THE USE OF REFERENCE OBJECTIVE LEVELS IN GROUP ASSESSMENT OF SOLUTIONS OF MULTIOBJECTIVE OPTIMIZATION

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SUMMARY

Many papers devoted to problems of group assessment of Pareto-optimal solutions or of compromise reaching in cooperative games were based on notions of utility functions or preference ordering identification. However, there is strong evidence that individual decision makers are apt to think in terms of goals or desirable levels of objectives rather than in terms of utility and preferences. Since reference objective levels can be used instead of weighting coefficients and utility functions to derive basic conditions for Pareto-optimality, they can also be applied to construct compromise-aiding procedures for cooperative games or for group assessment of Pareto-optimal solutions.

Several variants of such compromise-aiding procedures are investigated in the paper, together with deadlock situations and deadlock-resolving procedures.

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1. INTRODUCTION

Basic theory of multiobjective optimization has been developed in a strong relation to economic theory. Starting from the work of Pareto [17] through market theory and general equilibrium theory, multiobjective optimization has been always related to weighting coefficients, preference orderings and utility functions -- see, for example, Debreu 1959, [3]. Most of the research on multiobjective optimization and decision theory is, therefore, related to these basic notions.

While a utility function describes well an average behavior of an agent in an economic process, individuals do not think in terms of their utility preferences when they make decisions. In fact, experimental attempts to identify utility functions for individuals often show discrepancies between theory and experimental results -- see, for example, the paper of Tversky in Bell, Keeney and Raiffa 1977, [1]. Moreover, procedures and questions related to utility function identification are lengthy and time-consuming. Individual decision makers are seldom sufficiently interested in their own utility functions to take part in such experiments; they are rather used to think in terms of goals and desirable levels of various objectives when making everyday decisions.

This observation motivated several researchers on multiobjective optimization and decision making. Dyer 1972 [4], Kornbluth 1973 [11] used attainable levels of objective functions for an approach to multiobjective optimization called goal programming. Sakluvadze 1971 [19], 1974 [20], Yu and Leitmann 1974 [26] used sufficiently far unattainable levels of objective functions for utopia point programming. Wierzbicki 1975 - 1979 [21,22,24,25] has shown that any reference point in objective space -- attainable or not, utopia-point type or not -- can be used to consistently scalarize a multiobjective problem via socalled penalty scalarizing functions, which combine and refine the approaches of goal programming and utopia point programming. Moreover, reference objective levels and penalty scalarization can be used instead of weighting coefficients and utility functions to derive a full set of basic conditions of Pareto-optimality. They can also be applied to construct fast interactive procedures for multiobjective decision making, for dynamic multiobjective optimization, etc. All these results have been obtained earlier [21,22,24,25], and only a short summary of them is presented here. This paper is devoted to a study of group decision making procedures where individual decision makers have partly conflicting goals and the bargaining between them proceeds in terms of desirable objective levels.

2. REVIEW OF PROPERTIES OF PENALTY SCALARIZING FUNCTIONS

Consider a simple case of a multiobjective optimization problem, where several objective functions $(f_1(x),\dots,f_n(x)) = f(x)$ are all to be minimized in the Pareto sense. Let $x \in X_0$ be called admissible decisions and $q = f(x) \in Q_0 = f(x_0)$ be called objectives. The set $Q_0 \subset R^n$ is the set of attainable objectives, and its points $\hat{q} \in \hat{Q}_0$ such that $(\hat{q} - \tilde{R}^n_+) \cap Q_0 = \phi$ are Pareto-optimal objectives, where $\tilde{R}^n_+ = \{q \in R^n : q \neq 0, q_1 \geq 0, q_2 \geq 0, \dots, q_n \geq 0\}$. It is known -- see, for example, [3] -- that Pareto-optimal objectives correspond to minimal points over Q_0 of any strictly order-preserving function $s : Q_0 \to R^1$, that is, of a function s such that $q^2 - q^1 \in \tilde{R}^n_+$ implies $s(q^2) > s(q^1)$. Order-preserving properties are basic requirements for utility functions; the simplest strictly order-preserving function is the linear function $s(q) = \sum_{i=1}^{n} \lambda_i q_i$, where $(\lambda_1,\ldots,\lambda_n) = \lambda \in \overset{\circ}{R}{}^n_+ = \{\lambda \in R^n : \lambda_1 > 0,\ldots,\lambda_n > 0\}$ is a vector of weighting coefficients. However, the use of weighting coefficients in multiobjective optimization has known drawbacks.

If any reference objective point $\bar{q} \in R^n$ is given, then a typical penalty scalarizing function has the form:

$$s(q - \bar{q}) = -\|q - \bar{q}\|^2 + \rho\|(q - \bar{q})_+\|^2 \qquad (1)$$

where $(q - \bar{q})_+$ denotes the vector with components $\max(0, q_i - \bar{q}_i)$, and ρ is a scalar penalty coefficient. Many other forms of similar penalty scalarizing functions, with analogous properties, have been specified in [22,24].

The penalty scalarizing function (1) has the following basic properties:

A. For *any* $\bar{q} \in R^n$ and *any* $\rho > 1$, the function (1) is strictly order-preserving in q, if an Euclidean norm or a sum of absolute values norm is used, and order-preserving in q, if the maximum norm is used. Thus, for the Pareto-optimality of some $\hat{x} \in X_0$ and $\hat{q} = f(\hat{x})$ it is sufficient that $\hat{x} = \arg\min_{x \in X_0} s(f(x) - \bar{q})$, $\hat{q} = \arg\min_{q \in Q_0} s(q - \bar{q})$.

