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SOME THEORETICAL CONSIDERATIONS ON LINKAGE PROBLEMS

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PREFACE

There is a growing need for the linkage of different models in applied projects at the Institute. A detailed discussion of the difficulties met in work on this problem might be a theme for independent research, but here we will restrict the discussion to the problem of linking two or more linear programming models.

If in a linkage problem we merge two or more subproblems into a united one, then correspondingly in a decomposition approach we try to split an initial large-scale problem into a number of small subproblems gaining computational or other advantages. So far as we want to preserve the individuality of every model throughout the whole linkage process and restrict computations to those performed with submodels separately, linkage and decomposition become in fact the same problem. A number of methods are known for decomposition of large-scale linear programming problems. The most widely known are the Dantzig-Wolf decomposition method (Dantzig and Wolf 1961, Dantzig 1963) and the Benders decomposition method (Benders 1962) which for linear programming problems are duals of each other. While Benders' scheme considers the case of common or binding variables, the Dantzig-Wolf technique is proposed for the case of common constraints. So far as the latter form can be easily transformed into a problem with common variables we will be concerned only with this problem.

The general idea of a Benders decomposition consists of

(a) an intrinsic representation of the feasible set of linking variables by means of extreme rays of the cones defined by subproblems

(b) a representation of the infinite system of inequalities which defines the solution of the whole problem by means of extreme points of the feasible sets of subproblems.

Both (a) and (b) use an auxilliary problem of linear programming to generate ray or point when it becomes binding. Potentially there is an enormous number of such points or rays to be generated but actually only a small part of them is usually generated during the solution. Computer experiments (Ho 1977) show a good performance comparable or better than a direct simplex method of other advanced techniques such as basis factorization (Ho and Loute 1978).

However, it would be desirable to decrease the number of constraints implicitly inherited in a decomposition principle so far as it decreases an estimate of a number of cycles between master and slave problems. In this paper this is achieved in a manner based on ideas of nondifferential optimization. This idea produces a finite convergent monotonic descent algorithm which might be used in a number of cases. A promising direction would be, for instance, mixed integer-continuous linear problems.

In a mathematical sense the proposed approach relates to some extent to the limiting duality theory (Blair, Duffin, and Jeroslaw 1979, Borwein 1979 (1980), Duffin and Jeroslaw 1979). Among related IIASA publications we can mention Kallio, Orchard-Hays, and Propoi (1979).

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ABSTRACT

A new decomposition approach is proposed for solving large linear programming problems. This paper contains mainly theoretical results connected with this idea and a small numerical example illustrating the solution. The algorithm is based on ideas of nondifferential optimization and the solution is obtained in a finite number of steps monotonically decreasing an objective function with computational efforts restricted to those with subproblems forming original large-scale problems.

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1. MATHEMATICAL FORMULATION AND PRELIMINARY RESULTS

We will consider the problem of linear programming which has the following structure:

$$\min \{c_A z_A + c_B z_B\}$$

$$A_A z_A + B_A x \leq b_A \tag{1}$$

$$A_B z_B + B_B x \leq b_B \quad .$$

It is clear that for a fixed x this problem splits into two subproblems

$$\min c_A z_A = f_A(x)$$

$$A_A z_A \leq b_A - B_A x \quad , \tag{PA}$$

and

$$\min c_B z_B = f_B(x)$$

$$A_B z_B \leq b_B - B_B x \quad . \tag{PB}$$

The variables z_A and z_B might be called <u>internal</u> variables of subproblems A and B correspondingly, variable(s) x is a linking variable which binds them together. The original problem (1) is therefore equivalent to the problem of finding the minimum of function $f_A(x) + f_B(x)$:

$$\min_x \{f_A(x) + f_B(x)\} \quad , \tag{2}$$

where x might be constrained as well. The different variations of this problem are customarily called <u>master</u> problems and we will also use this terminology.

The analysis of the behavior of functions $f_A(x)$, $f_B(x)$ in the neighborhood of some fixed point can be performed with the help of parametric programming (Walkup and Wets 1969) and the expected type of behavior is that for each function there is a set of linking variables x which provides a nonempty feasible set of internal variables in subproblems (PA) and (PB). Otherwise we should subscribe the value $+\infty$ to either (PA) or (PB) with infinites in dual variables providing information about directions of possible changes in the values of the linking variables.

