LINKAGE OF OPTIMIZATION MODELS

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September 1979
WP-79-83

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ACKNOWLEDGEMENT

The authors wish to thank Dr. Etienne Loute of the Center for Operations Research and Econometrics, Universite Catholique de Louvain, Belgium, for carrying out the computational experiments with the Dantzig-Wolfe decomposition technique reported in Section 10.

ABSTRACT

The general goal of this article is to investigate the question of how to carry out analysis when a set of mathematical models being used are interdependent. We seek systematic ways of linking such models to each other. The linking approaches should preserve the structure of the original models so that their interpretation during the analysis does not get increasingly complicated. Although the emphasis is on linking two interdependent linear programming models, extensions to multimodel, nonlinear, and stochastic cases can, in principle, be straightforward.

The article has been divided into two parts. In the first part we give a precise statement of our interdependent systems. As well, we offer three typical examples of such systems: energy supply--economy, manpower--economy, and forestry--wood processing industry interaction systems. In the second part we consider alternative approaches: classical decomposition principles, approaches derived from nondifferentiable optimization techniques, application of paremetric programming techniques as well as the simplex method combined with a partitioning technique. By no means does the paper provide a final solution to our linkage problem. However, our computational experiments indicate that some of the approaches give rise to optimism, while others remain inconclusive.

CONTENTS

INTRODUCTION 1

Part I: MODELS

1. Statememt of the Problem 4
2. Energy-Economy Interaction Model 8
3. Skilled Labor Supply-Economy Model 14
4. Forestry-Wood Processing Industry Model 18

Part II: METHODS

5. Summary of Alternative Linking Approaches 23
6. Dantzig-Wolfe Decomposition Principle 24
7. Nondifferentiable Optimization Approach 27
8. Parametric Programming 31
9. A Basis Factorization Approach 38
10. Computational Experiments 44
11. Some Extensions and Other Applications 48

SUMMARY AND CONCLUSIONS 53

REFERENCES 56

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INTRODUCTION

Mathematical modeling is widely used in operations research and systems analysis. Among them, optimization models are probably the most common type [7,19]. Examples are energy, water and other resource supply models; models for national settlement planning, industrial or agricultural production planning, and manpower and educational planning; resource allocation models; etc., (see for example [5,6,11,16,17]).

However, at present models are, for the most part, analyzed independently, without linking them into a system. This approach has limited possibilities because many important features of the real systems, those involving interactions, are missing. Hence, the investigation of interrelations among separate models is becoming more and more urgent. Examples are such linkages as

energy supply - economy
water supply - agricultural production
manpower - economy

and so on.

When investigating interrelations between models, two basic approaches can be singled out. First, separate models may be integrated into a single optimization model with a joint objective function. Practically, however, this often means to build the combined model anew. The second approach is to link the already existing models, considering the models on some independent basis each with its own objectives.

Naturally, both approaches have their own advantages and disadvantages. The major advantage of the first approach is that it allows one to combine all the constraints and variables influencing the joint behavior of the submodels in the most expedient way. However, building an integrated model often leads to a very large optimization problem, which, even if possible to solve, is always difficult to interpret.

The second or "manual" approach, in which information obtained from one model is interpreted by an analyst and provided as input to another model, is more attractive but also much more time consuming and may lead to uncertainty whether the "true optimal" solution for the whole system has been obtained. Therefore, we would like to combine features of both approaches, that is, to develop decomposition schemes which:

- maintain the structure of the subproblems independently, thus permitting a sequence of subproblem solutions which are easy to interpret
- may easily permit an analyst's interference in the linkage process
- lead finally to optimization of the whole integrated problem.

This paper presents different approaches for linkage of models, both finite step and iterative. Actually, these approaches are based on some decomposition scheme as applied to an integrated model. However, in a decomposition approach we begin with an integrated model and a particular decomposition scheme implies a particular partition of the model. In linkage, on the other hand, we begin with the submodels which already exist (and are operational) and a particular linkage scheme defines the structure of the resulting integrated model. Therefore, in the linkage approach we have the possibility of utilizing information available from the operational submodels and, in fact, should take care to do so.

We limit ourselves to the case where the model can be formulated in the framework of linear programming (LP) or, in particular, dynamic linear programming (DLP), though the approach can, in principle, be extended to nonlinear or stochastic cases. We also limit ourselves to the case where the common objective for the integrated model can be expressed in explicit form.

The paper consists of two parts. In the first, we state the problem in general terms and describe several linkage models (energy-economy, manpower-production, forestry - wood processing).

The second part presents methods. Since we are dealing with partitioning schemes, we first review the Dantzig-Wolfe decomposition principle (D-W) as applied to an integrated model. Then we describe briefly iterative methods which amount to a nondifferentiable optimization technique. The main attention, however, is given to finite step methods, particularly to a basis factorization scheme for the simplex method as applied to the problem in question. Finally, we consider some computational tests, extend the approach when the number of submodels is arbitrary, and give examples of how two-stage stochastic linear programming and dynamic linear programming can be treated by the linkage approach.

Part I: MODELS

1. Statement of the Problem

Let us consider two LP problems P_i, $i = 1$ and 2, in the form:

$$\underset{x_i}{\text{maximize}}\ c_i' x_i$$

subject to

$$(P_i) \qquad A_i x_i = b_i \tag{1.1}$$

$$D_i x_i = y_i \tag{1.2}$$

$$x_i \geq 0$$

In problem P_i, x_i is an n_i-vector, b_i is an m_i-vector and y_i is a k-vector. In vector products, the left factor is a row, the right a column.

If we are considering models P_1 and P_2 separately then there is no distinction between constraints (1.1) and (1.2) and both vectors b_i and y_i are given exogenously (for i = 1 and 2). However, when we start to analyze interaction between the two models, we have to consider variables y_1 and y_2 as endogenous, subject to some coupling constraints.

We assume that the integrated model has an objective function F which is a weighted sum of "local" objective function:

$$F = \alpha_1 c_1' x_1 + \alpha_2 c_2' x_2$$

where α_1 and α_2 are some weight coefficients, and that the coupling constraints are given in the form

$$R_1 y_1 + R_2 y_2 = r \tag{1.3}$$

where r is a given vector. (There may also be nonnegativity constraints on the y_i.)

Thus the integrated model (P) can be stated as

$$\underset{x_i, y_i}{\text{maximize}} \sum_{i=1}^{2} c_i x_i$$

subject to

$$(\text{P}) \quad \begin{aligned} A_1 x_1 &= b_1 \\ D_1 x_1 &= y_1 \\ A_2 x_2 &= b_2 \\ D_2 x_2 &= y_2 \\ R_1 y_1 + R_2 y_2 &= r \end{aligned} \qquad \begin{aligned} & x_1 \geq 0 \\ \\ & x_2 \geq 0 \end{aligned}$$

where $c_i = \alpha_i c_i'$. We shall hereafter use only c_i.

Problem P has a special block-angular structure (see Figure 1) where I_1 and I_2 are identity matrices of appropriate dimension. It can easily be reduced to the conventional block-angular structure (Figure 2). In fact, substituting (1.2) into (1.3), one obtains

$$R_1 D_1 x_1 + R_2 D_2 x_2 = r$$

or

$$\bar{R}_1 x_1 + \bar{R}_2 x_2 = r \quad . \tag{1.4}$$

On the other hand, conventional block-angular structure (Figure 2) which is written

$$\text{maximize } c_1 x_1 + c_2 x_2$$

subject to

$$\begin{aligned} A_1 x_1 &= b_1 \\ A_2 x_2 &= b_2 \\ \bar{R}_1 x_1 + \bar{R}_2 x_2 &= r \\ x_1 \geq 0 \quad &, \quad x_2 \geq 0 \end{aligned} \tag{1.5}$$

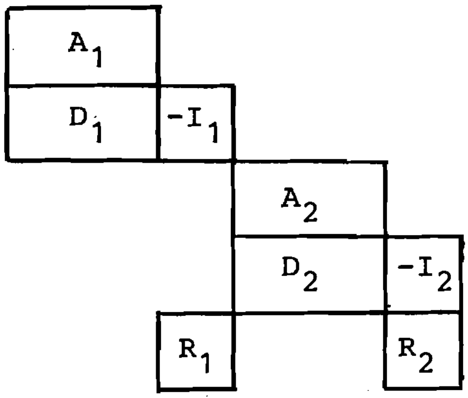

Figure 1. Constraint matrix of Problem P.

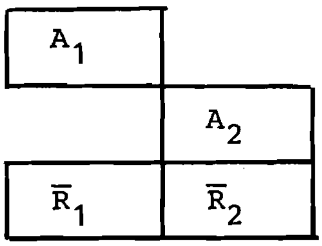

Figure 2. Conventional block-angular structure.

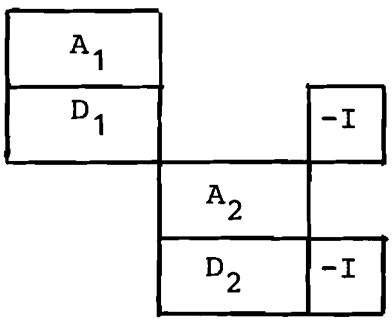

Figure 3. Constraint matrix of Problem $\bar{P}$.

can be easily transformed into the structure of Problem P. In fact, if we denote

$$\overline{R}_i x_i = y_i \tag{1.6}$$

then the constraint (1.4) appearing in (1.5) takes the form $y_1 + y_2 = r$. Hence problems P and (1.5) are equivalent in this sense. For our purposes, however, the formalization of the problem in the form P, rather than in the conventional form (1.5), is preferable. The reason for this is that separate models, which can be singled out from the integrated model (1.5)

$$\begin{aligned} &\text{maximize } c_i x_i \\ &\text{subject to} \\ &A_i x_i = b_i \qquad (i = 1 \text{ or } 2) \\ &x_i \geq 0 \end{aligned} \tag{1.7}$$

do not reflect, as a rule, all features of the individual subsystems P_i. Therefore we shall consider further only the integrated model P and associated "local" submodels P_1 and P_2.

Our prime interest lies in two types of interactions between the subsystems: (i) supply from one subsystem is demand for the other, and (ii) the subsystems share joint resources. As will be illustrated below, examples of the first type are energy supply from the energy sector to the rest of the economy, skilled labor supply from the educational subsystems to other sectors, and raw wood supply from forestry to wood processing industry. Examples of the second type are joint labor, land, water and financial resources between the agricultural subsystem and the rest of the economy.

Both types of interactions may be taken into account through a k-vector y of linking variables so that $y_i = y + r_i$. For the first type, components of r_i are equal to zero and y thus denotes supply for one system and demand for the other. For the second type of interaction, y may refer to the demand of joint resources for subsystem 1. Thus, if r is the total availability of resources and $y_2 = -D_2 x_2$ is the usage by the second subsystem, then the amount left for the first subsystem is $y_1 = r + D_2x_2 = r - y_2$ rather than $y_1 = y_2$ as in the first type of condition. Note, however, that there is then a bound on y_2, i.e. $y_2 \leq r$.