B. If $\hat{q} = \arg\min_{q \in Q_0} s(q - \bar{q})$ and $\hat{q} \neq \bar{q}$, then the (normalized) weighting coefficients $\hat{\lambda}$ corresponding to the point $\hat{q}$ can be *a posteriori* determined by

$$\hat{\lambda} = \frac{\bar{q} - \hat{q} + \rho(\hat{q} - \bar{q})_+}{\|\bar{q} - \hat{q} + \rho(\hat{q} - \bar{q})_+\|} \qquad (2)$$

C. If q is an ε-Pareto-optimal objective (that is, all normalized weighting coefficients $\hat{\lambda}_i$ correspond to $\hat{q}$ are greater than ε and $\rho > \max(1, \varepsilon^{-2})$, then

$$\min_{q \in Q_0} s(q - \hat{q}) = 0 \quad . \qquad (3)$$

In other words, the following property holds for any $\bar{q} \in R^n$ and $\rho > \max(1, \varepsilon^{-2})$:

$$\bar{q} - R^n_+ \subset S_0 = \{q \in R^n : s(q - \bar{q}) < 0\} \subset \bar{q} - R^n_{\varepsilon+} \qquad (4)$$

where $R^n_{\varepsilon+} = \{q \in R^n : \mathrm{dist}(q, R^n_+) \leq \varepsilon \|q\|\}$. The property (4) of the level set S_0 of $s(q - \bar{q})$ is called an order-approximation property; its importance is explained in Figure 1 in terms of supporting the set Q_0 at $\hat{q}$ as necessary condition of Pareto-optimality.

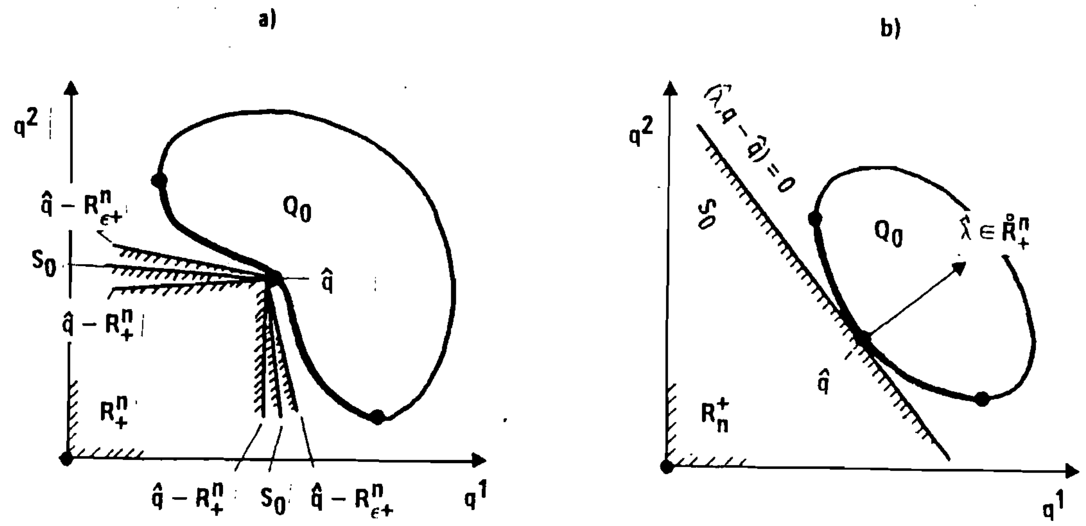

Figure 1. Supporting the set Q_0 as necessary condition of Pareto-optimality: a) general case, by the level set S_0 of the penalty scalarizing function; b) convex case, but the level set S_0 of the linear combination with weighting coefficients $\hat{\lambda}$.

Another useful property of penalty scalarizing functions of the type (1) is the following:

D. If $\bar{q} \notin Q_0 + R^n_+$ and the Pareto-set $\hat{Q}_0$ is compact, then $\hat{q} = \arg\min_{q \in Q_0} s(q - \bar{q})$ is also the closest point in $\hat{Q}_0$ to $\bar{q}$,

$$\hat{q} = \arg\min_{q \in \hat{Q}_0} \|q - \bar{q}\|.$$

The following condition of the existence of Pareto-optimal objectives is also related to the penalty scalarizing function (1):

E. If there exists a $\bar{q} \in Q_0$ such that the set $Q_0 \cap (\bar{q} - R^n_+)$ is nonempty and compact, then there exist Pareto-optimal objectives $\hat{q}$ in this set.

It should be stressed that the Pareto-optimal point $\hat{q}$, corresponding to a given reference objective point $\bar{q}$ obtained via minimization of $s(q-\bar{q})$, depends not only on $\bar{q}$ but also on the choice of the norm, the scales or ranges for separate objectives, and on the penalty coefficient ρ. However, the scaling of separate objectives, the choice of the norm and penalty coefficient play a rather technical role: the Pareto-optimal point $\hat{q}$ depends primarily on $\bar{q}$. By changing $\bar{q}$, the corresponding $\hat{q}$ can be moved to any point of the Pareto-set $\hat{Q}_0$, whatever choice of scaling, norm, and penalty coefficient has been made.