The set of linking variables which guarantees the finite values of subproblems (PA) and (PB) has an implicit description which makes many operations connected with this set (such as projection, finding a feasible direction, etc.) difficult to perform. Another difficulty occuring with this direct consideration is that problem (2) is a problem of nondifferentiable optimization under implicit linear constraints and currently there are not many algorithms for this problem. Finally, it would also be desirable to make use of a piece-wise linear structure of functions $f_A(x)$ and $f_B(x)$ to get a finite convergent algorithm.

Having these aims in mind, we develop an approach based on particular approximations of objective functions of the master problem. This approximation preserves the minimum of the oritinal problem while simplifying it.

For further development we will use the notation

$$\Phi(x) = f_A(x) + f_B(x) \quad .$$

The general idea is to construct another function $\hat{\Phi}(x)$ which would have as simple a structure as possible and on the other hand preserve the essential feature of the initial function $\Phi(x)$. We define the function $\hat{\Phi}(x)$ as follows:

$$\hat{\Phi}(x) = \sup_{\substack{\pi(y-x)+\mu \le \Phi(y) \\ \pi \in \Pi,\ \forall y}} \mu \qquad = \sup_{\pi\in\Pi} \inf_y \{\Phi(y)-\pi(y-x)\} \ ,$$

$$= \sup_{\pi\in\Pi} \{\inf_y\{\Phi(y)-\pi y\} + \pi x\} =$$

$$= \sup_{\pi\in\Pi} \{\pi x-\Phi^*(\pi)\} \ ,$$

where

$$\Phi^*(\pi) = \sup_y\{\pi y-\Phi(y)\} \quad ,$$

is a conjugate function (Rockafellar 1970).

If Π is a compact set then function $\hat{\Phi}(x)$ is finite everywhere. On the other hand, between $\hat{\Phi}(x)$ and $\Phi(x)$ there are a few similarities which are formulated in the following theorems.

Theorem 1. If $\Phi(x)$ is bounded from below and zero belongs to Π then

$$\inf_x \Phi(x) = \inf_x \hat{\Phi}(x) \quad .$$

Proof. For any x

$$\hat{\Phi}(x) = \sup_{\pi\in\Pi}\{\pi x + \inf_y\{\Phi(y) - \pi y\} \le$$

$$\leq \sup_{\pi \in \Pi}\{\pi x + \Phi(x) - \pi x\} = \Phi(x) \quad .$$

On the other hand for any x

$$\hat{\Phi}(x) \geq 0 \cdot x + \inf_y\{\Phi(y) - 0 \cdot y\} = \inf_y \Phi(y),$$

which proves the statement.

Theorem 2. If in addition to the assumptions of Theorem 1 $\Phi(x)$ is a convex function which attains infinium at least at one point and zero belongs to the interior of set Π, then every minimum of the function $\hat{\Phi}(x)$ is a minimum of $P(x)$.

Proof. Due to Theorem 1, $\hat{\Phi}(x)$ has a nonempty set of minimas as well. We denote this set as $\hat{X}^*$ and let X^* be a set of minimas of the function $\Phi(x)$. Theorem 1 states that

$$\hat{X}^* \supset X^* \quad .$$

However, if we assume that there is a point $\hat{x}^*$ which belongs to $\hat{X}^*$ but that $\hat{x}^* \notin X^*$, then in an extended space point $(\hat{x}^*, \inf_x \Phi(x))$ and epi $\Phi(x)$ of the function $\Phi(x)$ are separable with the help of some hyperplane. This means that there exists vector p and scalar w such that

$$\text{(i)} \quad w > \inf_x \Phi(x)$$

$$\text{(ii)} \quad p(x-\hat{x}^*) + w \leq \Phi(x), \text{ for all } x \quad .$$