Problem P will now be transformed (as noted above, without loss of generality) into problem $\bar{P}$:

$$\text{maximize } c_1x_1 + c_2x_2$$

subject to

$$(\bar{P}) \qquad \begin{aligned} A_1x_1 &= b_1 \\ D_1x_1 &= y + r_1 \qquad x_1 \geq 0 \\ A_2x_2 &= b_2 \\ D_2x_2 &= y + r_2 \qquad x_2 \geq 0 \quad . \end{aligned}$$

The structure of the constraint matrix for $\bar{P}$ is depicted in Figure 3.

Our task is to consider methods for solution of integrated models. In this we can use different decomposition or partitioning schemes. But, among possible schemes, we shall select such methods as preserve the structure of local submodels, P_1 and P_2, and use the information which is available from solutions of these local submodels.

Before considering solution methods, we will describe some typical examples of system interactions.

2. Energy-Economy Interaction Model

For analyzing long-range energy policy in a country (region), first the so-called energy supply system (ESS) has been described (see [12,16] and references there). This model can be formulated verbally as follows: for existing initial structure of the secondary energy production capacities and under given supply constraints for primary energy resources and nonenergy resources (labor, capital, etc.) which are needed for development of the ESS, find a transition to such a mix of secondary energy production options (fossil, nuclear, solar, etc.) which satisfies the projected energy demand and minimizes the total cost of such transition. In the ESS model, (endogenous) decision variables are annual increases of secondary energy production capacities. There are two major exogenous variables in the ESS model which represent basic links with the rest of the economy: final demand for energy (which is the output of an economy model) and the supply of nonenergy resources required by the energy sector (which is the output of the ESS model).

Clearly, the efficiency of modeling can be extended to a great degree if the ESS model is linked with the economy model. In this case, one can analyze not only the energy policies but possible changes in the structure of the economy as well, in order to influence the demand for energy and the supply of non-energy resources to the energy sector. Thus the analysis of interrelations between the ESS and economy models is currently of great practical importance.

We will start the discussion of energy-economy interaction by combining the ESS model and the economy model. For a uniform representation we assume that both the industrial processes of the economy and the energy sector are described in terms of physical flows. Furthermore, in the model we omit, for simplicity, time lags in construction of production capacities. The integrated model is just a dynamic whole-economy model where special attention is paid to the energy sector.

A major part of the model is an input-output model of the economy. Let $x(t)$ be the vector of (levels of) production activities including those for both the energy sector E and nonenergy sectors NE. Accordingly, we will partition x into $\binom{x_E}{x_{NE}}$ below. If $A(t)$ is the matrix of input-output coefficients, then $(I - A(t))x(t)$ is the vector of net production. This is used for (net) export $s(t)$, consumption $w(t)$ and investments $B(t)v(t)$ where $v(t)$ is the vector of investment activities for increasing production capacity and $B(t)$ is a matrix transforming investments into usage of products of various types. In this notation, the input-output model may be given as

$$(I - A(t))x(t) = B(t)v(t) + w(t) + s(t) \quad . \tag{2.1}$$

Production $x(t)$ is restricted both in the energy sector and other sectors through capacity availability $y(t)$:

$$x(t) \leq y(t) \quad . \tag{2.2}$$

For the capacity vector $y(t)$ we have the state equation

$$y(t+1) = (I - \Delta(t))y(t) + v(t) \tag{2.3}$$

where $\Delta(t)$ is a diagonal matrix of depreciation and $v(t)$, as indicated above, refers to investments.

Labor availability as well as other constraints for resources which are external with respect to the whole system (land, water, etc.), the so-called WELMM factors [9], may be written as

$$R(t)x(t) \leq r(t) \tag{2.4}$$

where $R = (R_{ij})$ is a matrix defining the usage of resource i per unit of production j, and $r(t)$ is the vector of resources available.

The accumulated consumption of primary energy resources by the beginning of period t is denoted by a vector $z(t)$ for which we have the state equation

$$z(t+1) = z(t) + Q(t)x_E(t) \quad . \tag{2.5}$$

Here the matrix $Q = (Q_{ij})$ shows the amount of primary energy resource i extracted per unit of energy production activity j. Primary energy resources are available up to an amount given by vector $\overline{z}(t)$:

$$z(t) \leq \overline{z}(t) \quad . \tag{2.6}$$

Consumption of goods is assumed to occur according to an externally given profile vector (or distribution) $g(t)$ so that consumption $w(t)$ is given by

$$w(t) = g(t)u(t) \quad , \tag{2.7}$$

where $u(t)$ (for each t) is an endogenous variable determining the level of consumption.

To state the complete integrated energy-economy model, we use for illustrative purposes the total discounted consumption as an objective function. (Many other objectives are of interest for this integrated model, of course.) If the discounting factor for period t is $\beta(t)$, the problem is to find nonnegative vectors $x(t)$, $y(t)$, $v(t)$, $w(t)$, $z(t)$, and a scalar $u(t)$, for all t, to

$$\text{maximize } \sum_t \beta(t)u(t)$$

subject to (2.1) - (2.7), for all t, and with initial state $y(0)$ and $z(0)$.

This problem is a dynamic linear programming (DLP) model [17]. It allows us to investigate the interactions between (a detailed) energy sector and nonenergy sectors of an economy. Such an integrated model has been developed by G. Dantzig and S. Parikh at the Stanford University (PILOT model). It describes, in physical terms, technological interactions within the sectors of the U.S. economy including a detailed energy sector [6].

We shall now turn our discussion to two separate models, the ESS model and the economy model, which were integrated above. For this purpose we shall partition the input-output matrix $A(t)$ into

$$A = \begin{pmatrix} A_E^E & A_{NE}^E \\ A_E^{NE} & A_{NE}^{NE} \end{pmatrix} \tag{2.8}$$

where A_E^E is the coefficient matrix within the energy sector and

A_{NE}^{NE} that one within the rest of the economy. Similarly we partition all vectors and matrices of the integrated model to correspond to the energy sector and the nonenergy sector for which we shall use suffices E and NE, respectively. For instance, x_E refers to the energy production while x_{NE} refers to the production of all other goods.

Let $d_E(t)$ be the sum of the energy supply to nonenergy sectors. to consumption and to export, and let $d_{NE}(t)$ be the demand of goods from other sectors to the energy sector. Then (2.1) yields

$$d_E(t) = (I - A_E^E(t))x_E(t) - B_E^E(t)v_E(t) \qquad (2.9)$$

and

$$d_{NE}(t) = A_E^{NE}(t)x_E(t) + B_E^{NE}(t)v_E(t) \quad . \qquad (2.10)$$

Denote by $r_E(t)$ the usage of the WELMM resources in the energy sector; i.e.

$$r_E(t) = R_E(t)x_E(t) \quad . \qquad (2.11)$$

As an example, we may want to minimize the total discounted cost for maintanence and construction of energy production capacity. If this cost for period t is given by $c^1(t)x_E(t) + c^2(t)v_E(t)$, and $\beta(t)$ is the discounting factor, then the ESS model may be stated as follows:

find nonnegative vectors $y_E(t)$, $v_E(t)$, $x_E(t)$, and $z(t)$, for all t to

$$\text{minimize} \sum_t \beta(t)(c^1(t)x_E(t) + c^2(t)\, v_E(t))$$

subject to (2.5), (2.6), (2.9) - (2.11), the energy sector part of (2.2) and (2.3), and with the initial state $y_E(0)$ and $z(0)$.

Similarly, we state the economy model for nonenergy sectors. In this case (2.1) yields for $d_E(t)$ (the demand of energy) and for $d_{NE}(t)$ (the supply of goods to the energy sector)

$$d_E(t) = A_{NE}^E(t)x_{NE}(t) + B_{NE}^E(t)v_{NE}(t) + w_E(t) + s_E(t) \quad , \qquad (2.12)$$

and

$$d_{NE}(t) = (I - A_{NE}^{NE}(t))x_{NE}(t) - B_{NE}^{NE}(t)v_{NE}(t) - w_{NE}(t) - s_{NE}(t) \quad . \qquad (2.13)$$

Given the WELMM resources $r_E(t)$ used by the energy sector, (2.4) yields for the nonenergy sector

$$R_{NE}(t)x_{NE}(t) \leq r(t) - r_E(t) \quad . \qquad (2.14)$$

If the objective function is adapted from the integrated model, the economy model may be stated as follows:

find nonnegative vectors $y_{NE}(t)$, $v_{NE}(t)$, $x_{NE}(t)$, $w(t)$, and $u(t)$,

for all t, to

maximize $\sum_t \beta(t)u(t)$,

subject to (2.7), (2.12) - (2.14), the nonenergy part of

constraints (2.2) and (2.3), and with the initial state $y_{NE}(0)$.

If the exogeneous supply and demand vectors $d_E(t)$ and $d_{NE}(t)$ and the WELMM usage $r_E(t)$ of the latter two models are considered as endogeneous coupling variables, then the ESS model and the economy model jointly comprise an equivalent model with our integrated energy-economy model.

Thus, we have three models: the ESS model and the economy model, (which represent energy and nonenergy sectors of an economy), and an integrated model. Furthermore, the latter model can be written in such a way that it will contain the two other models as submodels. Clearly, such an integrated model has the structure of Problem $\overline{P}$ (Fig. 3). In this case, block A_1 is associated with the energy submodel, matrix D_1 represents demand constraints for secondary energy, supply constraints of nonenergy resources, and the WELMM constraint, while block A_2 and D_2 are associated with the economy submodel.

3. Skilled Labor Supply-Economy Model

The separate educational model aims to find such enrollments to different educational institutions as will both satisfy the availability constraints on educational capacities (e.g. teachers, buildings, etc.) and be as close as possible to the projected manpower demand. Hence, in this model available educational capacities and demand for labor are exogenous variables.

When the interaction between manpower and economic development is analyzed, two major options should be taken into account: development of some sectors in an economy in order to absorb the projected surplus in manpower of certain types and development of educational capacities in order to fill up possible shortages in manpower for other sectors of an economy. We may also have to consider the possibility of labor force migration into and out of the system.

In addition, the model should be disaggregated on major economic activities (various industrial sectors, agriculture, construction, transportation, public administration and other services) and on the levels of education (primary, secondary, higher) [15].

Thus one can see that, methodologically, a skilled labor supply-economy interaction model is close to the energy supply-economy model described above. Below we shall consider first a simple integrated model.

