence

$$\underset{\sim}{K}(t) \{\underset{\sim}{x}^*(t)\} = [\underset{\sim}{Q} + \underset{\sim}{G}'\underset{\sim}{K}(t + 1) \underset{\sim}{V}^{-1}(t) \underset{\sim}{G}] \{\underset{\sim}{x}^*(t)\} \quad .$$

$$\tag{A.12}$$

This equation is a result of the necessary conditions for an optimum. Therefore, it must hold for any initial condition $\{\underset{\sim}{x}_0\}$. Since only $\{\underset{\sim}{x}^*(t)\}$ depends on the initial condition, (A.12) must hold for any $\{\underset{\sim}{x}^*(t)\}$, and we must have

$$\underset{\sim}{K}(t) = \underset{\sim}{Q} + \underset{\sim}{G}'\underset{\sim}{K}(t + 1)\ \underset{\sim}{V}^{-1}(t)\ \underset{\sim}{G}\ , \qquad \text{for all } t$$

$$= \underset{\sim}{Q} + \underset{\sim}{G}'\underset{\sim}{K}(t + 1)\ [\underset{\sim}{I} + \underset{\sim}{B}\underset{\sim}{R}^{-1}\underset{\sim}{B}'\underset{\sim}{K}(t + 1)]^{-1}\underset{\sim}{G}\ .$$

(A.13)

Equation (A.13) is the discrete Riccati equation. The Riccati equation may be solved backwards, starting with the boundary condition

$$\underset{\sim}{K}(T) = \underset{\sim}{F}\ .$$

(A.14)

That this boundary condition holds follows from substituting (A.5) into (A.4)

$$\underset{\sim}{K}(T)\ \{\underset{\sim}{x}^*(T)\} = \underset{\sim}{F}\{\underset{\sim}{x}^*(T)\}$$

or

$$\underset{\sim}{K}(T) = \underset{\sim}{F}\ .$$

If $\{\underset{\sim}{x}(T)\}'\ \underset{\sim}{F}\{\underset{\sim}{x}(T)\} = 0$, i.e., $\underset{\sim}{F} = \underset{\sim}{0}$, then

$$\{\underset{\sim}{\lambda}(T)\} = \{\underset{\sim}{0}\}\ \text{and}\ \underset{\sim}{K}(T) = \underset{\sim}{0}\ .$$

It is easy to show that $\underset{\sim}{K}(t)$ is positive semi-definite for all t. Since $\underset{\sim}{F}$ was assumed to be positive semi-definite, $\underset{\sim}{K}(T)$ is positive semi-definite. The matrix $\underset{\sim}{K}(T - 1)$ may be found by the relation

$$\underset{\sim}{K}(T - 1) = \underset{\sim}{Q} + \underset{\sim}{G}'\underset{\sim}{K}(T)\ [\underset{\sim}{I} + \underset{\sim}{B}\underset{\sim}{R}^{-1}\underset{\sim}{B}'\underset{\sim}{K}(T)]\ \underset{\sim}{G}$$

(A.15)

$$= \underset{\sim}{Q} + \underset{\sim}{G}'\underset{\sim}{K}(T)\ \underset{\sim}{G} + \underset{\sim}{G}'\underset{\sim}{K}(T)\ \underset{\sim}{B}\underset{\sim}{R}^{-1}\underset{\sim}{B}'\underset{\sim}{K}(T)\ \underset{\sim}{G}'\ .$$

Since Q is positive semi-definite and R^{-1} is positive definite, $K(T - 1)$ must be positive semi-definite, and so all $K(t)$. Now since all of the $K(t)$ are positive semi-definite, the matrix

$$V(t) = I + BR^{-1}B'K(t + 1)$$

is nonsingular, and (A.9) has a unique solution. It is also clear from (A.15) that $K(t)$ is symmetric if Q, F and R are symmetric. Since R is nonsingular $K(t)$ is also unique.

The solution of the Riccati equation requires the inversion of $V(t)$ at each time period. $V(t)$ is an N x N matrix, N being the number of elements in the state vector $\{x(t)\}$, i.e., the number of targets. However, it is possible to reduce the dimension of the matrix to be inverted. It is known from matrix algebra that

$$[I_N + WZ']^{-1} = I_N - W[I_K + Z'W]^{-1} Z' \qquad (A.16)$$

where W and Z' are N x K and K x N respectively, with $K \leq N$. Let

$$W = B$$

$$Z' = R^{-1}B'K(t + 1)$$

then

$$V^{-1}(t) = I_N - B[I_K + R^{-1}B'K(t + 1) B]^{-1} R^{-1}B'K(t + 1)$$

$$= I_N - B\left[R[I_K + R^{-1}B'K(t + 1)\ B]\right]^{-1}\ B'K(t + 1)$$

$$(A.17)$$

$$= I_N - B[R + B'K(t + 1)\ B]^{-1}\ B'K(t + 1)$$

where the matrix to be inverted

$$[R + B'K(t + 1)\ B]$$

is K x K. K is the number of instruments, and is generally much smaller than N.