Clear $p \neq 0$. Also trivially

$$\inf_x \Phi(x) \leq \Phi(x) \quad ,$$

and consequently for any $0 \leq \alpha \leq 1$

$$\text{(j)} \quad \alpha p(x-\hat{x}^*) + \alpha w + (1-\alpha)\inf_x (x) \geq \Phi(x)$$

$$\text{(jj)} \quad \alpha w + (1-\alpha)\inf_x\Phi(x) \geq \inf_x\Phi(x) \quad .$$

If $0 \in \text{int}\,\Pi$ then there is $\alpha > 0$ such that $p^1 = \alpha p \in \Pi$. Therefore

$$\hat{\Phi}(\hat{x}^*) = \sup_{\substack{\pi(x-\hat{x}^*) + \mu \le \Phi(x) \\ \pi\in\Pi}} \mu \ge \sup_{p^1(\hat{x}-x^*) + \mu \le \Phi(x)} \mu \ge$$

$$\ge \alpha w + (1-\alpha)\inf_x \Phi(x) > \inf_x \Phi(x) = \inf_x \hat{\Phi}(x) \quad ,$$

which contradicts the definition of $\hat{X}^*$ and completes the proof.

Theorem 3. If $\Phi(x)$ is a convex function which attains its minimum at a point x^* such that

$$0 \in \text{int}\{\partial\Phi(x^*)\} \quad ,$$

where $\partial\Phi(x^*)$ is a set of subgradients at point x^* and Π is such that

(i) $0 \in \text{int}\,\Pi$

(ii) $\Pi \subset \partial\Phi(x)$,

then

$$\hat{\Phi}(x) = \hat{\Phi}(x^*) + \sup_{\pi\in\Pi} \pi(x-x^*) \quad .$$

Clearly

$$\hat{\Phi}(x^*) = \inf \Phi(x) \quad .$$

Proof.

$$\hat{\Phi}(x) = \sup_{\pi\in\Pi} \inf_{z}\{\Phi(z) - \pi(z - x)\} \leq$$

$$\leq \sup_{\pi\in\Pi} \{\Phi(x^*) - \pi(x^* - x)\} =$$

$$= \Phi(x^*) + \sup_{\pi\in\Pi} \pi(x - x^*) \quad .$$

On the other hand by definition

$$\hat{\Phi}(x) = \sup_{\substack{\pi(z-x) + \mu \leq \Phi(z) \\ \pi\in\Pi}} \mu = \sup_{\pi \in \Pi} \sup_{\substack{\pi(z-x) + \mu \leq \Phi(z) \\ \forall z}} \mu \geq$$

$$\geq \sup_{\pi\in\Pi} \sup_{\mu(x^* - x) + \mu' \leq \Phi(x^*)} \mu' = \sup_{\pi\in\Pi}\{\Phi(x^*) + \pi(x-x^*) \quad .$$

The inequalities above follow from the fact that if μ' satisfies constraint

$$\pi(x^*-x) + \mu' \leq \Phi(x^*) \quad ,$$

then μ' satisfies constraints

$$\pi(z-x) + \mu' \leq \Phi(z) \quad , \quad \text{for all } z \quad , \quad \pi \in \partial\Phi(x^*)$$

as well. For $\pi \in \Pi \subset \partial\Phi(x^*)$

$$\pi(z-x) + \mu' \leq \pi(z-x) + \Phi(x^*) + \pi(x-x^*) =$$

$$= \Phi(x^*) + \pi(z-x^*) \leq \Phi(z) \quad ,$$

due to convexity of the function $\Phi(z)$.

Theorem 3 together with others tells us that if the initial problem of finding

$$\min_x \Phi(x) = \min_x\{f_A(x) + f_B(x)\} = \Phi(x^*) \quad ,$$

has certain properties of nondegeneracy; that is, if the minimum $\Phi(x)$ is unique then this problem can be reformulated as a problem of finding the minimum of the function with a very simple structure:

$$\hat{\Phi}(x) = \Phi(x^*) + \sup_{\pi \in \Pi} \pi(x-x^*) \quad . \tag{3}$$

If set Π has a simple structure like polyhedra with a finite number of extreme points then $\hat{\Phi}(x)$ might be optimized in a finite number of steps by steepest-descent method. The problem now is to find a means for computing differential characteristics of $\hat{\Phi}(x)$ in an efficient way maximally using the structure of initial problem.