Let m(t) be the vector of skilled manpower at time t, n(t) be the vector (of the same dimension) of the manpower increase during time period t, and $P(t)$ be the transition matrix. Then the state equations for the manpower/educational subsystem will be the following for each time period t:

$$m(t + 1) = P(t)m(t) + n(t) \quad . \tag{3.1}$$

The training of people requires resources, first of all, teachers. We define a square matrix $F(t)$ such that $F(t)n(t)$ is the need for skilled labor for training those determined by $n(t)$. Second, training requires physical resources; i.e. buildings and equipment. Let $y_e(t)$ be the upper limit on trainees set by physical resources (during period t) so that

$$n(t) \leq y_e(t) \quad . \tag{3.2}$$

The development of resources $y_e(t)$ can be expressed as part of the development of the vector $y(t)$ of production resources (capacity) of the whole economy. Let $v(t)$ be the vector of investment activities for increasing production capacity, and let $\Delta(t)$ denote the matrix of depreciation. Then we have

$$y(t + 1) = (I - \Delta(t))y(t) + v(t) \quad . \tag{3.3}$$

The balance of goods production and consumption for the whole economy is given as usual:

$$(I - A(t))x(t) = B(t)v(t) + w(t) \quad . \tag{3.4}$$

Here $x(t)$ is the vector of production activities, $A(t)$ is the matrix of input-output coefficients, matrix $B(t)$ defines the consumption of goods per unit of investment activities, and $w(t)$ is the vector of final consumption of goods.

Production $x(t)$ is restricted by production capacity:

$$x(t) \leq y_{ne}(t) \quad , \tag{3.5}$$

where $y_{ne}(t)$ is the noneducational part of the physical resource vector $y(t)$. Production is also restricted by skilled labor availability/demand relations so that

$$F(t)n(t) + L(t)x(t) = m(t) + s^{-}(t) - s^{+}(t) \quad , \tag{3.6}$$

where the matrix $L(t)$ specifies requirement of skilled labor for each sector of the economy and vectors $s^{-}(t)$ and $s^{+}(t)$ express the shortage and surplus (respectively) of skilled labor.

The consumption vector $w(t)$ may again be given as

$$w(t) = g(t)u(t) \quad , \tag{3.7}$$

where $g(t)$ is an exogenously given vector of consumption profile, and the scalar $u(t)$ expresses the consumption level (for each t).

With the above model optimal policies with different objective functions can be analyzed. For illustrative purposes, we shall define vectors of weights $\lambda^{-}(t)$ and $\lambda^{+}(t)$, and, as part of the objective, the weighted sum I_1 of the labor shortage and surplus as follows:

$$I_1 = \sum_t (\lambda^{-}(t)s^{-}(t) + \lambda^{+}(t)s^{+}(t)) \quad . \tag{3.8}$$

The other part of the objective may again consist of the total discounted consumption I_2:

$$I_2 = \sum \beta(t)u(t) \quad , \tag{3.9}$$

where $\beta(t)$ is the discounting factor.

The integrated problem is now to find nonnegative vectors $m(t)$, $n(t)$, $y(t)$, $x(t)$, $v(t)$, $s^{+}(t)$ and $s^{-}(t)$, and scalar $u(t)$, for all t, to

$$\text{maximize } -I_1 + I_2 \quad ,$$

subject to (3.1) - (3.9), for all t, and

with the initial state y(0), m(0).

This problem has the block-angular structure of Figure 2 where block A_1 is now associated with the educational/manpower model and block A_2 with the economy/production model. In order to transform this model into the form of Problem P with explicit separation of educational and economy submodels, we partition (3.4) into

$$(I - A(t))x(t) = B_{ne}(t)v_{ne}(t) + f(t) + w(t) \quad , \qquad (3.10)$$

and

$$f(t) = B_e(t)v_e(t) \quad , \qquad (3.11)$$

where e and ne refer to educational and noneducational parts (of investment activities) and f(t) is the supply of resources for development of an educational subsystem. Furthermore, (3.6) is partitioned into

$$d(t) = L(t)x(t) \quad , \qquad (3.12)$$

and

$$d(t) = m(t) - F(t)n(t) + s^-(t) - s^+(t) \quad , \qquad (3.13)$$

where d(t) is the demand of labor for all but educational purposes.

In this notation, the educational subproblem is to find nonnegative vectors m(t), n(t), $y_e(t)$, $v_e(t)$, $s^-(t)$ and $s^+(t)$, for all t, to minimize I_1 defined in (3.8) subject to (3.1), (3.2), (3.11), (3,13) and the educational part of (3.3), as well as with the initial state m(0) and $y_e(0)$, and with externally given f(t) and d(t).

Similarly, the problem of the economy subsystem is to find nonnegative vectors $y_{ne}(t)$, $v_{ne}(t)$, $x(t)$ and scalar $u(t)$, for all t, to maximize total consumption I_2 subject to (3.5), (3.7), (3.10), (3.12) and the noneducational part of (3.3), as well as with the initial state $y_{ne}(0)$ and with externally given coupling variables $f(t)$ and $d(t)$.

4. Forestry-Wood Processing Industry

We shall now consider the interdependent systems of wood supply (forest development) and wood processing; i.e. forestry and wood based industry [11]. The discussion begins with the wood supplying part describing the growth of the forest given harvesting and planting activities as well as land availability over time. The wood consuming part consists of an input-output model describing the production process as well as production capacity and financial resource considerations.

Let $w(t)$ be a vector determining the number of trees of various types (say pine, spruce and birch) in different age categories at the beginning of time period t. We define a square transition (or growth) matrix Q so that $Qw(t)$ is the number of trees at the beginning of period $t + 1$ given that nothing is harvested nor planted. Thus, matrix Q describes aging and death of the trees. Let $p(t)$ and $h(t)$ be vectors of planting and harvesting activities, respectively, of different kinds (e.g. planting of different types of trees and terminal harvesting or thinning), and let the matrices P and H be defined so that $Pp(t)$ and $-Hh(t)$ are the incremental change in the tree quantity caused by the planting and harvesting activities. Then, for the state vector $w(t)$ of the number of trees we have the following equation:

$$w(t+1) = Qw(t) + Pp(t) - Hh(t) \quad . \tag{4.1}$$

Planting is restricted through land availability. We may formulate this so that the total stem volume of trees in forests cannot exceed a given volume $L(t)$ during t. Thus, if w is a vector of stem volume per tree for different types of trees in various age groups, then the land availability restriction may be stated as

$$Ww(t) \leq L(t) \quad . \tag{4.2}$$

Given the level of harvesting activity $h(t)$, there is a minimum requirement for the planting activity $p(t)$:

$$p(t) \geq Nh(t) \quad , \tag{4.3}$$

where N is the matrix transforming harvesting activity to the planting requirements. Such a requirement may be enforced by law, for instance.

In our simple formulation we shall leave out other restrictions, such as harvesting labor or capacity. Finally, the wood supply $y(t)$, given the level of harvesting activities $h(t)$, is given for period t as

$$y(t) = SHh(t) \quad . \tag{4.4}$$

Here the matrix $S = (S_{ij})$ transforms a tree of a certain type and age combination j into a volume of type i of raw wood (e.g. pine log, spruce pulpwood, etc.).

For illustrative purposes we may choose as an objective to maximize the discounted total profit I_F of forestry. If this profit for period t is given as $c^1(t)y(t) - c^2(t)h(t) - c^3(t)p(t)$, and $\beta(t)$ is the discounting factor, we may state the forestry model as:

find nonnegative vectors $w(t)$, $p(t)$, and $h(t)$, for all t, to

$$\text{Maximize } I_F = \sum_t \beta(t)\left(c^1(t)y(t) - c^2(t)h(t) - c^3(t)p(t)\right)$$

subject to (4.1) - (4.4), and with the initial state

$w(0)$ and specified wood supplies $y(t)$.

For the industrial side, let $x(t)$ be the vector of production activities for period t (such as the production of sawn goods, panels, pulp, paper, and converted wood products), and let U be the matrix of wood usage per unit of production activity. The wood demand for period t is then given as

$$y(t) = Ux(t) \quad . \tag{4.5}$$

Note that the matrix U may also have negative elements. For instance, sawmill activity consumes logs but produces pulpwood as a byproduct.

Let A be an input-output table so that $(I - A)x(t)$ is the vector of wood product supply to the final market. If $D(t)$ is the corresponding (maximum) external demand, we require

$$(I - A)x(t) \leq D(t) \quad . \tag{4.6}$$

Production is restricted by the capacity $c(t)$ available:

$$x(t) \leq c(t) \quad . \tag{4.7}$$

The vector $c(t)$ in turn has to satisfy the state equation

$$c(t + 1) = (I - \Delta)c(t) + v(t) \quad , \tag{4.8}$$

where Δ is a diagonal matrix accounting for depreciation and $v(t)$ is the increment from investments during period t. The vector $v(t)$ of investment activities is restricted through financial considerations. To specify this, let $m(t)$ be the state variable of cash at the beginning of period t, let $G(t)$ be the vector of sales revenue less direct production costs per unit of production, let $F(t)$ be the vector of monetary fixed costs per unit of capacity, let $\ell(t)$ be the amount of external financing employed by the industry at the beginning of period t, let δ be the interest rate for external financing per period, let $\ell^+(t)$ be new loans made during period t, let $\ell^-(t)$ be loan repayments during t, and let $E(t)$ be the vector of cash expenditure per unit of increase in the production capacity. Then, the state equation for cash may be written as

$$m(t+1) = m(t) + G(t)x(t) - F(t)c(t) - \delta\ell(t) - \ell^{-}(t) + \ell^{+}(t) - E(t)v(t) \quad . \tag{4.9}$$

Finally for the industrial model, we may write the state equation for external financing as follows:

$$\ell(t+1) = \ell(t) - \ell^{-}(t) + \ell^{+}(t) \quad . \tag{4.10}$$

Again, for illustrative purposes, we may choose the total discounted profit (before taxes), denoted by I_p, as an objective function for the industrial model. This then is given as follows: find nonnegative vectors $x(t)$, $c(t)$, $v(t)$, and scalars $m(t)$, $\ell(t)$, $\ell^{+}(t)$, and $\ell^{-}(t)$, for all t, to

$$\text{maximize } I_p = \sum_t \beta(t)[G(t)x(t)-(F(t)+E(t)\Delta)c(t)-\delta\ell(t)] \quad ,$$

subject to (4.5) - (4.10) and

with the initial state $c(0)$, $m(0)$, and $\ell(0)$.

For both of the models above, the wood supply $y(t)$ from the foresty model to the industrial model is considered as exogeneous. For the integrated model we consider $y(t)$ as an endogeneous vector of coupling variables, for all t. Thus, if the total discounted profit $I_F + I_p$ is chosen as our objective, our integrated model for forestry and forest industry may be stated as follows: find nonnegative vectors $w(t)$, $p(t)$, $h(t)$, $y(t)$, $x(t)$, $c(t)$, $v(t)$, and scalars $m(t)$, $\ell(t)$, $\ell^{+}(t)$ and $\ell^{-}(t)$, for all t, to

$$\text{maximize } I_F + I_p \quad ,$$

subject to (4.1) - (4.10), and

with initial state $w(0)$, $c(0)$, $m(0)$ and $\ell(0)$.

