OPTIMAL FUND DISTRIBUTION

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Optimal Fund Distribution

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From D. Bell's paper WP-74-55, I received the impression that some members of our Methodology group are interested in the following optimization problem:

$$u(x) = \max \left[u_1(x_1) + \cdots + u_n(x_4)\right]$$

under constraints

$$\sum_i x_i = x(x_i \geq 0) \quad .$$

Suggested below is a simple result which gives a very clear description of maximum accumulation as a function of growing x in the case of the concave utility functions $u_i(\cdot)$.

Note that u(x) is a concave function because for any

$$\lambda', \lambda'' > 0 \quad ; \quad \lambda' + \lambda'' = 1 \quad .$$

We have

$$u(\lambda' x' + \lambda'' x'') = \max \left[\sum_i u_i(\lambda' x_i' + \lambda'' x_i'')\right]$$

$$\geq \max \left[\lambda' \sum_i u_i(x_i') + \lambda'' \sum_i u_i(x_i'')\right]$$

$$\text{s.t.} \quad \sum_i (\lambda' x_i' + \lambda'' x_i'') = \lambda' x' + \lambda'' x''$$

$$\geq \max \left[\lambda' \sum_i u_i(x_i') + \lambda'' \sum_i u_i(x_i'')\right]$$

$$\text{s.t.} \quad \sum_i x_i' = x' \quad , \quad \sum_i x_i'' = x''$$

$$= \lambda' \max \left[\sum_i u_i(x_i')\right] + \lambda'' \max \left[\sum_i u_i(x_i'')\right]$$

$$\text{s.t.} \ \sum_i x_i' = x' \qquad \text{s.t.} \sum x_i'' = x''$$

$$= \lambda' u(x') + \lambda'' u(x'')$$

Suppose a total fund x is distributed in units Δx.
Let $\bar{x} = \{x_i^o\}_{i=\overline{1,n}}$ denote an optimal distribution vector:

$$u(x) = \sum_i u_i(x_i^o) \quad .$$

<u>Theorem</u>. The following property of maximum accumulation holds true:

$$\overline{x + \Delta x} = \{x_i^o + \delta_{ij} \ \Delta x\}_{i=\overline{1,n}}$$

where δ_{ij} is the Kronecker symbol and the corresponding j is determined by a condition

$$u_j(x_j^o + \Delta x) - u_j(x_j^o) = \max_i \{u_i(x_i^o + \Delta x) - u_i(x_i^o)\} \quad .$$

Particularly,

$$u(x + \Delta x) - u(x) = \max_i \{u_i(x_i^o + \Delta x) - u_i(x_i^o)\} \quad .$$

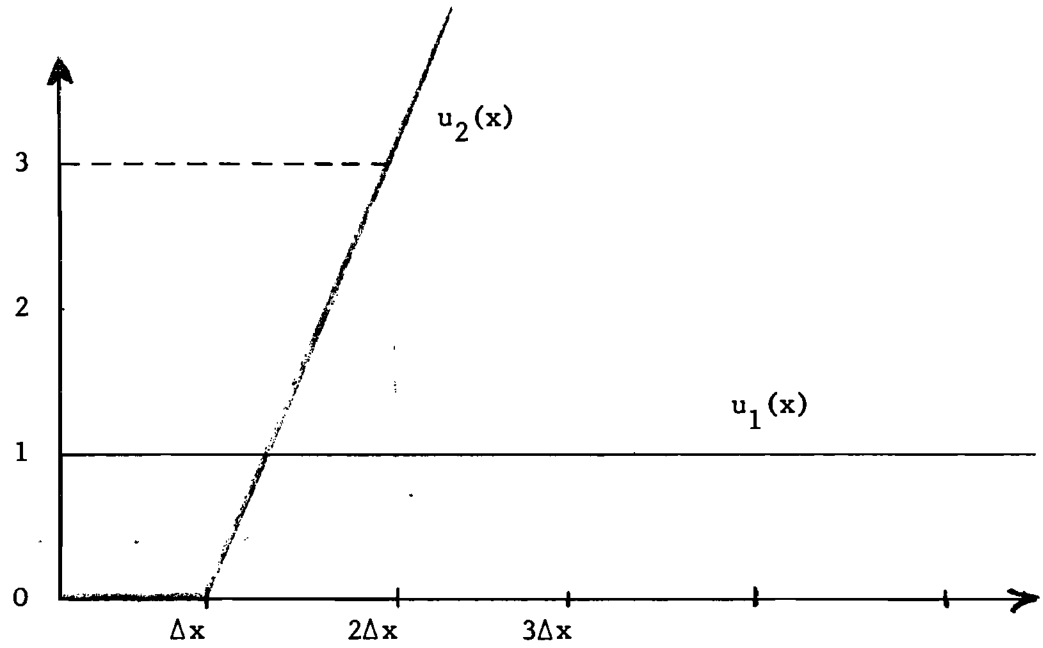
<u>Note that this property is not valid for non-concave functions $u_i(\cdot)$</u> -- see the following figure where

$$\overline{\Delta x} = \{\Delta x, 0\}$$

but

$$\overline{2\Delta x} = \{0, 2\Delta x\}$$

$u(x) = u_1(x) + u_2(x)$



The theorem itself is almost obvious.

Indeed, let

$$\overline{x + \Delta x} = \{y_i^o\}$$

be an optimal distribution vector so

$$u(x + \Delta x) = \sum_i u_i(y_i) \quad , \quad \sum_i y_i^o = x + \Delta x \quad .$$

For at least one component it has to be $y_j^o > x_j^o$ because otherwise

$$\sum_j y_j^o \leq \sum_j x_j^o = x \quad .$$

Let us set

$$x_j = y_j^o - \Delta x \quad , \quad y_j = x_j^o + \Delta x \qquad \text{for some} \quad y_j > x_j^o$$

and

$$x_i = y_i^o \quad , \quad y_i = x_i^o \quad \text{for} \quad i \neq j \quad .$$

Because $u_j(\cdot)$ is a concave function, and $x_j \geq x_j^o$, we have

$$u(x + \Delta x) - \sum_i u_i(x_i) = \sum_i u_i(y_i^o) - \sum_i u_i(x_i) =$$

$$= u_j(x_j + \Delta x) - u_j(x_j) \leq u_j(x_j^o + \Delta x) - u_j(x_j^o) =$$

$$= \sum_i u_i(y_i) - \sum_i u_i(x_i^o) =$$

$$= \sum_i u_i(y_i) - u(x) \quad .$$

where

$$u(x + \Delta x) \geq \sum_i u_i\ (y_i) \quad , \; u(x) \geq \sum_i u_i(x_i)$$

and it may be only if

$$\sum_i u_i(y_i) = u(x + \Delta x)$$

i.e., $\{y_i\}$ is the optimal distribution vector. Remember that

$$y_j = x_j^o + \Delta x \quad , \quad y_i = x_i^o \;\text{ for } i \neq j \quad !$$