2. COMPUTATION OF A SUBGRADIENT SET

The central problem in running the steepest descent method for minimizing function (3) is the problem of finding the steepest descent direction. So far as this function is a nondifferentiable one, it is necessary to compute the shortest subgradient in a convex hull of some extreme points of the polyhedra Π. This computation might be a relatively easy problem providing this subset of extreme points is known, but the major computational difficulty consists in determining it.

Without any loss of generality we consider the calculation of the subgradient set of function (3) at point $x = 0$.

By definition

$$\hat{\Phi}(0) = \sup_{\pi \ \Pi} \inf_{x \in X}\{\Phi(x) - \pi x\} =$$

$$= -\inf_{\pi \in \Pi} \sup_{x \in X}\{\pi x - \Phi(x)\}$$

$$= -\inf_{\pi\in\Pi}\sup_{x\in X}\{\pi x-\Phi(x)\}$$

$$= \inf_{\pi\in\Pi}\sup_{x_A=x_B}\{\tfrac{1}{2}\pi x_A+\tfrac{1}{2}\pi x_B-f_A(x_A)-f_B(x_B)\} =$$

$$= -\inf_{\pi\in\Pi}\sup_{x_A,x_B}\inf_{\lambda}\{(\tfrac{1}{2}\pi+\lambda)x_A-f_A(x_A)+(\tfrac{1}{2}\pi-\lambda)x_B-f_B(x_B)\} =$$

$$= -\inf_{\pi\in\Pi,\lambda}\{\sup_{x_A}\{(\tfrac{1}{2}\pi+\lambda)x_A-f_A(x_A)\}+\sup_{x_B}\{(\tfrac{1}{2}\pi-\lambda)x_B-f_B(x_B)\}=$$

$$= -\inf_{\pi_A+\pi_B\in\Pi}\{\sup_{x_A}\{\pi_A x_A-f_A(x_A)\}+\sup_{x_B}\{\pi_B x_B-f_B(x_B)\}\} =$$

$$= -\inf_{\pi_A+\pi_B\in\Pi}\{f_A^*(\pi_A)+f_B^*(\pi_B)\} ,$$

where we use the traditional notations for conjugate functions and substituted variables

$$\pi_A = \tfrac{1}{2}\pi + \lambda$$

$$\pi_B = \tfrac{1}{2}\pi - \lambda .$$

We also use a saddle point equality which is valid for convex functions f_A and f_B.

In fact, we are more interested in the subgradient set of function (3) than in its numerical value. It is remarkable, however, that if the function value is given by a solution of the extremum problem

$$-\hat{\Phi}(0) = \inf_{\pi_A+\pi_B\in\Pi}\{f_A^*(\pi_A)+f_B^*(\pi_B)\} , \qquad (4)$$

then the subgradient set $\partial\hat{\Phi}(0)$ is given by

$$\partial\hat{\Phi}(0) = \pi_A^* + \pi_B^* , \qquad (5)$$

where π_A^*, π_B^* are the solutions of problem (4).

For further consideration we transform problem (4) even more. Let us assume that set Π is a sum of two sets Π_A and Π_B:

$$\Pi = \Pi_A + \Pi_B \quad .$$

Then the original problem (4) can be rewritten as

$$-\hat{\Phi}(0) = \inf_{\substack{\pi_A \in \Pi_A \\ \pi_B \in \Pi_B}} \inf_{p} \{f_A^*(\pi_A+p)+f_B^*(\pi_B-p)\} \tag{6}$$

and subgradient set (5) has the same representation:

$$\partial\hat{\Phi}(0) = \pi_A^* + \pi_B^* \quad ,$$

where $\pi_A^* \in \Pi_A$, $\pi_B^* \in \Pi_B$--are solutions of problem (6).