The structure of constraints for period t of the integrated model is shown in Figure 4.

$w(t+1)$	$w(t)$	$p(t)$	$h(t)$	$y(t)$	$x(t)$	$c(t)$	$v(t)$	$m(t)$	$\ell(t)$	$\ell^{+}(t)$	$\ell^{-}(t)$	$c(t+1)$	$m(t+1)$	$\ell(t+1)$	
I	-Q	-P	H												= 0
		I	-N												≥ 0
	W														≤ L
			-SH	I											= 0
				I	-U										= 0
					I-A										≤ D
					I	-I									≤ 0
						-(I-Δ)	-I					I			= 0
					-G	F	E	-1	δ	-1	1		1		= 0
									-1	-1	1			1	= 0

Figure 4. Constraint matrix of the forest sector model for period t.

Part II: METHODS

5. Summary of Alternative Linking Approaches

In this part we consider different methods for solution of our integrated problem

$$\underset{x_i, y}{\text{maximize}} \sum_{i=1}^{2} c_i x_i$$

subject to

$$(\overline{P}) \qquad \begin{aligned} A_i x_i &= b_i \\ D_i x_i &= y + r_i \\ x_i &\geq 0 \quad , \text{ for } i = 1, 2 \end{aligned}$$

via linking through the coupling vector y the associated local subproblems P_i $(i = 1, 2)$:

$$\underset{x_i}{\text{maximize}}\ c_i x_i$$

subject to

$$(P_i) \qquad \begin{aligned} A_i x_i &= b_i \\ D_i x_i &= y + r_i \\ x_i &\geq 0 \quad . \end{aligned}$$

A conventional way of solving the integrated problem $\overline{P}$ is to apply the Dantzig-Wolfe decomposition principle [7]. Application of this principle eventually transforms problem $\overline{P}$ into a block-angular structure where the coupling constraints imply that vector y in both local problems are equal. The approach leads to a price mechanism which coordinates the usage of this common vector of resources. We shall briefly review this approach in Section 6.

The dual approach of the decomposition principle, Benders' decomposition [3], applies directly to the structure of problem

$\overline{P}$. The method results in a resource allocation scheme for the coupling vector y so that eventually an optimal allocation y for $\overline{P}$ is obtained. This approach might also be of interest for our linking problem. However, it can be interpreted as an application of Dantzig-Wolfe decomposition to the dual of ($\overline{P}$), and, therefore, in the following we shall not consider Benders' decomposition separately as a linking tool.

The optimal value of the local problem P_i is a nondifferentiable (piecewise linear and concave) function of the coupling vector y. Nondifferentiable optimization techniques yield an approach for coordinating y in such a way as to obtain the maximal value for the sum of the optimal values of the local problems. This approach, which is discussed in Section 7, does not have finite convergence, and convergence rate is important here. In Section 8 we aim at preserving the favorable properties of the nondifferentiable optimization approach while seeking for faster convergence. The resulting method is a heuristic approach based on a parametric programming technique applied to the coupling vector y.

In Section 9 we consider the simplex method as a linking technique. This approach may be interpreted as a basis factorization scheme applied to the integrated problem $\overline{P}$. The resulting method deals with local problems P_i having simple side constraints on the coupling vector y.

Finally, some computational experiments will be reported. The decomposition principle and the simplex method (resulting from our basis factorization) are applied to link moderate sized models of forestry and wood processing industry, such as described in Section 4 above.

6. Dantzig-Wolfe Decomposition Principle

The Dantzig-Wolfe Decomposition principle--D-W for short--is so well known that it is unnecessary to describe it. It is clear that a D-W approach can be applied to our problems. Procedurally, a D-W algorithm is less intricate than most other schemes--at least if one ignores the problem of getting final results in terms of the original variables. The amount of information to be transmitted appears modest--one pricing row must be sent to each subproblem, each of which sends back one or more candidate columns (which, however, can be massive in the aggregate). So why look for some other approach? There are valid reasons.

First, the technique of driving a model exclusively with prices is inappropriate for some cases. (Although the existence of unboundedness in a subproblem is inconclusive in a D-W approach, this does not appear to be a serious consideration in the present context). In fact, the real problem is frequently to find appropriate allocations of resources and a more direct approach than a pricing mechanism seems desirable.

Second, D-W algorithms may have poor convergence properties. While it may be true that a mathematically proven optimal solution is frequently not required, the potentially large number of grand cycles before reaching an acceptable solution is a disadvantage.

Third, the highly composite nature of a solution cannot be ignored. The local center performing the calculations for a subproblem must recompute the actual final solution which can be a sizeable job in itself. In order to do this, either this center must keep track of all candidate columns and be given back final composite variable values with suitable identification, or the controlling center must keep all candidate columns, multiply them by the final composite variable values and send their sum back to the subproblem center. Even this rigamarole may actually be the lesser of two evils. The composite candidate columns hide the meaning of their component columns, i.e., the true variables, so that no intuition can be brought to bear in adjusting the derived problem. Furthermore, these columns are almost always dense and tend to be nearly linearly dependent, the more so as optimality is approached. This is not only the cause of slow convergence but contributes to digital instability in the derived problem.*

Against these and one or two other disadvantages of a D-W approach can be set a number of potential advantages. Perhaps the greatest one is that the interpretation of prices by the subproblem and the returned mappings are strictly local matters. That is, the controlling center does not need to know how the subproblem is solved or what meaning is attributed to the preferred candidate columns. When this degree of decentralization is desirable, a D-W approach may be found appropriate.

It is perhaps worth pointing out that the real problem can be interpreted as a resource allocation problem as well. Let the coupling constraints have the form

$$\sum_{p=0}^{P} A_p X_p = b_0 \quad , \tag{6.1}$$

and the subproblem constraints

$$B_p X_p = b_p \quad , \quad X_p \geq 0 \quad , \quad p > 0 \quad . \tag{6.2}$$

If one knew the optimal contribution to b_0 of the products $b_{0p} \equiv A_p X_p$, the whole problem would decompose into P + 1 independent problems. If this were done, the optimal price vectors π_{0p} for the common resources b_{0p} (computed by the subproblems p, for all p) would not necessarily have the same values compared with each other and compared with the price vector π_0 for the coupling constraints (6.1) computed directly as an optimal price vector for the integrated problem. Hence the concept of uniform prices is to some degree arbitrary.

With respect to the foregoing, the dependence of D-W on the prices as driving forces for the subsystems creates another

*We distinguish between the original master problem (coupling constraints) and the derived problem of mappings used in the D-W approach.

difficulty in making adjustments at the control level. For example, if by chance an optimal set of prices were submitted to each subproblem on the first cycle, this would remain undetected and many grand cycles would still be carried out, with various alterations in the pricing row. An optimal set of allocations b_{0p}, on the other hand, would immediately be recognized as such.

7. Nondifferentiable Optimization Approach

A straightforward approach to the solution of problem $\bar{P}$ leads to the following constructions:

Let us define (i = 1, 2):

$$f_i(y) = \max \{c_i x_i | A_i x_i = b_i;\ D_i x_i = y + r_i;\ x_i \geq 0\} \tag{7.1}$$

and

$$X_i(y) = \{x_i | A_i x_i = b_i;\ D_i x_i = y + r_i;\ x_i \geq 0\} \quad . \tag{7.2}$$

$X_i(y)$ is the feasible set for subproblem P_i, if y is fixed; $f_i(y)$ is the optimal value of subproblem P_i for this y:

$$f_i(y) = \max_{x_i \in X_i(y)} c_i x_i \quad .$$

Define also Y as a set of all y, for which $X_1(y) \cap X_2(y)$ is nonempty. For all $y \in Y$ both subproblems P_i have a solution. We shall also call the set Y feasible.

Clearly, the integrated problem $\bar{P}$ is equivalent to:

$$\max_{y \in Y} [f_1(y) + f_2(y)] = \max_{y \in Y} \sum_{i=1}^{2} \max_{x_i \in X_i(y)} c_i x_i \quad . \tag{7.3}$$

Therefore the solution of problem $\bar{P}$ is reduced to the solution of local subproblems P_i and then optimization of linkage (coupling) variable y. However, there are major difficulties in such an approach: first, the functions $f_i(y)$ are nondifferentiable and, second, the feasible set Y is not given explicitly. Based on the theory of linear programming, and nondifferentiable optimization methods, the following lemma states the result concerning nondifferentiability of $f_i(y)$:

Lemma 7.1 Let Y be bounded and let $X_i(y)$ be bounded for all $y \in Y \subset R^k$. Then $f_i(y)$ is a continous concave piece-wise linear function. The derivative of this function in direction g is given by

$$f'_{ig}(y) = \min_{v_i \in V_i(y)} v_i g \quad , \tag{7.4}$$

where $V_i(y)$ is the set of optimal dual variables for problem P_i, associated with the constraint $D_i x_i = y + r_i$.

These constructions give a basis for developing different linkage methods based on the nondifferentiable technique. The most simple for realization is the general gradient method [18]. For problem $\overline{P}$ it yields:

$$y^{\nu+1} = P_Y[y^\nu + \rho_\nu(v_1^\nu + v_2^\nu)] \quad , \tag{7.5}$$

where ν is the iteration count, $P_Y(z)$ is projection operator of vector z on set Y, v_i^ν is an optimal dual price for the constraints $D_i x_i = y^\nu + r_i$ in subproblem P_i $(i = 1,2)$.

Theorem 7.1 [18] Let

$$\sum_{\nu=0}^{\infty} \rho_\nu = 0; \quad \rho_\nu \to 0, \quad \nu \to \infty \quad ,$$

then

$$y^\nu \to y^* \quad , \quad x_i^\nu \to x_i^* \quad ,$$

where x_i^ν is an optimal solution of P_i for $y = y^\nu$, and $\{y^*, x_1^*, x_2^*\}$ is an optimal solution of the integrated problem $\overline{P}$.

The major advantages of the algorithm (7.5) is that it requires minimal information from solutions of subproblems (practically only optimal dual variables v_i^ν for coupling resources) and thus, it is simple in realization. Another advantage of this approach is that it can be easily generalized to nonlinear and stochastic cases [8].

However, convergence of the algorithm (7.5) may be rather slow; i.e. it requires many subsequent solutions of subproblems P_i with different y^ν and thereby many projection operations as well. The latter difficulty may be overcome by taking into account the fact that projection should be done subsequently for different vectors y^ν, which are very close to each other.

Another difficulty which should be underlined here is that the algorithm (7.5) is hard to control manually in the sense that a concrete value of y^ν in fact does not mean much as only average tendency is important.