Therefore, it is the reference objective point $\bar{q}$ that takes and, in a sense, generalizes the role of weighting coefficients λ in fundamental theory of multiobjective optimization. The penalty scalarizing function, in a sense, takes and generalizes the role of a utility function. It satisfies less axiomatic requirements than a utility function, but has nevertheless some stronger properties: besides being order-preserving, it is also order-approximating, which provides for the general and easily applicable form of the necessary condition of Pareto-optimality (3). However, the scalarizing function (1) is not a utility function, and is not used to find "the optimal" $\hat{q}$ out of the Pareto-set. Its only purpose is to generate Pareto-optimal $\hat{q}$ which is in some sense close to the given $\bar{q}$ if $\bar{q} \notin Q_0$, or in some sense satisfying the reference levels expressed by $\bar{q}$ if $\bar{q} \in Q_0$, see e.g. [21]. Therefore, the penalty scalarizing function expresses rather a pragmatical behavior of a decision maker than his utility function. If the decision maker is not satisfied with the obtained results $\hat{q} = f(\hat{x})$, he can change $\bar{q}$ and, by doing it, very fast learns to obtain any desirable point $\hat{q} \in \hat{Q}_0$, see [25].

3. COMPROMISE-AIDING PROCEDURES; THE CASE OF SINGLE OBJECTIVES FOR INDIVIDUAL DECISION MAKERS

Consider a partly conflicting situation in a pure strategy game, where several agents or decision makers have separate objectives $q_1,\ldots,q_n$ which they would like to minimize. Here we avoid consciously the description "players" since the stress is rather put on compromise reaching than on playing a game. The decision makers can make independent decisions $x_1 \in X_{01}, \ldots, x_n \in X_{0n}$, but the decisions influence not only their own objectives:

$$q_1 = f_1(x_1,\ldots,x_n),\ldots,q_n = f_n(x_1,\ldots,x_n) \quad . \tag{5}$$

Assume that the decision makers form a committee to agree upon a joint decision $x = (x_1,\ldots,x_n)$ which would result in an outcome $q = f(x)$ in a sense satisfactory to all of them. How can one devise pragmatical procedures to help them attaining a compromise in their decisions?

One way of constructing such compromise-aiding procedures is to refer to basic economic theory and to aggregate the utility functions of the decision makers. Several difficult methodological problems are encountered when proceeding along this way. In this paper, however, it is assumed that the decision makers express their goals in terms of objective levels $\bar{q}_i$, and the bargaining takes place in the objective space. It is also assumed that the committee is aided by an optimization procedure, which defines the decision $\hat{x}$ needed to obtain a Pareto-optimal outcome $\hat{q}$, in a sense close to the desired objectives.

The simplest form of the compromise-aiding procedure was proposed by Kallio and Lewandowski in 1979 [12]. It was assumed that each decision maker specifies only his own desirable level $\bar{q}_i$ and is not necessarily fully informed about other objectives. The iterative procedure is as follows (iteration number j):

Step 1. Given $\bar{q}^j = (\bar{q}_1^j,\ldots,\bar{q}_n^j)$, a penalty scalarizing function of the type (1) is minimized to obtain a Pareto-optimal $\hat{q}^j$. The decision makers are informed about this feasible outcome and the decisions needed to obtain it.

Step 2. Each decision maker is required to move his reference objective level towards $\hat{q}_i^j$, at least β-times the entire distance:

$$\bar{q}_i^{j+1} = \bar{q}_i^j + \beta_i^j(\hat{q}_i^j - \bar{q}_i^j) \quad ; \qquad \beta \le \beta_i^j \le 1 \tag{6}$$

where $\beta \in (0;1]$ is a prespecified number. If they all agree to do so, Step 1 is repeated with $j \leftarrow j+1$. If at least one of them does not agree, the situation is called a deadlock and calls for special deadlock-breaking procedures.

The convergence of this procedure, if no deadlocks occur, is self-evident: the distance between $\bar{q}^j$ and $\hat{q}^j$ must converge to zero, and practically the decision makers would soon agree on $\bar{q}^{j+1} = \hat{q}^j$. This is shown in Figure 2 for the case of two decision makers.

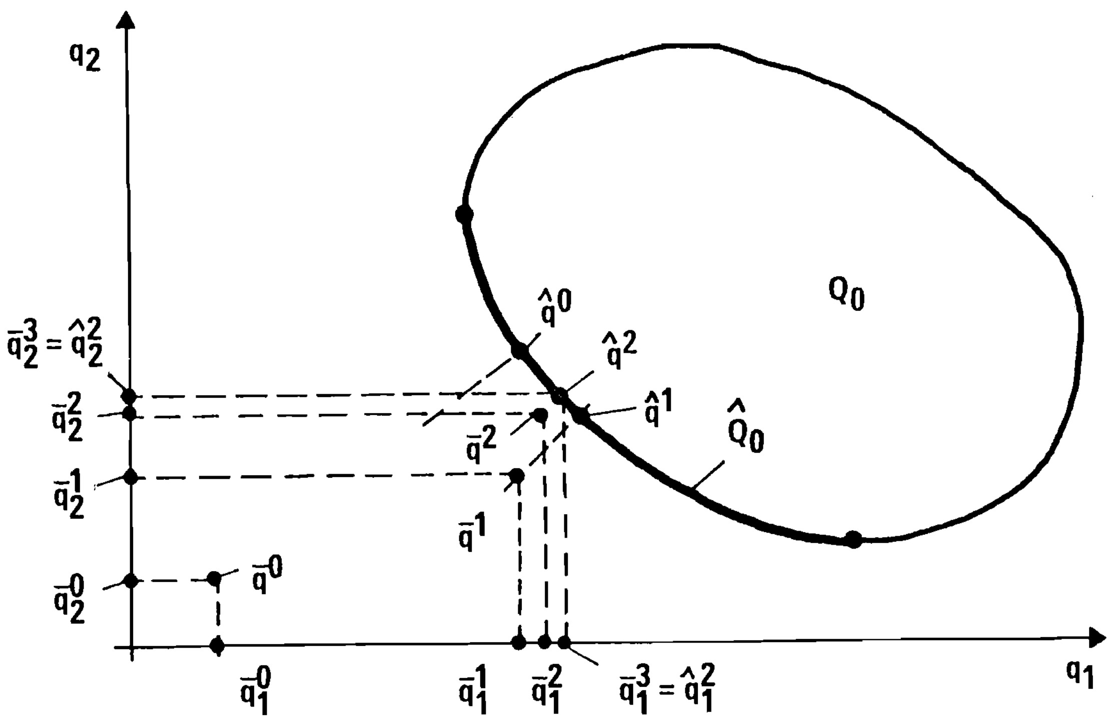

Figure 2. Illustration of the simplest compromise-aiding procedure.