It is worth noting that set Π has to satisfy certain assumptions to justify this transformation. The conditions of theorems 1 - 3 demonstrate that it would be desirable to have set Π first, small enough, and secondly, still containing some neighborhood at the origin. For computational convenience it has to have the least number of extreme points as possible. If it is chosen in an appropriate way, then according to (3) solutions of (6) are extreme points of set Π, and hence, the external <u>inf</u> can be replaced by finite <u>min</u>:

$$-\hat{\Phi}(0) = \min_{\substack{\hat{\pi}_A \in \hat{\Pi}_A \\ \hat{\pi}_B \in \hat{\Pi}_B}} \inf_{p} \{f_A^*(\hat{\pi}_A+p) + f_B^*(\hat{\pi}_B-p)\} \quad , \tag{7}$$

where we denote the set of $\hat{\Pi}_A$ extreme points of the polyhedra $\Pi_A(\hat{\Pi}_B$ and $\hat{\Pi}_B$ correspondingly).

To find such $\hat{\pi}_A^*$, $\hat{\pi}_B^*$ which provides the minimim generally speaking, we have to solve a finite number of linear programming problems originating from (7). The number of these problems is the order of number of linking variables and it is reasonable to consider this number a relatively small one.

Let us, however, examine the corresponding LP problems in a more detailed way:

(LP) For fixed $\hat{\pi}_A \in \hat{\Pi}_A$, $\hat{\pi}_B \in \hat{\Pi}_B$ find

$$\inf_p f_A^*(\hat{\pi}_A + p) + f_B^*(\hat{\pi}_B - p) = \psi(\hat{\pi}_A, \hat{\pi}_B)$$

$$= \inf_p \sup_{\substack{(x_A, z_A) \in V_A \\ (x_B, z_B) \in V_B}} \{(p+\hat{\pi}_A)x_A - c_A z_A - (p-\hat{\Pi}_B)x_B - c_B z_B\}$$

where V_A and V_B are sets given by inequalities:

$$V_A = \{(x_A, z_A) : A_A z_A + B_A x_A \leq b_A\}$$

$$V_B = \{(x_B, z_B) : A_B z_B + B_B x_B \leq b_B\} .$$

As these sets are independent of each other, it is natural to assume their nonemptiness, otherwise, some of the subsystems are not feasible for <u>any</u> input values of linking variables. For convenience we assume the boundness of V_A and V_B.

Due to linearity of the objective function with respect to x_A, x_B, z_A, z_B we may replace internal <u>sup</u> with finite min over the sets of extreme points V_A and V_B:

$$\psi(\hat{\pi}_A, \hat{\pi}_B) = \inf_p \sup_{\substack{(\hat{x}_A, \hat{z}_A) \in \hat{V}_A \\ (\hat{x}_B, \hat{z}_B) \in \hat{V}_B}} \{(p+\hat{\Pi}_A)\hat{x}_A - c_A \hat{z}_A - (p-\hat{\Pi}_B)\hat{x}_B - c_B \hat{z}_B\} .$$

The value of this program has an equivalent representation:

$$\psi(\hat{\pi}_A, \hat{\pi}_B) = \inf_{\substack{(p+\hat{\pi}_A)\hat{x}_A c_A \hat{z}_A - (p-\hat{\pi}_B)\hat{x}_B - c_B \hat{z}_B \leq V \\ (\hat{x}_A, \hat{z}_A) \in \hat{V}_A \\ (\hat{x}_B, \hat{z}_B) \in \hat{V}_B}} V \qquad (8)$$

This problem has a small number of variables and a large number of rows and consequently can be efficiently solved by different modification of the row generation technique. These methods require the solution of the linear programming problems of the kind:

For given $\bar{p}$, $\bar{v}$ find

$$\sup_{\substack{(x_A, z_A)\in V_A \\ (x_B, z_B)\in V_B}} \{(\bar{p}+\hat{\pi}_A)x_A - c_A z_A - (\bar{p}-\hat{\pi}_B)x_B - c_B z_B\} - \bar{v} \quad ,$$

which is obviously decomposed into two separate subproblems:

$$\sup_{A_A z_A + B_A x_A \le b_A} \{(\bar{p}+\hat{\pi}_A)x_A - c_A z_A\} = \bar{v}_A \quad , \quad \sup_{A_B z_B + B_B x_B \le b_B} \{-(\bar{p}-\hat{\pi}_B) - c_B z_B\} = \bar{v}_B \quad .$$

If $\bar{v}_A + \bar{v}_B > \bar{v}$, then a new row will be generated and a new intermediate solution $\bar{p}$, $\bar{v}$ will be obtained. After a finite number of steps we obtain the solution of problem (2).