Another implementation of nondifferentiable technique is based on the idea of feasible directions. Suppose that $\{x_{iB}^\nu, x_{iN}^\nu\}$ is an optimal basic solution of subproblem P_i for $y = y^\nu$ and B_i is the basis, associated with this solution. Then

$$x_{iB}^\nu = B_i^{-1}\begin{pmatrix} b_i \\ y^\nu + r_i \end{pmatrix} \doteq \bar{x}_{iB}^\nu + \phi_i^\nu y^\nu \qquad (i = 1.2) \qquad (7.6)$$

and

$$x_{iB}^\nu \geq 0 \quad , \quad x_{iN}^\nu = 0 \quad .$$

As $\{x_{iB}^\nu, x_{iN}^\nu\}$ is optimal solution of subproblem P_i (for both i) for $y = y^\nu$, then the solution of the integrated problem $\bar{P}$ can be improved only by changing y^ν. Let g^ν be a vector in R^k and define, for a scalar θ_ν, $y^{\nu+1}$ as follows:

$$y^{\nu+1} = y^\nu + \theta_\nu g^\nu \quad . \qquad (7.7)$$

The problem is to find an appropriate direction g^ν for changing y^ν and the corresponding step size θ_ν.

Using (7.4) and (7.6) the general problem for defining an optimal g^ν can be stated as

$$\max_{g} \min_{v_i \in V_i(y^\nu)} \sum_i v_i g$$

$$[\phi_i^\nu g]_j \geq 0 \qquad j \in J_i^\nu, \quad i = 1, 2 \quad . \tag{7.8}$$

Here J_i^ν is the index set of active constraints in (7.6); that is

$$J_i^\nu = \{j; [\bar{x}_{iB}^\nu + \phi_i^\nu y^\nu]_j = 0\} \quad ,$$

and vector g should be normalized in order to avoid unbounded solutions.

The length of step is defined then straightforwardly:

$$\theta^\nu = \min \{\theta_1, \theta_2\} \quad ,$$

$$\theta_i = \max \quad \{\theta | \bar{x}_{iB}^\nu + \phi_i^\nu(y^\nu + \theta g^\nu) \geq 0\} \ , \quad i = 1, 2.$$

This algorithm gives, in principle, monotonic convergence and requires less iteration than (7.6). However, implementation of particular steps of the algorithm (especially choice of feasible direction (7.8) is a rather complicated problem. Other approaches to solution of problem $\bar{P}$ based on nondifferentiable technique are described in [2,8].

As one can see above, both these schemes consist of subsequent solution of local subproblems P_i with fixed y, then improving y and so on. This approach leads immediately to the necessity of handling nondifferentiable functions $f_i(y)$. In order to avoid this difficulty we may try to change simultaneously y and x_i. Such an approach is the simplex method discussed in Section 9, for instance.

8. Parametric Programming

The two primary forms of parametric programming--changes to the right-hand-side column (RHS) and changes to the objective row (OBJ)--are the oldest extensions to the simplex method, dating back to 1953-4. Later, application of both forms under a single parameter was developed (usually called RIM for the "rim" of the model) as well as parametrization of a structural column or structural row. These five forms essentially exhaust the practicable possibilities since more general sets of parametrized coefficients lead to virtually intractable computational problems [13].

In spite of the fact that elaborate computer programs for all these forms have existed for about fifteen years, and longer for the first two, they seem to be used relatively infrequently. One reason for this may be the numerical instability often encountered. Except possibly for the OBJ form, the usual difficulties of pivot selection and threshhold tolerances are compounded in the parametrics. Also, by their nature, they will ultimately push a model toward a point of either extreme degeneracy or singularity. Nevertheless, the parametrics and their twin ranging procedures produce more information per iteration than any other solution algorithms. This can also be a difficulty: there is so much information that it is difficult to comprehend and utilize. However, it is not necessary to drive parametrics to ultimate limits in order to make good use of them.

The ranging procedures compute the maximal changes possible in objective, demand, availability, or technological coefficients--individually or by unmixed vector amounts--which do not require a change in the set of active variables, i.e. a change of basis. They also give incremental costs and may indicate the change of basis which would be required to move further. The parametrics compute exactly the same sort of information but use it to drive the parametrized part of the model to new values, step by step.

These two forms of post-optimal analysis--they both work properly only from initial optimal solutions--can be used together. A ranging procedure can be used selectively to indicate desirable directions of change for critical commodities and, roughly, weights to be applied to each. Combining these weighted directions into a single vector, the latter can be considered a parametric change and the twin parametric algorithm used to achieve a nontrivial amount of change. At least this is conceptually possible. A practical difficulty is that a single direction which the feasible space limits to a zero amount can result in a zero change for all directions. This is not quite the same as the usual problem of degeneracy. Variables corresponding to critical values are already at a limit (or they would not be critical) and this may or may not engender degeneracy. The more likely question to be resolved is whether the current limit values are arbitrary or absolute. For example, if the amount of a fuel resource allowed to supply model limits the availability of useful energy to satisfy demand, the amount of resource is not usually the point at issue, but whether the proper mix of secondary forms has been produced. Of course, if new requirements imply still more fuel, the resulting amount of change will probably be zero. Often, however, one is looking for the best scedule of activities to convert primary resources to useful forms. It is the variability of the latter which must be adjusted.

Suppose a feasible allocation is known and two submodels have been solved with this allocation. Suppose one model represents supply and the other demand. The supply model will likely have certain resources left over--it would be nonoptimal to use more resources than necessary just because they are (arbitrarily) available. The demand model will tend to use all commodities allowed to it in the most profitable way. Generally speaking, each such commodity (input to demand) will have a dual value indicating the incremental value of having more (conceivably less in some situations such as undesireable byproducts which must be utilized). Some judgment may have to be applied to these dual values; a figure of 1000, say, is not necessarily 100 times better than a figure of 10. Essentially, however, these values may be used as weights to form a linear vector sum which is treated as a change column to the original allocation. Each submodel then makes a parametric RHS run. In doing so, each model generates a piecewise linear objective function in which a vertex corresponds to a change of basis. Such piecewise linear functions are continuous and convex. However, some dual values are discountinous at a change of basis while others are constant. The initial slope of the objective function may be positive or negative. For an infeasible step size, the functional value is defined to be $-\infty$. These functions can be plotted as follows:

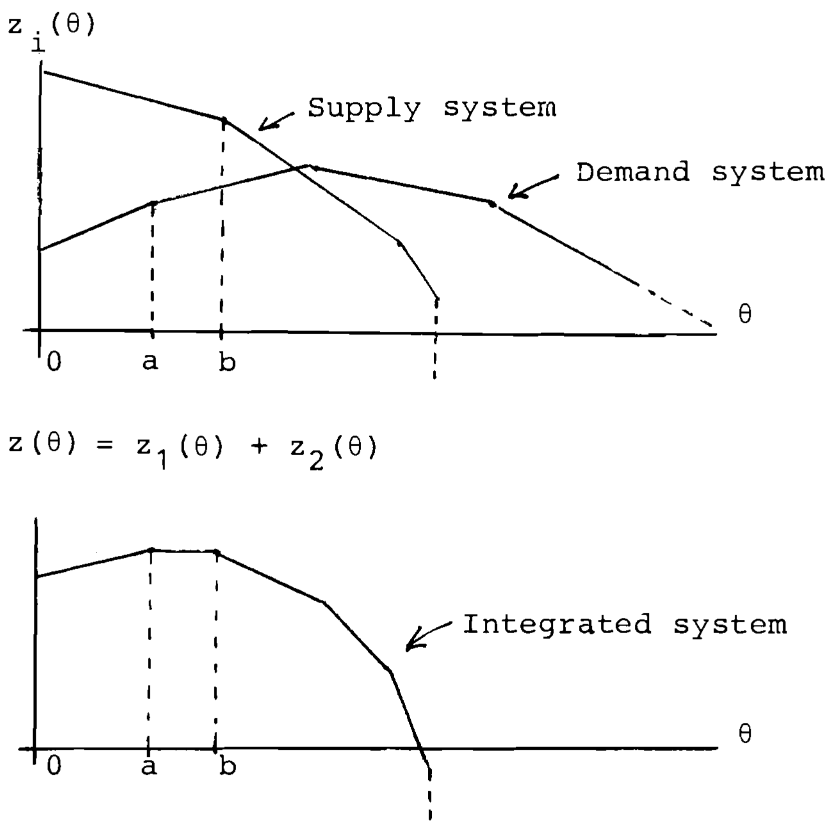

Figure 5. Optimal values z_i of the subsystems as a function of parameter θ.

A maximum of the sum in the illustration occurs for any value of the parameter θ between a and b. The maximum must always occur on such a segment or at a point where one model changes basis. As shown, the demand basis changes at a and the supply basis at b and one pair of bases is optimal for the two models. This process can be repeated giving, presumably, different weighted vector sums to parametrize. There is no problem of improvement so long as the initial slope for the sum is positive and the parameter achieves a positive value with this slope, since then the global functional must improve (maximization being assumed). However, considerable care must be taken that the effective initial slope is in fact positive. What can happen is that an immediate change of basis changes a (discountinuous) dual enough to cause a positive slope to turn negative before any actual movement is made. To guard against this, the submodels should invoke the RHS ranging procedure before reporting dual values. Changes in an allocation which would require an immediate change of basis must be reported with the less favorable dual value. The effective incremental rate of change in the functional cannot then be overstated, though it may be understated due to cancellations. As the optimal allocation is approached, the sums of dual values may not give good change ratios and the procedure may terminate suboptimal.

Perhaps the easiest way to describe both the procedure and its difficulties is with a tiny numerical example:

DEMAND	SUPPLY
maximize $2w_1 + w_2$	minimize $x_1 + 1.8x_2$
subject to	subject to
$.5w_1 + .3w_2 \geq 5$	$0 \leq x_1 \leq 9$
$.2w_1 + .7w_2 \geq 6$	$0 \leq x_2 \leq 9$
$0 \leq w_1 \leq y_1$	$.2x_1 + .6x_2 \geq y_1$
$0 \leq w_2 \leq y_2$	$.6x_1 + .3x_2 \geq y_2$

A feasible allocation for both models is $y_1 = 6$, $y_2 = 8$. The initial and optimal simplex tableaux, set up as usual with slacks and for minimizing, for both models are as follows:

u_0	u_1	u_2	u_3	u_4	w_1	w_2	β_1
1					-2	-1	0
	1				-.5	-.3	-5
		1			-.2	-.7	-6
			1		1		6
				1		1	8
1			2	1			20
	1		.5	.3			.4
		1	.2	.7			.8
			1		1		6
				1		1	8

v_0	v_1	v_2	v_3	v_4	x_1	x_2	β_2
1					1	1.8	9
	1				1		9
		1				1	9
			1		-.2	-.6	-6
				1	-.6	-.3	-8
1	2.6			6			-24.6
	1				1		9
	2	1		10/3			1/3
	-1		1	-2			1
	2			-10/3		1	26/3

Table 1. Initial and optimal simplex tableau for the DEMAND and SUPPLY problems.