Another question is whether this procedure is sufficiently flexible, that is, whether arbitrary points of the Pareto-set $\hat{Q}_0$ are attainable by this procedure, provided the decision makers are willing to cooperate in achieving this point. Since the starting objective levels $\bar{q}^0 = (\bar{q}_1^0, \ldots, \bar{q}_n^0)$ are arbitrary, there is no doubt about the possibility of reaching an arbitrary Pareto-optimal point. But once the starting point $\bar{q}^0$ is specified, the final Pareto-optimal points are limited. In the case of $n = 2$ and convex Q_0, it is possible to show by simple geometrical consideration that a point $\hat{q} \in \hat{Q}_0$ can be obtained as the limit of the compromise-aiding procedure, if

$$\frac{\hat{\lambda}_2}{\hat{\lambda}_1} > \frac{\hat{q}_2 - \bar{q}_2^0}{\hat{q}_1 - \bar{q}_1^0}\beta \quad \text{and} \quad \frac{\hat{\lambda}_1}{\hat{\lambda}_2} > \frac{\hat{q}_1 - \bar{q}_1^0}{\hat{q}_2 - \bar{q}_2^0}\beta \qquad (7)$$

where $\hat{\lambda}_2$ and $\hat{\lambda}_2$ are the weighting coefficients corresponding to $\hat{q}$. Therefore, as long as both $\hat{q}_1 - \hat{q}_1^0 > 0$, $\hat{q}_2 - \hat{q}_2^0 > 0$, the choice of q_1^0, q_2^0 and β limits possible trade-off ratios $\frac{\hat{\lambda}_1}{\hat{\lambda}_2}$ and $\frac{\hat{\lambda}_2}{\hat{\lambda}_1}$ at $\hat{q}$.

Since the main goal of constructing compromise-aiding procedures is to help decision makers and not to replace them in actual decision making, it is useful to construct procedures based on the assumption that more information is possessed by and presented to the decision makers. Suppose that each decision maker knows the entire problem sufficiently well to judge upon reference objective levels for other decision makers. Thus, he can specify an entire vector $\bar{q}^{k,j} = (\bar{q}_1^{k,j},\ldots,\bar{q}_n^{k,j})$ of reference objective levels at jth iteration of the procedure, including his own objective level $\bar{q}_k^{k,j}$ and the other objective levels $\bar{q}_i^{k,j}$, $i \neq k$. Since we do not assume anything but the equity of decision makers at this point, the corresponding iterative procedure can be constructed as follows:

Step 1. Given n reference objective vectors $\bar{q}^{k,j}$, the corresponding Pareto-optimal points $\hat{q}^{k,j}$ are obtained by the minimization of a penalty scalarizing function. An average reference objective vector $\bar{q}^j = \frac{1}{n}\sum_{k=1}^{n} \bar{q}^{k,j}$ and the corresponding Pareto-optimal point $\hat{q}^j$ is also determined. All information about the outcomes and the decisions needed to obtain them is presented to the decision makers.

Step 2. Each decision maker is asked to move his reference objective vector towards $\hat{q}^j$, at least β-times the entire distance:

$$\bar{q}^{k,j+1} = \bar{q}^{k,j} + \beta^{k,j}(\hat{q}^j - \bar{q}^{k,j}); \quad \beta \leq \beta^{k,j} \leq 1 \quad , \qquad (8)$$

where $\beta \in (0;1]$ is prespecified. Again, the decision makers can either all agree to do so, and then the iterations proceed, or disagree, which results in a deadlock.

If no deadlocks occur, it is natural to expect that the procedure is convergent. However, the convergence is not self-evident and has not been proved yet. A convergence proof for a slightly modified variant of this procedure is given in the Appendix.

4. COMPROMISE-AIDING PROCEDURES: THE CASE OF MULTIPLE OBJECTIVES FOR INDIVIDUAL DECISION MAKERS AND OTHER EXTENSIONS

The procedures described in the previous paragraph can be easily extended to the case when each decision maker has more than one objective. The simplest procedure, however, can be used only if the decision makers have strictly disjoint objectives. If some of the objectives are common for several decision makers, the space of all objectives must be considered as common for all, and the second, more complicated procedure can be applied.

There are also cases of hierarchical decision making, when one or more decision makers have certain prerogatives over others. There are many possible models and procedures to represent such a situation. One of them is the following.

Suppose a higher-level decision maker can influence by his decisions, denoted y, not only the outcomes, $q_k = f_k(x_1,\ldots,x_n,y)$, but also the constraints of other decision makers, $X_{0k} = X_{0k}(y)$. The higher-level decision maker has also his own objective, $q_0 = f_0(x_1,\ldots,x_n,y)$. Since the model-decisions are made by an optimization procedure, he can be represented by his desirable level of objective, $\bar{q}_0$, only. However, to express his priorities, two changes in the general procedures can be made. First, the penalty scalarizing function can be modified to $s(q-\bar{q}) + \rho_0(\bar{q}_0 - q_0)_+$, where $\rho_0 \gg \rho$ represents the priority in attaining the higher-level objective (this function is also order-preserving and order-approximating, see [24]).