3. NUMERICAL EXPERIMENTS

To demonstrate the method described in the previous chapters let us consider the following LP problem

$$\min -x_1 + 2x_2 - 0.5x_3 - x_4 + 4x_6 - x_7 - 5x_8 + x_9$$

$$x_1 + 2x_2 + 5x_3 + x_4 + 5x_5 \le 10$$

$$10x_1 + x_2 + 4x_3 + 5x_4 - 4x_5 \le 20$$

$$-x_1 - 5x_2 + x_3 + x_4 \le 30$$

$$x_5 + 10x_6 + x_7 + x_8 + 10x_9 \le 40$$

$$0{,}1x_5 + x_6 \le 3$$

$$-x_6 + 5x_7 + x_8 + 5x_9 \leq 20$$

$$x_i \geq 0 \quad .$$

By introducing linking variable $x = x_5$ and denoting variables x_1, x_2, x_3, x_4 as internal variables z_A^i, $i = 1,4$ of problem A and x_6, x_7, x_8, x_9 as internal variables z_B^i, $i = 1,\ldots,4$ of problem B we can transform the initial problem into the appropriate form:

$$\min_{x \geq 0}\{f_A(x) + f_B(x)\}$$

where

$$f_A(x) = \min\{-z_A+2z_A^2-0.5z_A^3-z_A^4\}$$

$$z_A^1 + 2z_A^2 + 5z_A^3 + z_A^4 + 5x \leq 10$$

$$10z_A^1 + z_A^2 + 4z_A^3 + 5z_A^4 - 4x \leq 20$$

$$-z_A^1 - 5z_A^2 + z_A^3 + z_A^4 \leq 30$$

$$z_A^i \geq 0 \quad .$$

$$f_B(x) = \min\{4z_B^1-z_B^2-5z_B^3+z_B^4\}$$

$$10z_B^1 + z_B^2 + z_B^3 + 10z_B^4 + x \leq 40$$

$$z_B^1 + 0.1x \leq 3$$

$$-z_B^1 + 5z_B^2 + z_B^3 + 5z_B^4 \leq 20$$

$$z_B^i \geq 0 \quad .$$

The separate solutions of problems A and B with $x \geq 0$ as independent variables provide a solution:

$$\min_{x \geq 0} f_A(x) = f_A(x_A^*) = -4.8276$$

$$x_A^* = 1.0345$$

and

$$\min_{x \geq 0} f_B(x) = f_B(x_B^*) = -101.818$$

$$x_B^* = 0.0 \quad .$$

The following table shows the values of functions f_A and f_B at these points.

Table 1. Results of separate solutions of subproblems A and B.

	x_A^*	x_B^*
$f_A(x)$	-4.8276	-4.0
$f_B(x)$	-101.724	-101.818

The solution of problem (2) started from point $x^0 = 0$. Set Π was taken as the difference of the two simplexes

$$\Pi = \Pi_s - \Pi_s \quad , \quad \Pi_s = \{0 \leq \pi \leq 0.1\} \quad .$$

The simplex tableau for the master problem of finding $\hat{\Phi}(0)$ and its subgradient has the following form.

Table 2. Simplex tableau for the master problem.

	π_S^A	$-\pi_S^B$	p^+	p^-	v^+	v^-	
	$\hat{x}_A$	$-\hat{x}_B$	$\hat{x}_A-\hat{x}_B$	$\hat{x}_B-\hat{x}_A$	-1	1	v_A+v_B
	⋮	⋮	⋮	⋮			
	$\hat{x}_A$	$-\hat{x}_B$	$\hat{x}_A-\hat{x}_B$	$\hat{x}_A-\hat{x}_B$	-1	1	v_A+v_B
cost row					1	-1	

where we deleted separate constraints for π_S and introduced artificial variables p^+, p^-, v^+, v^- to take care of the nonpositivity of vectors $p = p^+ - p^-$ and $v = v^+ - v^-$,

$$v = \inf \{c_A z_A + \pi_S^A x_A + c_B z_B - \pi_S^B x_B + p(\hat{x}_A - \hat{x}_B)\}$$

$$A_A z_A + B_A x_A \leq b_A$$

$$A_B z_B + B_B x_B \leq b_B \quad .$$

In this tableau $\hat{x}_A$, $\hat{x}_B$ are the solutions of the auxiliary problems and v_A, v_B are the optimal costs of these solutions. The tableau for problems for generating new x_A and v_A has the following structure.