The slacks u_3, u_4 correspond to y_1, y_2 for DEMAND, and the slacks v_3, v_4 correspond to y_1, y_2 for SUPPLY. β_1, β_2 are the basic solution vectors for the two models and the dual values appear in the top lines. The functional z_1 for DEMAND is 20, z_2 for SUPPLY is -24.6, giving a total functional $z = z_1 + z_2 = -4.6$.

DEMAND would like to increase both y_1 and y_2, with y_1 preferred by a ratio of 2 to 1, as indicated by the dual values for u_3 and u_4. Both can increase indefinitely but y_1 can decrease only by .8 (.4/.5) and y_2 by 1.1428 (.8/.7). However, changes in either direction are finite. SUPPLY on the other hand, would like to decrease y_2 with a weight of 6 but is indifferent about y_1 up to an increase of 1 unit. (The reversal of signs for the third and fourth constraints in the tableau must be taken into account in interpreting directions).

If we set up a parametric RHS change column reflecting both desires, we are assured of a nonzero step size. Let Δ represent this column and γ_1, γ_2 be its representation in both models (tableaux) with due regard to signs (which are contrary). Then

$$\Delta \equiv \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \\ -5 \end{bmatrix}, \quad \gamma_1 = -B_1^{-1}\Delta = \begin{bmatrix} 1 \\ .5 \\ 3.1 \\ -2 \\ 5 \end{bmatrix}, \quad \gamma_2 = B_2^{-1}\Delta = \begin{bmatrix} -30 \\ 0 \\ -16.66 \\ 12 \\ 16.66 \end{bmatrix} .$$

Taking ratios with the β-columns gives $\theta_1 = .8/3.1 = .258$ for DEMAND and $\theta_2 = 1/12 = .0833$ for SUPPLY. Thus up to $.0833\Delta$, the rate of improvement in z is $-(\gamma_{01} + \gamma_{02}) = 29$ or a change of $29/12 = 2.4166$ is obtained. DEMAND is free to move further but SUPPLY must make a change of basis at this point, with v_1 replacing v_3. When it does so, the new value of γ_{02} is 1.2 so that, since γ_{01} is positive, any further change would decrease z rather than increase it.

If the indicated change of basis is made in SUPPLY, its new tableau is:

$$\begin{array}{c} \{v_0 \quad v_1 \quad v_2 \quad v_3 \quad v_4 \quad | \quad x_1 \quad x_2\} \\ \left[\begin{array}{ccccc|cc} 1 & & & 2.6 & .8 & & \\ \hline & & & 1 & -2 & 1 & \\ & & 1 & 2 & -.66 & & \\ & 1 & & -1 & 2 & & \\ & & & 2 & -7.33 & & 1 \end{array}\right] \end{array} \begin{array}{c} \beta_2 \\ \begin{bmatrix} -22.1 \\ 9 \\ 1.722 \\ 0 \\ 7.277 \end{bmatrix} \end{array}$$

Now it wants to decrease y_1 and y_2; say, in the ratios 2.6 to .8. The desired change for the DEMAND system remains unchanged, on the other hand. Thus, the new direction of change Δ in RHS and the updated columns ν_1 and ν_2 may be given as follows:

$$\Delta \equiv \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -2.6 \\ -.8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -.6 \\ .2 \end{bmatrix}, \quad \gamma_1 = \begin{bmatrix} 1 \\ .24 \\ .05 \\ .6 \\ -.2 \end{bmatrix}, \quad \gamma_2 = \begin{bmatrix} -1.4 \\ -1.0 \\ -2.33 \\ 1 \\ -2.66 \end{bmatrix} .$$

Although $\gamma_{01} + \gamma_{02}$ has the correct sign, $\theta_2 = 0$ and no movement can be made. To move further, SUPPLY must make another change of basis with v_3 replacing v_1, i.e. cancelling the previous change but γ_{02} then changes to 1.2 and would lead to a degradation in z. This impasse could have been predicted by a ranging procedure on the nonbasic slacks v_3 and v_4. However, the best rule to adopt here is that, since DEMAND is not binding and SUPPLY is, the reduction requested by SUPPLY should be honored to the next change of basis, provided the overall effect is favorable. (It can be shown that no combination of the requests of both models is both feasible and favorable at this point but this much information is not readily available to a controlling center). Taking SUPPLY prices with the first optimal basis we have the following:

$$\Delta = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -6 \end{bmatrix}, \quad \gamma_1 = \begin{bmatrix} 6 \\ 1.8 \\ 4.2 \\ 0 \\ 6 \end{bmatrix}, \quad \gamma_2 = \begin{bmatrix} -36 \\ 0 \\ -20 \\ 12 \\ 20 \end{bmatrix}, \quad \theta_1 = .1314, \quad \theta_2 = .3806$$

At this point, the prices which previously led to an impasse again hold but now permit a change of .0079 limited by θ_1. Cumulative results so far are as follows:

$$y_1 = 5.8619, \; y_2 = 6.89683, \; z_1 = 18.62063, \; z_3 = -20.75847$$

giving a combined value of $z = -2.13784$, a considerable improvement but not optimal. The main variables have values

$$w_1 = 5.8619, \; w_2 = 6.89683, \; x_1 = 7.93171, \; x_2 = 7.12598.$$

If the combined problem is solved, the optimal solution is:

$$w_1 \doteq 5.61538,\ w_2 = 7.30769,\ x_1 = 9.0,\ x_2 = 6.35897$$

with $z = -1.90769$ and implied values for y

$$y_1 = w_1 \quad , \quad y_2 = w_2 \quad .$$

Note that the first step above gave a significant improvement with no difficulty. Thereafter, either the step size was small or the proper choice for Δ somewhat problematical. Even in this tiny problem, it is not clear how to proceed further toward optimality.

9. A Basis Factorization Approach

We shall now develop a version of the revised simplex method for $\overline{P}$, which may be characterized as follows: At each iteration all the columns of y are basic and the nonbasic variables of one or the other of the vectors x_i, but not both, are considered for entering the basis. One set is considered as long as it can provide an entering variable. Thereafter the roles of x_1 and x_2 are interchanged. Switches back and forth will be made until $\overline{P}$ has been solved. While allowing components from one of the x_i vectors to enter, the nonnegativity constraints of the currently basic variables in the other x_i vector will be treated in a special way. In effect, this will allow us to work with what we call the local basis matrices of the dimensions $(m_i + k) \times (m_i + k)$.
We aim at developing for this approach an implementation which can be done easily given that we have an existing computer code for the simplex method utilizing the product form for the inverse. While doing so, we may assume that a feasible allocation for the coupling variabley y is known in advance. In practice, such knowledge may be available from past experience, for instance.

Let the current basis B for $\bar{P}$ have the following partitioning

$$B = \begin{bmatrix} A_1^B & & \\ D_1^B & -I & \\ & & A_2^B \\ & -I & D_2^B \end{bmatrix} ,$$

where A_i^B and D_i^B correspond to basic columns in x_i, for $i = 1, 2$, and the rest of the columns in B refer to y. We shall refer to B as a <u>global basis</u>. Without loss of generality, assume that the columns of y are permuted so that the square submatrix

$$B_2 \equiv \begin{bmatrix} 0 & A_2^B \\ -I_2 & D_2^B \end{bmatrix} ,$$

in B is nonsingular. Here we denote by I_i an $k_i \times k_i$ identity matrix for some $0 \leq k_i \leq k$, such that $k_1 + k_2 = k$. We say that k_2 of the columns of y in B belong to the $(k + m_2) \times (k + m_2)$ subbasis B_2. Accordingly, we partition y into

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} ,$$

where $y_i \in R^{k_i}$.

Let A_i^N and D_i^N be the columns in A_i and D_i corresponding to the nonbasic variables in x_i and partition x_i and c_i similarly into $x_i = \begin{pmatrix} x_i^B \\ x_i^N \end{pmatrix}$ and $c_i = (c_i^B, c_i^N)$. In this notation $\bar{P}$ may be depicted as in Figure 6 below.

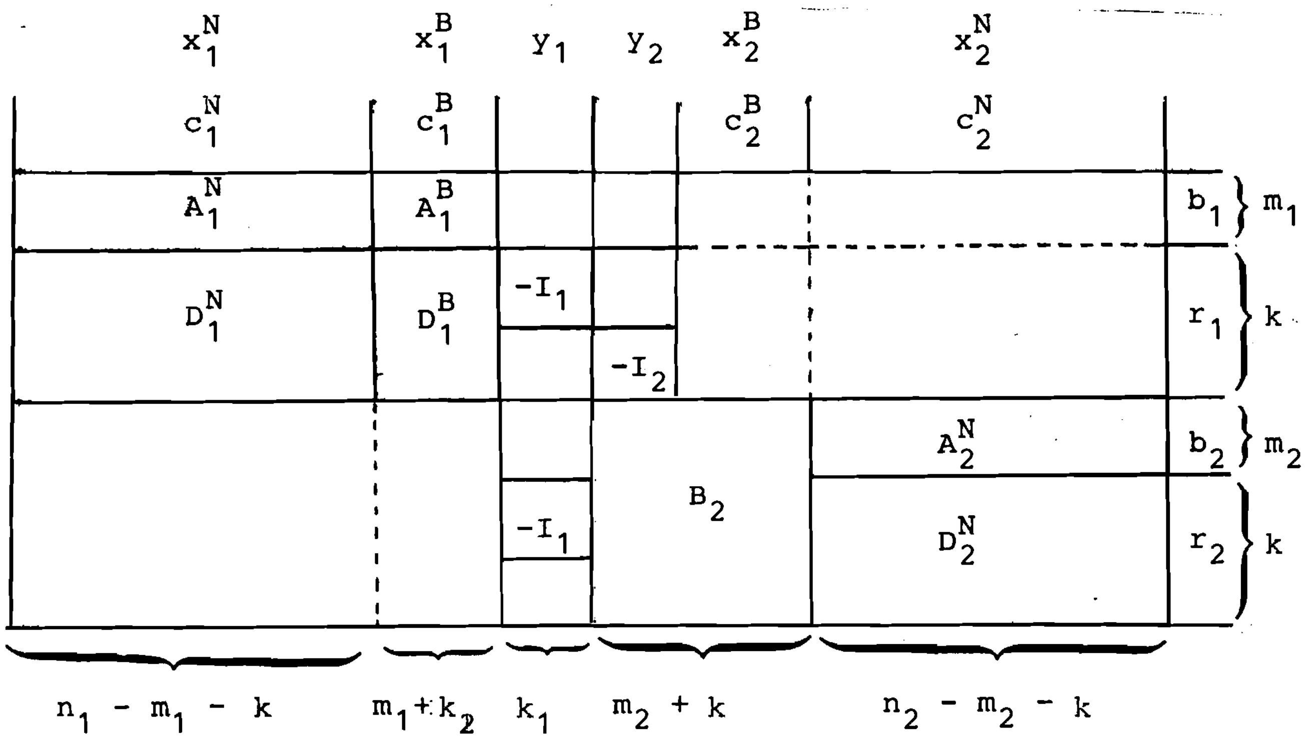

Figure 6. Problem $\overline{P}$.