Secondly, he can specify both $\bar{q}_0^0$ and $\bar{q}^0 = (\bar{q}_1^0, \ldots, \bar{q}_n^0)$; he is supposed to attain a compromise on the $\bar{q}^0$ together with other decision makers, but not on $\bar{q}^0$, his own objective, which is depending on him alone.

Several other possibilities of compromise-aiding procedures in the hierarchical and multiobjective cases are investigated by Kallio and Lewandowski, 1979 [12]. An interesting application to the planning of possible developments of the Finnish forestry industrial sector is also described there.

5. SPECIAL FORMS OF PENALTY SCALARIZING FUNCTIONS FOR COMPROMISE-AIDING PROCEDURES

It is a known fact in mathematical psychology--see, for example, Tversky in [1]--that decision-makers do not take similar attitudes to the possibility of not attaining their goals as compared to the possibility of exceeding them. In other words, if $\hat{q} - \bar{q} \in R_+^n$ and the postulated levels $\bar{q}$ of (minimized) objectives $\hat{q}$ are not attained, a reasonable procedure should get $\hat{q}$ as close to $\bar{q}$ as possible. On the other hand, if $\hat{q} - \bar{q} \in -R_+^n$, the postulated levels are exceeded, the additional gains should be allocated between various procedures reasonably fair. The precise meaning of this fairness is not of basic importance in the context of reference objectives $\bar{q}$ being modified and thus influencing $\hat{q}$. However, a certain reasonability and fairness of the allocation of gains does help the compromise-aiding procedures in preventing unnecessary deadlocks.

The penalty scalarizing function (1), although it has the required property of $\hat{q}$ being close to $\bar{q}$ if $\hat{q} - \bar{q} \in R_+^n$, does not result in a reasonable allocation of gains if $\hat{q} - \bar{q} \in -R_+^n$. This is because (see Figure 3a) the function corresponds to the *norm maximization* of gain under the soft constraint $q - \bar{q} \in -R_n$, expressed by the penalty term.

By adapting the ideas of Nash 1950 [16] and Ho 1970 [8], the following penalty scalarizing function has been proposed by Majchrzak 1978 [15]:

$$s(q - \bar{q}) = -\prod_{i=1}^{n} (\bar{q}_i - q_i)_+ + g\|(q - \bar{q})_+\|_{E^n} \quad , \qquad (9)$$

based on product of gains, if $q - \bar{q} \in -R_n$, and on the penalty term with Euclidean norm in the opposite case. It is easy to show that this function is order-preserving for any $\rho > 0$, since both product and norm preserve order for positive components. This function is also strictly order-approximating, since $S_0 = \{q \in R^n: s(q - \bar{q}) \leq 0\} = \bar{q} - R^n_+$. The function is also quasi-convex, while the function (1) is quasi-convex only if the sum of absolute values norm is used (and convex if, additionally, $\rho > 2$). The product of gains, as proposed by Ho, expresses some degree of fairness of gain allocation (see Figure 3b).

A differentiable version of the function (9)

$$s(q - \bar{q}) = -(\prod_{i=1}^{n} (\bar{q}_i - q_i)^2_+ + \rho\| (q - \bar{q})_+\|^2_{E^n} , \quad (10)$$

nas, however, inflection points along the entire boundary of $\bar{q} - R^n_+$.

A more sophisticated concept of the fairness of gain allocation can be expressed by the following function:

$$s(q - \bar{q}) = \rho\| (q - \bar{q})_+\| - \quad (11)$$

$$- \min(\rho \min_{1\leq i\leq n} (\bar{q} - q_i)_+ \; , \; \sqrt{\prod_{i=1}^{n} (\bar{q}_i - q)_+}) \; .$$

The level sets of this function are given in Figure 3c. The function, while being differentiable at the boundary of $\bar{q} - R^n_+$ except at the point $q = \bar{q}$, is not differentiable and switches from $-\rho \min_{1\leq i\leq n} (\bar{q}_i - q_i)_+$ to $-\prod_{i=1}^{n} (\bar{q}_i - q_i)_+$ along the set

$P\rho = \{q \in \bar{q} - R^n_+: \; \rho^2 \min_{1\leq i\leq n} (\bar{q}_i - q_i)^2_+ = \prod_{i=1} (\bar{q}_i - q_i)_+\}$, representing the boundary of a cone in $\bar{q} - R^n_+$. It has a simple interpretation in two-dimensional case: while the gain allocation is guided by the product of gains, at least ρ^2-times the larger gain is guaranteed for the smaller one. The function is quasi-convex, order-preserving (since not only the Euclidean norm and the product but also the minimum norm preserve order for positive components) and strictly order-approximating (since $S_0 = \bar{q} - R^n_+$).