Table 3. Simplex tableau for subproblem A.

	z_A^1	z_A^2	z_A^3	z_A^4	x_A	b_A
	1	2	5	1	5	10
	10	1	4	5	-4	20
	-1	-5	1	1		30
cost row	-1	2	-0.5	-1	$-(\pi_S^A+p)$	

The corresponding tableau for $\hat{x}_B$ has a similar form but with different numbers.

Table 4. Simplex tableau for subproblem B.

	z_B^1	z_B^2	z_B^3	z_B^4	x_B	b_B
	10	1	1	10	1	40
	1				.1	3
	-1	5	1	5		20
cost row	4	-1	-5	1	π_B+p	

In different runs these tableaux differ only in cost rows which makes it possible to obtain new solutions from those generated on previous iterations by simplified methods of parametric programming. However, in a rather amateurish code written by the author to solve the problem under consideration, these possibilities have been omitted.

The results of sequential solving in master and slave problems are presented in the following table.

Table 5. Successive solutions of the master problem and subproblems A and B at initial point $x^{-} = 0$.

p	$\pi^A-\pi^B$	v	$\hat{x}_A$	$\hat{x}_B$	v_A	v_B
10	-	-	2	-2	0	-102
-10	-	-	-2	30	-2.4	-50
-1.467	-.1	96.33	-2.0	20	-2.4	-100
-0.069	-.1	101.923	1.0345	-2	-4.8276	-102
-0.265	-.1	106.224	1.0345	20	-4.8276	-100

The last run of the master problem resulted in

$$v = 106.448$$

$$\pi^A - \pi^B = -0.1 \quad ,$$

and at this point execution has been suspended because no new constraint was generated.

The result of these computations means that $\hat{\Phi}(x)$ has a negative slope at point $x = 0$, therefore the next calculations were made at point $x^0 = 2$. The following table presents the results of these calculations.

Table 6. Successive solutions of the master problem and subproblems A and B at initial point x = 2.

p	$\pi^A-\pi^B$	v	x_A	x_B	v_A	v_B
10	-	-	2.0	0.0	0.0	-101.818
-10	-	-	0.0	30	-4.0	-50
-1.501	0.1	98.817	0.0	20	-4.0	-100
0.090	0.1	101.998	1.0345	0.0	-4.8276	-101.818
-0.131	0.1	106.414	1.0345	20	-4.8276	-100

The last run of the master problem resulted in

$$v = 106.455$$

$$\pi^A - \pi^B = 0.1 \quad ,$$

and execution has been stopped because no new constraint was generated.

This means that $\hat{\Phi}(x)$ has a positive slope at $x = 2$ and hence the optimal value of the linking variable has to satisfy the equation

$$\hat{\Phi}(0) - 0.1x^* = \hat{\Phi}(2) + 0.1(x^*-2) \quad .$$

So far as

$$\hat{\Phi}(0) = -106.448 \quad ,$$

$$\hat{\Phi}(2) = -106.455 \quad ,$$

we obtain

$$x^* = 1.035 \quad ,$$

which coincides with good accuracy with an optimal value of a linking variable for problem A. Here are some computational results for several values of linking variables.

Table 7. Values of objectives for different values of the linking variable.

x	v_A	v_B	$v_A + v_B$	
0.0	-4.0	-101.818	-105.818	
0.035	-4.08	-101.815	-105.843	
1.034	-4.8275	-101.724	-106.552^*	optimum
2.0	0.	-101.636	-101.636	

It is clear that the value of problem A is influenced by a linking variable to a much greater extent, therefore it is no surprise that the optimal value of the linking variable is determined by this subproblem.

However, it is a matter of some specific properties of data and, generally speaking, we may obtain some intermediate point which lies somewhere within interval [0,2].

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