Because B_2 is assumed to be nonsingular, we obtain for y_2 and x_2^B

$$\begin{pmatrix} y_2 \\ x_2^B \end{pmatrix} = B_2^{-1}\left[\begin{pmatrix} b_2 \\ r_2 \end{pmatrix} + \begin{pmatrix} 0 \\ I_1 \\ 0 \end{pmatrix} y_1 - \begin{pmatrix} A_2^N \\ D_2^N \end{pmatrix} x_2^N\right] . \tag{9.1}$$

We shall now consider a simplex iteration which starts with the basis B and allow only nonbasic variables in x_1 to enter the basis. For this purpose we may set x_2^N in (9.1) to zero to obtain

$$y_2 = \overline{y}_2 - F\, y_1 , \tag{9.2}$$

$$x_2^B = \overline{x}_2^B - G\, y_1 , \tag{9.3}$$

where $\overline{y}_2$, $\overline{x}_2^B$, F and G are defined according to (9.1) with $x_2^N = 0$. Substituting y_2 and x_2^B from (9.2) and (9.3) into $\overline{P}$ yields the restricted problem, called subproblem (S_1), as follows:

find x_1 and y_1 to

$$\begin{array}{llr} (S_1^1) & \text{maximize} & c_1x_1 + d_1y_1 \\ (S_1^2) & \text{s.t.} & A_1x_1 = b_1 \\ (S_1^3) & & D_1x_1 + Hy_1 = b_0^1 \\ (S_1^4) & & x_1 \geq 0 \\ (S_1^5) & & Gy_1 \leq \bar{x}_2^B \quad , \end{array}$$

and subject to an additional requirement which prevents a component of x_2^B from re-entering after it has left the basis of $(\overline{P})$. According to our previous notation, we have in (S_1), $d_1 = -c_2^B G$, $H = \begin{pmatrix} I_1 \\ -F \end{pmatrix}$, and $b_0^1 = r_1 - \begin{pmatrix} 0 \\ \bar{y}_2 \end{pmatrix}$. The term d_1y_1 in (S_1^1) accounts for the change in the objective function of $\overline{P}$ which is caused by a change in the basic variables x_2^B. (S_1^5) is the nonnegativity requirement for the basic components x_2^B of x_2. The final additional requirement in (S_1) is taken into account so that a component of y_1 is eliminated from (S_1) each time a constraint (S_1^5) becomes binding. The elimination is carried out via solving a component of y_1 from the binding constraints. This approach allows a special treatment for the constraints (S_1^5) thereby yielding a particularly attractive solution technique for (S_1). In the following, we shall discuss in detail how (S_1) is solved while carrying out simplex iterations for $\overline{P}$.

According to our definitions, the matrix

$$B_1 \equiv \begin{bmatrix} A_1^B & 0 \\ D_1^B & H \end{bmatrix} , \tag{9.4}$$

comprises a basis for (S_1^2) and (S_1^3). We shall consider B_1 as an initial basis while solving (S_1). We treat (S_1^5) as side-constraints which we may forget for a moment. The pricing operation for problem $(S_1^1 - S_1^4)$ is carried out as usual in the simplex method. An entering variable e is chosen among the nonbasic variables x_1^N, the alpha-column $\alpha = B_1^{-1}a$, where a is the column for e in

$\begin{bmatrix} A_1^N \\ D_1^N \end{bmatrix}$, is computed, and the minimum ratio test subject to the constraint $x_1^B \geq 0$ is carried out to determine a step size $\bar{\theta}$ in the usual way.

At this point we shall return to the side constraints (S_1^5). We partition α into $\begin{pmatrix} \alpha^x \\ \alpha^y \end{pmatrix}$, where α^y refers to the change in y_1. Thus, if $\bar{y}_1$ is the current value for y_1, the maximum step size $\bar{\bar{\theta}}$ allowed by the side constraints (S_1^5) (the nonnegativity of the variables x_2^B) can be computed as the maximizer of the following minimum ratio test:

$$\max_{\theta} \quad \theta(G\alpha^y) \leq \bar{x}_2^B - G\bar{y}_1 \quad . \tag{9.5}$$

If $\bar{\theta} < \bar{\bar{\theta}}$, a component of x_1^B leaves the basis B_1 as well as the global basis B, and column a of the entering variable replaces the column of the leaving variable in both of these bases.

In relatively rare cases $\bar{\theta} \geq \bar{\bar{\theta}}$. In this case, a component of x_2^B leaves the global basis B, and a slightly more complicated update is needed for the inverse of the local basis B_1. Using the binding constraint, which by the last requirement in (S_1) must be an equality, we eliminate one component of y_1 from (S_1). The following results show that updating the inverse of the local basis B_1 now requires pre-multiplication by two elementary matrices instead of only by one which is the case if $\bar{\theta} < \bar{\bar{\theta}}$.

<u>Lemma 9.1</u> Let g be the row of G corresponding to the binding constraint in (S_5^1). Then $g\alpha^y \neq 0$, and, in particular, there exists p such that $g_p \neq 0$.

For updating B_1^{-1} we shall define two elementary matrices R and E such that ERB_1^{-1} is the updated local basis inverse. We define

$$\rho_j = \begin{cases} g_k & \text{for } j = m_1 + k_2 + i \text{ and for } 1 \leq i \leq k_1 \\ 0 & \text{otherwise} \quad , \end{cases} \tag{9.6}$$

and, because $g\alpha^y \neq 0$, we may define

$$\eta_j = \begin{cases} 1/g\alpha^y & \text{of } j = m_1 + k_2 + p \\ -\alpha_j/g\alpha^y & \text{otherwise} \quad , \end{cases} \tag{9.7}$$

where p is defined according to Lemma 1. In this notation we have the following result:

<u>Theorem 9.2</u> Consider the case $\bar{\theta} \geq \bar{\bar{\theta}}$, and define p according to Lemma 1. Let R and E be $(k + m_1)\times(k + m_1)$ elementary matrices, whose $(m_1 + k_2 + p)^{th}$ row (for R) and column (for E) are defined by (9.6) and (9.7), respectively. Then the updated local basis inverse for subproblem (S_1) is given by

$$ERB_1^{-1} \quad ,$$

where B_1^{-1} is the current local basis inverse for (S_1). Furthermore, the updated objective function coefficient vector for y_1 is obtained if the i^{th} element of the vector d_1R^{-1} is omitted.

An interpretation for this partitioning scheme can be found from the block-product algorithm described in [14] or from other approaches which can be viewed as extensions of the generalized upper bounding technique (see e.g. [10]). In this framework, our approach does not allow a nonbasic variable from a subblock to enter the basis. As a result, only the most simple cases of the block-product algorithm can occur. When only the subblock variables allow improvement for the objective function, we interchange the roles of the two subsystems; i.e., the current subblock becomes the master block. Also the proofs of Lemma 9.1 and Theorem 9.2 may be found as special cases of the results derived in [10,14], for instance.

It is worth noting that a parametric approach is intermediate between sole reliance on prices, as in D-W, and careful control of feasibility, as in our partitioning scheme (for carrying out what amounts to a variant of the simplex mehtod on the global problem). In all three cases, prices or reduced costs are only indicative and never guarantee the best move. However the assumptions regarding initial feasibility are somewhat different in the three approaches:

D-W: Enough candidate columns must be gleaned from the submodels to provide a feasible basis for the derived problem. It is assumed that the submodels are themselves feasible but a phase 1 pricing must be used until a feasible derived basis is obtained. There is no way to bypass this except by utilizing a previous solution to essentially the same problem.

Parametrics: It is assumed that a common feasible allocation exists for the submodels and that such an allocation can be prescribed ab initio.

Partitioned Simplex: Although in principle a phase 1 could be carried out, in reality it is assumed that a feasible allocation can be made ab initio (see Section 10 below).

For a more extensive discussion of this, see Chapter 12.8 in [13].

10. Computational Experiments

We shall now report a few computational tests which were carried out using the version of the simplex method developed above and the Dantzig-Wolfe decomposition principle. Our simplex method was not actually implemented but rather standard features of the SESAME mathematical programming system (implemented in IBM/370 and operating under VM/CMS) were used to carry out an accurate simulation of what our simplex approach does. Basically the interactive system was used to control the set of variables which were allowed to enter the basis at each simplex iteration. The experiments with the D-W decomposition technique were carried out by Dr. Etienne Loute with his implementation in the MPSX/370 system. In both cases experiments were performed on a dynamic forestry-industry interaction model, such as described in Section 4.

For the test with the simplex method the forestry model consists of 260 rows and 280 columns while the model of wood processing industry has 350 rows and 541 columns. Both of these include 20 columns corresponding to the linking variables of the integrated model. For linking the two models we may assume that a value for the coupling vector is known which is feasible for both of the models. Such a vector may be available from past experience, for instance. (In our case this vector shows the wood supply form forestry to the industry for all time periods). In our experiment, however, we ran the Phase I of an ordinary simplex method in order to get such an initial solution to start with. We may note in Table 2 that our initial solution was fairly poor: the objective function which represents the increment of the forest sector to the GNP was initially negative.

Given the initial solution, we first solved the industrial model (i.e., we ran the simplex method on the integrated model without letting the activities form the forestry to enter the basis). After no industrial activity is able to improve the solution, we switch to solve the forestry model similarly. Thereafter we switch back to the industrial model, and so on, until the global model is solved.

Results of our experiment are reported in Table 2. Both submodels (industry and forestry) were visited three times until global optimality was detected. The number of iterations needed to reoptimize (after preceeding visit) is reported. Here the number zero for the last visit of the forestry model indicates that only the optimality test has been carried out. It may be interesting to note that the very first cycles (visits to local problems) bring the global objective very close to its optimal value. Although our sample is the smallest possible, our conjecture is that this is a general phenomenon; i.e., very few switches from one local problem to another are needed in order to get close to the global optimum.

In experimenting with the D-W decomposition technique a slight variation of the previous forestry-industry model was

Table 2. An experiment using the simplex method for linking models.