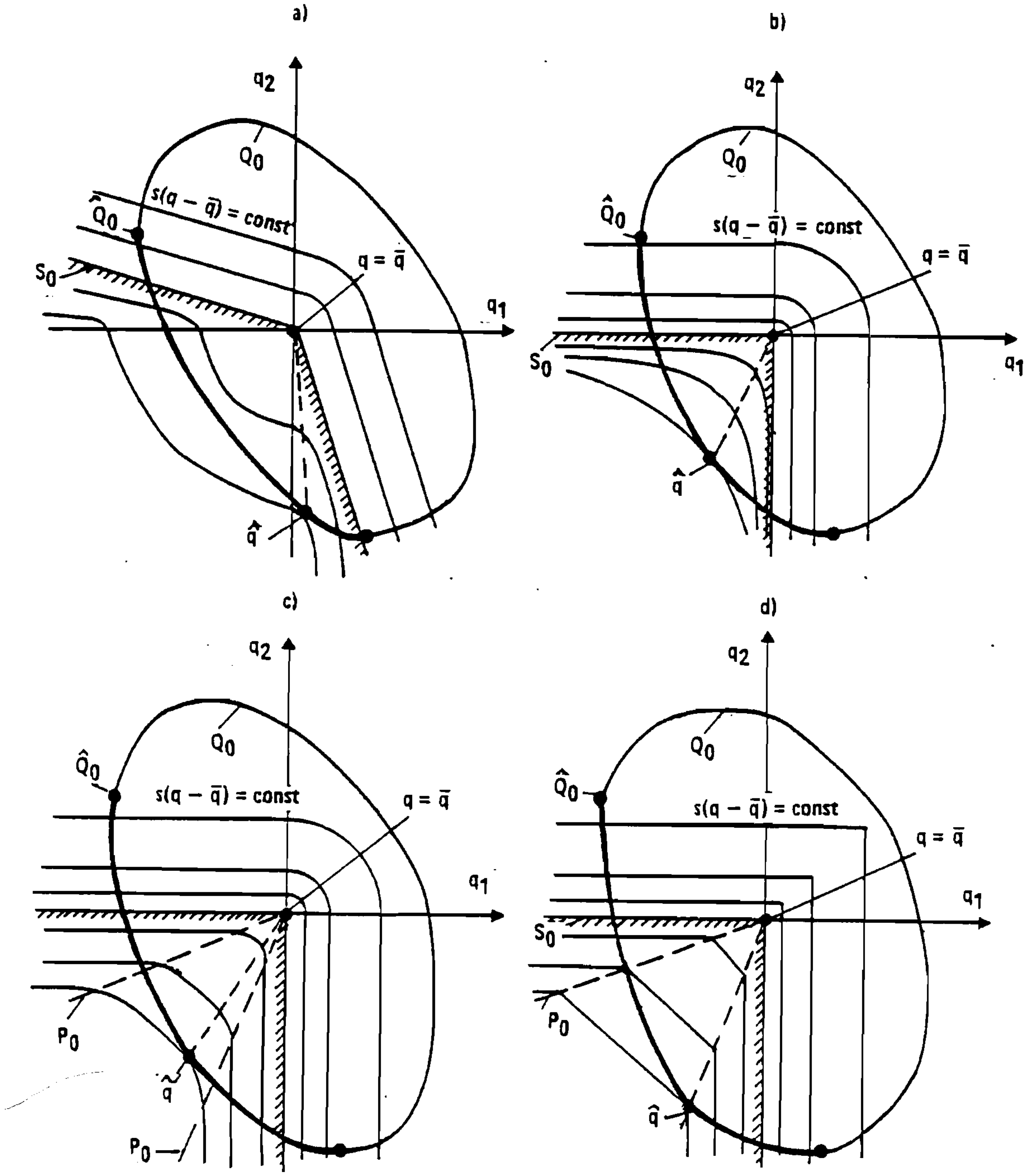

Figure 3. Level sets and minimal points in Q_0 of various penalty scalarizing functions (with the origin shifted to $\bar{q}$): a) the function (1); b) the functions (9), (10); c) the function (11); d) the function (12).

Another useful piecewise linear penalty scalarizing function is the following:

$$s(q - \bar{q}) = \max(\rho \max_{1 \leq i \leq n} (q_i - \bar{q}_i), \sum_{i=1}^{n} (q_i - \bar{q}_i)) \quad . \tag{12}$$

It is not quite easy to see that this function is order-preserving. However, observe that the set where $\sum_{i=1}^{n} (q_i - \bar{q}_i) \geq \rho \max_{1 \leq i \leq n} (q_i - \bar{q}_i)$, if $\rho > n$, is a cone in $\bar{q} - R_+^n$ and does not have any points in $\bar{q} + R_+^n$. Outside of this cone the function is just $\rho \max_{1 \leq i \leq n} (q_i - \bar{q}_i)$, which is clearly order-preserving. Inside of the cone--see Figure 3d, where the boundary of the cone is denoted by P_0--the function corresponds to minus sums of absolute values, which is also order-preserving for all negative components. The combination of those functions preserves order, too, which is easy though tedious to check.

The function (12) is also strictly order-approximating, since $S_0 = \bar{q} - R_+^n$. It is also a convex function. Therefore, the minimization of this function can be represented by a linear programming problem--provided the set Q_0 is represented by linear inequalities:

$$\underset{q = (q_1, \ldots, q_n) \in Q_0 \;\; ; \;\; y \in Y_0(q - \bar{q})}{\text{minimize } y} \quad , \tag{13}$$

where

$$Y_0(q - \bar{q}) = \tag{14}$$

$$= \{y \in R^1 : y \geq \rho(q_i - \bar{q}_i) \text{ , all } i=1,\ldots,n;\ y \geq \sum_{i=1}^{n} (q_i - \bar{q}_i)\} \quad .$$

The function (12) represents another concept of a fair allocation of gains $\bar{q} - \hat{q}$: just the sum of the gains is important, provided that each individual gain is not smaller than $1/\rho$ times the sum of the gains, see Figure 3c. The minimal part of the gain guaranteed for each decision maker must be clearly smaller the $1/n$.

6. DEADLOCKS AND DEADLOCK-RESOLVING PROCEDURES

Deadlocks in compromise-reaching can occur for various reasons. Two classes of deadlocks are of primary interest here.