Once $K(t)$ is known for all t, the sequence of the state vector is computed from (A.3) and (A.18):

$$\{x^*(t + 1)\} = \left[I_N - B[R + B'K(t + 1)\ B]^{-1}\ B'K(t + 1)\right]\{x^*(t)\}$$

$$(A.18)$$

and the sequence of the control vector follows from (A.19)

$$\{u^*(t)\} = -\ R^{-1}B'K(t + 1)\ \{x^*(t + 1)\}\ .\qquad (A.19)$$

The co-state variables are computed by (A.20)

$$\{\lambda^*(t)\} = K(t)\ \{x^*(t)\}\ .\qquad (A.20)$$

The optimal cost functional is

$$J = \frac{1}{2}\ \{x^*(T)\}'\ F\{x^*(T)\} + \frac{1}{2}\left[\sum_{i=0}^{T-1}\{x^*(t)\}'\ Q\{x^*(t)\}\right.$$

$$(A.21)$$

$$\left. +\ \{u^*(t)\}'\ R\{u^*(t)\}\right]$$

Substituting for $\{u^*(t)\}$ yields

$$J = \frac{1}{2}\{x^*(T)\}' \; F\{x^*(T)\} + \frac{1}{2}\left[\sum_{i=0}^{T-1} \{x^*(t)\}' \; Q\{x^*(t)\}\right.$$

$$\left. + \{x^*(t+1)\}' \; K(t+1)' \; BR^{-1}B'K(t+1) \; \{x^*(t+1)\}\right] \quad .$$

Substituting $Q\{x^*(t)\}$ using (A.12) yields

$$J = \frac{1}{2}\{x^*(T)\}' \; F\{x^*(T)\} + \frac{1}{2}\left[\sum_{i=0}^{T-1} \{x^*(t)\}' \; K(t) \; \{x^*(t)\}\right.$$

$$- \{x^*(t)\}' \; G'K(t+1) \; V^{-1}(t) \; G\{x^*(t)\}$$

$$\left. + \{x^*(t+1)\}' \; K(t+1)' \; BR^{-1}B'K(t+1) \; \{x^*(t+1)\}\right] \quad .$$

But by (A.9)

$$V^{-1}(t) \; G\{x^*(t)\} = \{x^*(t+1)\}$$

and, therefore,

$$J = \frac{1}{2}\{x^*(T)\}' \; F\{x^*(T)\} + \frac{1}{2}\left[\sum_{i=0}^{T-1} \{x^*(t)\}' \; K(t) \; \{x^*(t)\}\right.$$

$$+ [\{x^*(t+1)\}' \; K(t+1)' \; BR^{-1}B' - \{x^*(t)\}' \; G']$$

$$\left. K(t+1) \; \{x^*(t+1)\}\right] \quad . \tag{A.22}$$

By (A.9)

$$\{x^*(t)\}' \; G' = \{x^*(t+1)\}' \; [I + BR^{-1}B'K(t+1)]' \quad .$$

And

$$[\{\underset{\sim}{x}{}^{*}(t + 1)\}' \ \underset{\sim}{K}(t + 1)' \ \underset{\sim\sim}{BR}{}^{-1}\underset{\sim}{B}' - \{\underset{\sim}{x}{}^{*}(t)\}' \ \underset{\sim}{G}']$$

$$= [\{\underset{\sim}{x}{}^{*}(t + 1)\}' \ \underset{\sim}{K}(t + 1)' \ \underset{\sim\sim}{BR}{}^{-1}\underset{\sim}{B}' - \{\underset{\sim}{x}{}^{*}(t + 1)\}$$

$$- \{\underset{\sim}{x}{}^{*}(t + 1)\}' \ \underset{\sim}{K}(t + 1)' \ \underset{\sim\sim}{BR}{}^{-1}\underset{\sim}{B}'] = [-\{\underset{\sim}{x}{}^{*}(t + 1)\}] \quad .$$

The objective function (A.22) becomes

$$J = \frac{1}{2} \{\underset{\sim}{x}{}^{*}(T)\}' \ \underset{\sim}{F}\{\underset{\sim}{x}{}^{*}(T)\} + \frac{1}{2}\left[\sum_{i=0}^{T-1} \{\underset{\sim}{x}{}^{*}(t)\}' \ \underset{\sim}{K}(t) \ \{\underset{\sim}{x}{}^{*}(t)\} \right.$$

$$\left. - \{\underset{\sim}{x}{}^{*}(t + 1)\} \ \underset{\sim}{K}(t + 1) \ \{\underset{\sim}{x}{}^{*}(t + 1)\} \right]$$

$$J = \frac{1}{2} \{\underset{\sim}{x}{}^{*}(T)\}' \ \underset{\sim}{F}\{\underset{\sim}{x}{}^{*}(T)\} + \frac{1}{2} \{\underset{\sim}{x}(0)\}' \ \underset{\sim}{K}(0) \ \{\underset{\sim}{x}(0)\}$$

$$- \frac{1}{2} \{\underset{\sim}{x}{}^{*}(T)\}' \ \underset{\sim}{K}(T) \ \{\underset{\sim}{x}{}^{*}(T)\} \quad .$$

But since $\underset{\sim}{K}(T) = \underset{\sim}{F}$, we have

$$J^{*} = \frac{1}{2} \{\underset{\sim}{x}(0)\}' \ \underset{\sim}{K}(0) \ \{\underset{\sim}{x}(0)\} \quad . \qquad ((A.23)$$

The optimal value of the objective functional depends on the initial condition $\{\underset{\sim}{x}(0)\}$ and on $\underset{\sim}{K}(0)$. The matrix $\underset{\sim}{K}(0)$ depends on the matrices $\underset{\sim}{G}$, $\underset{\sim}{F}$, $\underset{\sim}{Q}$, R and $\underset{\sim}{B}$ and on the feedback matrices $\underset{\sim}{K}(t)$, $t = 1,\ldots,T$.

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