Cycle	Iterations	Objective function	Problem solved
Initial solution	-	-19.67	-
1	445	98.98	Industry
2	24	99.51	Forestry
3	15	99.68	Industry
4	19	99.96	Forestry
5	2	100.00	Industry
6	0	100.00	Forestry
total	505		

used. For instance, a different global objective was used. Two runs were made and in both cases the D-W Phase I was carried out to start with.

For Run 1 the solution strategy was as follows: The two subproblems were first solved for optimality while no price was charged for the coupling constraints (i.e. wood). The composite columns corresponding to these optimal solutions were then forced to the basis of the master problem, which thereafter was optimized. At each cycle both subproblems were optimized and each improving solution found was brought to the (derived) master problem. After a feasible solution was obtained (for the master) an upper bound was computed for the global objective using the current dual solution.

Table 3. Experiments with the Dantzig-Wolfe decomposition algorithm.

RUN 1

Cycle	Iterations Master	Sub1	Sub2	Proposals generated Sub1	Sub2	Primal objective function	Upper bound	Solution
0	-	237	329	1	1			infeasible
1	3	13	13	6	3	-324		infeasible
2	3	6	9	3	5	-160		infeasible
3	4	28	28	4	2	-11200		infeasible
4	9	32	5	6	2	-37		infeasible
5	13	35	40	9	4	-1.7×10^{8}	2.2×10^{9}	feasible
6	4	0	18	0	1	-5.9×10^{7}	-5.8×10^{7}	feasible
7	0	0	0	0	0	-5.9×10^{6}	-5.9×10^{7}	optimal
total	36	351	442	29	18			

RUN 2

Cycle	Iterations Master	Sub1	Sub2	Proposals generated Sub1	Sub2	Primal objective function	Upper bound	Solution
0	-	273	378	1	1	-9		infeasible
1	1	25	33	1	1	-785		infeasible
2	3	36	5	1	1	-14		infeasible
3	5	0	111	0	1	-1.5×10^{9}	-5.9×10^{7}	feasible
4	3	15	10	1	1	-1.4×10^{9}	8.0×10^{9}	feasible
5	7	15	17	1	1	-9.4×10^{8}	6.7×10^{9}	feasible
6	7	23	20	1	1	-4.9×10^{8}	4.1×10^{9}	feasible
7	9	0	0	0	0	-5.9×10^{7}	-5.9×10^{7}	optimal
total	35	387	574					

The results of Run 1 are reported on the top of Table 3. Six cycles were needed to reach a feasible solution. Thereafter, only one cycle was needed to reach optimality which was proven by the following cycle. For each cycle, iterations needed to (re-)optimize the master and the subproblems, the number of columns generated by each of the subs, the global objective function value and the dual objective function value (when feasible) is reported.

For Run 2, the textbook strategy was followed. The master problem was first optimized. At each cycle, both subproblems were optimized and only the optimal solution was carried to the master problem (provided that it was improving). Again the dual solution value was computed for obtaining an upper bound for the global (primal) objective. In this case four cycles were needed to obtain a feasible solution. Simultaneously a dual optimal solution was found. However, four more cycles were needed in order to detect primal optimality. Again, the results are reported in Table 3.

Given that no prior information on the solution of the global problem was used here (such as the initial solution in our experiment with the simplex mehtod) the results obtained in Run 1 and 2 can be found very satisfactory. Because of the design of the experiments and limited experience we are not able, of course, to draw any conclusions on the relative performance of our simplex approach and the D-W decomposiiton as tools for linking linear programming models into an integrated system.

11. Some Extentions and Other Applications

The case above was considered when two blocks (linear programming submodels) are linked into an integrated model. In this section we discuss extension of this case to an arbitrary number of submodels and also describe some other possible applications of the approach.

Multiblock case. The extension of Problem P to the case with an arbitrary number N of submodels is straightforward:

$$\underset{\{x_i,y_i\}}{\text{maximize}} \sum_{i=1}^{N} c_i x_i$$

subject to

$$A_i x_i = b_i$$

$$D_i x_i = y_i \qquad\qquad x_i \geq 0 \quad , \quad (i = 1,\ldots,N) \qquad\qquad (11.1)$$

$$\sum_{i=1}^{N} R_i y_i = r \quad .$$

The constraints (11.1) are of general type and can express, for instance, both the case where the submodels have a coupling activity y (with $D_i y_i = y$), and the case where they share a common resource (e.g., $y_1+y_2+..+y_N = r$). Formally problem (11.1) can also be reduced to a straight extension of Problem $\overline{P}$ (of Section 1); i.e., to the following problem:

$$\underset{\{x_i y\}}{\text{maximize}} \sum_{i=1}^{N} c_i x_i$$

subject to

$$A_i x_i = b_i \quad ; \qquad\qquad (11.2)$$

$$D_i x_i = y+r_i \quad ; \quad x_i \geq 0 \quad .$$

(A redefinition of the matrices and vectors may be necessary for this transformation).

Different linkage methods described in Part II admit different "easiness" of extensions for Problem (11.2). We will not, however, discuss this, but will describe some possible applications of the extended problem.

Two-stage stochastic programming

In the two-stage approach for making decisions under uncertainty, one first makes a basic (master) decision x and then a correction z(w) which depends on the realization of the random "state of the nature" w; that is

$$\begin{aligned} &\underset{x,z(w)}{\text{maximize}}\ [cx + Ed(w)z(w) \\ &\text{subject to} \\ &Ax + D(w)z(w) = b(w) \\ &x, z(w) \geq 0 \ . \end{aligned} \tag{11.3}$$

If we have discrete probability distribution of w with N outcomes, problem (11.3) can be reduced to an equivalent deterministic problem [7]:

$$\begin{aligned} &\text{maximize } cx + \sum_{i=1}^{N} p_i d_i z_i \\ &-Ax = y \\ &D_i z_i = y + b_i \quad (i = 1,\ldots,N) \\ &x_i, y \geq 0 \ , \end{aligned} \tag{11.4}$$

where p_i is the probability for outcome i. Clearly, this problem is of the form of problem (11.2) with N+1 blocks. (This is natural, due to the fact that variable y is a decision, which is common for different realizations i of "the state of the nature"). Therefore, different linkage methods can be used for the solution of two-stage stochastic linear programming problems.

Dynamic linear programming. These problems represent another important class of optimization problems for which application of the linkage approach might be useful. The DLP problem in a rather general case can be written as [17]:

$$\underset{\{u_t,y_t\}}{\text{maximize}} \sum_{t=0}^{T-1} (a_t y_t + b_t u_t) + a_T y_T$$

subject to

$$A_t y_t + B_t u_t = y_{t+1} \tag{11.5}$$

$$G_t y_t + D_t u_t = f_t \quad , \quad u_t \geq 0 \quad .$$

There are different ways of applying a linkage approach to this problem. Let us consider, for example, static problems P_t (for all t):

$$\underset{u_t}{\text{maximize}}\ b_t u_t$$

subject to

$$(P_t) \qquad \begin{array}{l} B_t u_t = y_{y+1} - A_t y_t \\ D_t u_t = f_t - G_t y_t \quad , \end{array} \qquad u_t \geq 0 \tag{11.6}$$

where y_t and y_{t+1} are fixed and denote by $f_t(y_{t+1},y_t)$ the optimal value of (the objective function) in Problem (11.6). Then, the original Problem (11.5) is reduced to maximization of

$$\sum_{t=0}^{T-1} [a_t y_t + f_t(y_{t+1},y_t)] + a_T y_T \quad ,$$

over all feasible $\{y_t\}$. This approach was used in [3] for developing a decomposition algorithm for the solution of DLP problem (11.5).

In practice, a dynamic problem is sometimes analyzed using a single static model; i.e., one model is used for all static models P_t while the state vectors y_t and y_{t+1} are being held consistent with each other over time. Linking the static models P_t to each other via state vectors y_t results in this case in dealing with one local problem only (with varying initial and or terminal state).

Note, that frequently, due to uncertainty of the future data, it is desirable to formulate a dynamic multistage problem as a two-stage problem, in which the first period submodel represents a (detailed) program for the first time period and the second submodel describes a macroplan for the rest of the planning horizon. This approach was used in [1], for instance.

SUMMARY AND CONCLUSIONS

The general goal of this article is to investigate the question of how to carry out analysis when a set of mathematical models being used are interdependent. We seek systematic ways of linking such models to each other. The linking approaches should preserve the structure of the original models so that their interpretation during the analysis does not get increasingly complicated. Although the emphasis is on linking two interdependent linear programming models, extensions to multimodel, nonlinear, and stochastic cases can, in principle, be straightforward (as indicated in Section 11, for instance).

The article has been divided into two parts. In the first part we give a precise statement of our interdependent systems. As well we offer three typical examples of such systems: energy supply--economy, manpower--economy, and forestry--wood processing industry interaction systems. In the second part we consider alternative approaches: classical decomposition principles, approaches derived from nondifferentiable optimization techniques, application of parametric programming techniques as well as the simplex method combined with a partitioning technique. By no means does the paper provide a final solution to our linkage problem. However, some of the approaches give rise to optimism, while others remain inconclusive.

It seems that after its earlier disappointments, the Dantzig-Wolfe decomposition principle (and related approaches) would deserve reconsideration. Perhaps somewhat surprisingly, also the simplex method seems to yield a useful tool for linking models provided that a particular solution strategy (such as the one derived in Section 9) is applied. For both of these approaches some computational experiments are reported in Section 10 where a moderate sized forestry--industry interaction system is used as a test case. No conclusion has been made upon the relative performance of these two approaches. An obvious disadvantage with the decomposition principle is that a globally feasible solution is not readily available over the grand cycles (i.e., during the analysis of the interaction system). The number of grand cycles, however, does not seem to become extensive, and the data transmitted between the subsystems remains moderate. Also in the simplex approach, the number of times one has to deal with each of the subsystems (until global optimality has been reached) seems to remain low. In this case, a larger quantity of data is transmitted between the subsystems compared with the D-W approach. A relative advantage with the simplex approach is that the global solution is explicitly available during the analysis. One of the key issues in judging the applicability of these two approaches is the complexity of implementation of the approach. This, of course, depends on the available software, hardware, and the structure of the interdependent model system itself.

The advantage with a (nonfinite) iterative approach, such as the first approach presented in Section 7, is a minimal data transmission requirement between the subsystems. A major disadvantage, at least to our knowledge, is a very large number of cycles (iterations) of visiting each of the subsystems until a near optimal global solution is obtained. Also during such cycles, a globally feasible solution is not available in general.

The principle of a feasible direction method based on non-differentiable optimization techniques is presented in Section 7. However, further research remains to be done; in particular, in computing such feasible directions at least approximately. A solution for this problem may be found through parametric programming techniques. We discuss this attractive approach in general in Section 8. However, only heuristic rules for determining directions (for change in the coupling vector defining interdependence) is provided. Such rules do not in general guarantee global convergence for the integrated system, and practical difficulties have been demonstrated using a tiny example.

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