One type of reason for a deadlock might occur if a decision maker, while accepting the agreement-aiding procedure as fair, feels that his initial demands in terms of reference objective levels were modest when compared to other demands, which has put him into a disadvantageous situation. This type of deadlock is relatively easy to resolve. If all other decision makers agree, they can restart the procedure with new reference objective levels. If they disagree, they can use a mediator or referee, for example, a higher level decision maker in the hierarchical case.

Another, much more difficult type of deadlock might occur if a decision maker preceives that the agreement-aiding procedure is not fair because it gives equal weight to all decision makers, and he could influence the results much more when deciding on his own. For simplicity, such a decision maker will be called a dissident. The dissident can take two different attitudes: either he wants to cooperate further, but he would like more weight attached to his demands, he is a cooperative dissident, or he refuses to cooperate and wants to make his own decision, he is an adversary dissident. Naturally, if a dissident walks out of negotiations, no deadlock-resolving procedure can be of any use; but we shall consider here the situation where he stays in negotiations demanding simply that his decisions x_k must be made by him, not by the optimization procedure.

To devise a deadlock-resolving procedure, a gaming model of the problem must be constructed: the sequence of decision making must be specified, fairly representing the real-world situation simulated by the model. For example, depending on the real problem, a part of the decisions for the dissident decision-maker can be made by him first, then a part of other decisions can be specified, etc.; or the dissident must wait until other decisions are taken; or decisions can be made simultaneously, but a probable violation of constraints in the model must be expressed by a specified payment, a change of objective function. All these extentions of the model needed to transform it to a gaming model should be specified, presented to the decision-makers and agreed upon before the negotiations start; otherwise, no deadlock-resolving procedure can be usefully constructed.

If a gaming model of the problem is available and consistent with the optimization model, various types of dissident-deadlocks can be resolved. If the dissident is cooperative, he might be allowed to make his own decisions and introduce into the gaming model, while the optimization procedure represents the other decision-makers by not playing against the dissident but trying to keep the other objectives close to the last agreed average reference levels. The dissident decisions are then either taken as fixed, if he moves first, or predicted by an optimization procedure, if he moves last. The obtained level of the dissident's objective is then considered as a fixed reference level, similarly as in the case of a higher level decision maker, and used in a repetition of the optimization procedure in order to bring the results to the Pareto set. Thus, a cooperative dissident must agree that his decisions will be modified, while his attained objective level is guaranteed; if he does not agree, he puts himself in the adversary category.

If the dissident is adversary, another optimization procedure can be devised to play against him, just to show how much he can loose by putting himself into an adversary situation. Clearly, results of such a gaming exercise have only psychological value, since other objectives have to be sacrificed during this gaming. But the reason of this gaming is to convince the dissident that he should rather agree on cooperation--or to reveal that the problem is essentially of adversary nature.

7. POSSIBLE EXTENSIONS AND CONCLUSIONS

The aim of the paper was rather to show the possibilities of constructing pragmatical compromise-aiding procedures based on reference objective levels than to develop fully the related theory. Much can be done in this direction. Various compromise-aiding procedures must be checked against practical applications, convergence of these procedures analyzed, special deadlock-resolving procedures developed.

The only point stressed here is that penalty scalarizing functions based on reference objective levels are, on one hand, deeply related to the basic theory of multiobjective optimization and result, on the other had, in a pragmatical approach to group multiobjective decision making. Many forms of the objectives, even in terms of desired dynamic trajectories or desired probability distributions can be also considered by this approach [25].

APPENDIX: CONVERGENCE OF A MODIFIED COMPROMISE-AIDING PROCEDURE

Consider the following modified compromise-aiding procedure:

Step 1. Given n reference objective vectors $\bar{q}^{k,j}$, corresponding Pareto-optimal points $\hat{q}^{k,j}$ are obtained by the minimization of a penalty scalarizing function $s(q-\bar{q})$. An average objective vector $\bar{q}^j = \frac{1}{n}\sum_{k=1}^{n} \bar{q}^{k,j}$ and the corresponding Pareto-optimal point $\hat{q}^j$ is also determined. All information about the outcomes and the decisions needed to obtain them is presented to the decision makers.

Step 2. If $s(\hat{q}^j - \bar{q}^j) < 0$, then new reference points are automatically determined by

$$\bar{q}^{k,j+1} = \bar{q}^{k,j} + (2 - \beta)(\hat{q}^j - \bar{q}^j) \quad , \tag{A1}$$

where $\beta \in (0;1)$ is a given parameter. Set $j \leftarrow j+1$ and repeat Step 1.

Step 3. If $j > 1$, then condition $\| \hat{q}^{j-1} - \bar{q}^j \| \leq (1-\beta) \| \hat{q}^{j-1} - \bar{q}^{j-1} \|$ is checked. If this condition does not hold, then $\bar{q}^j$ is modified by:

$$\bar{q}^j_{new} = \hat{q}^{j-1} + (1-\beta)\frac{\|\hat{q}^{j-1}-\bar{q}^{j-1}\|}{\|\hat{q}^{j-1}-\bar{q}^j_{old}\|}(\bar{q}^j_{old} - \hat{q}^{j-1}) \quad , \quad \text{(A2)}$$

and a modified corresponding Pareto-optimal point $\hat{q}^j_{new}$ is also determined. The additional information is presented to the decision makers.

Step 4. Each decision-maker is asked to move his reference objective vector towards $\hat{q}^j$ ($\hat{q}^j_{new}$ if the modification is Step 3 was performed), at least β-times the entire distance:

$$\bar{q}^{k,j+1} = \bar{q}^{k,j} + \beta^{k,j}(\hat{q}^j - \bar{q}^{k,j}) \quad ; \quad \beta \leq \beta^{k,j} \leq 1 \quad . \text{(A3)}$$

If all decision makers agree to do so, then $j \leftarrow j+1$ is set and Step 1 is repeated. If some of them disagree, a deadlock occurs. If all agree to use $\beta^{k,j} = 1$, then the procedure stops.

The additional Step 2 changes automatically reference levels and the average objective vector to obtain $\bar{q}^{j+1} \notin Q_0 + R^n_+$ (see Figure A1). Since all objective levels are improved by this change, the decision makers are only informed and not asked to agree; they can, however, disagree in nest iteration. It is easy to check that Step 3 can be omitted if Step 2 has been performed just before, since the condition required by Step 3 is then automatically satisfied.

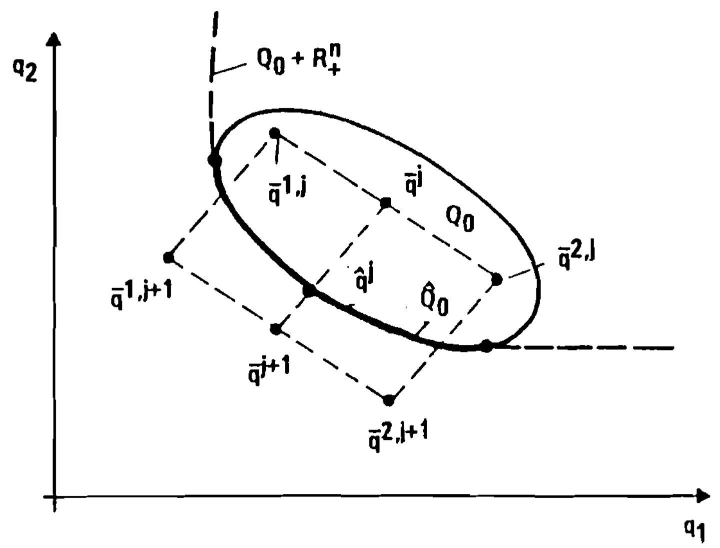

Figure A1. Interpretation of the reference level change performed in Step 2.

It is also possible to show that if Q_0 is convex and $\hat{Q}_0$ compact, whatever the initial $\bar{q}^{k,0}$, then Step 2 can be performed only a finite number of times to obtain $\bar{q}^j \notin Q_0 + R^n_+$ for all subsequent iterations. Therefore, the convergence analysis of the procedure can be limited to the case $\bar{q}^j \notin Q_0 + R^n$.

Lemma A1. If the set Q_0 is convex and $\hat{Q}_0$ is compact, and if no deadlocks occur, then the procedure described above is convergent in the sense that there exists $\hat{q}^\infty = \lim_{j\to\infty} \hat{q}^j$ and also $\lim_{j\to\infty} \bar{q}^{k,j} = \lim_{j\to\infty} \bar{q}^j = \hat{q}^\infty$.

Proof. Since $\hat{q}^j$ minimizes $s(q - \bar{q}^j)$ and thus also the distance for $\bar{q}^j$ to $\hat{Q}_0$, if $\bar{q}^j \notin \hat{Q}_0 + R^n_+$, hence

$$\| \hat{q}^j - \bar{q}^j \| \leq \| \hat{q}^{j-1} - \bar{q}^j \| \leq (1 - \beta) \| \hat{q}^{j-1} - \bar{q}^{j-1} \| ,$$

where the last inequality results from Step 2 (this inequality could also be proven, not forced algorithmically, but the necessary assumptions are much stronger in this case). Hence, $\lim_{j\to\infty} \| \hat{q}^j - \bar{q}^j \| = 0$; $\lim_{j\to\infty} \| \hat{q}^{j-1} - \bar{q}^j \| = 0$. However, $\| \hat{q}^j - \hat{q}^{j-1} \| \leq \| \hat{q}^j - \bar{q}^j \| + \| \hat{q}^{j-1} - \bar{q}^j \|$; hence also $\lim_{j\to\infty} \| \hat{q}^j - \hat{q}^{j-1} \| = 0$. Since $\{\hat{q}^j\}_{j=0}^{\infty} \subset \hat{Q}_0$ is compact, it has accumulation points; they cannot be distinct, since then $\lim_{j=\infty} \| \hat{q}^j - \hat{q}^{j-1} \|$ would not exist. Therefore, there is a unique accumulation point $\hat{q}^\infty = \lim_{j\to\infty} \hat{q}^j$. Clearly, $\lim_{j\to\infty} \bar{q}^j = \hat{q}^\infty$.

Equations (A3) imply also $\| \bar{q}^{k,j+1} - \hat{q}^j \| \leq (1 - \beta) \| \bar{q}^{k,j} - \hat{q}^j \|$, which can be rewritten as $\| \bar{q}^{k,j+1} - \hat{q}^\infty \| \leq (1-\beta) \| \bar{q}^{k,j} - \hat{q}^\infty \| + (2 - \beta) \| \hat{q}^j - \hat{q}^\infty \|$. If $\| \bar{q}^{k,j} - \hat{q}^\infty \|$ were not convergent to zero, then for arbitrarily small $\varepsilon > 0$, $\varepsilon < \beta$, there would be arbitrarily large j such that $(1-\varepsilon) \| \bar{q}^{k,j} - \hat{q}^\infty \| \leq \| \bar{q}^{k,j+1} - \hat{q}^\infty \|$ would hold; but this would imply

$(\beta - \varepsilon)\|\bar{q}^{k,j} - \hat{q}^{\infty}\| \leq (2 - \beta)\|\hat{q}^{j} - \hat{q}^{\infty}\|$ converging to zero, a contradiction. Hence $\lim_{j\to\infty}\|\bar{q}^{k,j} - \hat{q}^{\infty}\| = 0$.

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