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# SENSITIVITY OF WATER RESOURCE SYSTEMS UNDER UNCERTAINTY: ANALYSIS AND SYNTHESIS

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ABSTRACT

This paper discusses the problem of planning water resource systems to be robust with respect to uncertainties. The main goal is to submit several reasonable statements about optimization problems in this field. Some properties of water resource systems under uncertainty are discussed briefly. The paper by no means pretends to encompass the complete scope of the problem. At best it provides an introduction to the problem encountered by the static models in the planning of water resource systems with unknown deterministic parameters.

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# SENSITIVITY OF WATER RESOURCE SYSTEMS UNDER UNCERTAINTY: ANALYSIS AND SYNTHESIS

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## INTRODUCTION

Because uncertainty is such a broad concept it is necessary to say a few words about the kind of uncertainty we are dealing with here. As an abstraction of unknown reality, uncertainty can be caused by two factors [1]:

1. Deliberate decisions made by forces with wholly or partially opposite interests (strategic games).
2. Unknown states of nature treated both in the direct and figurative meaning (games against nature).

The latter under consideration here covers many practical situations and with respect to water resource systems can imply uncertainty in data on water requirements, hydrology, precipitation, capital and operating costs etc. With some accuracy we can describe the variations of these data in terms of uncertain parameters of a system. Following [2,3] we can distinguish two such kinds of uncertainty:

1. The parameters of the system are stochastic. Though their true values are unknown, the decision maker has, however, some knowledge of these parameters in terms of probability (type of distribution, mean, variance etc).
2. The system's parameters are deterministic, but unknown. In this case the decision maker is aware only of the domain of allowable parameters.

Only the second case, referred to as complete uncertainty in parameters, is discussed below for two reasons. Sometimes it is impossible to determine the probability distributions for some parameters for lack of either the requisite information or the time to analyze it all. On the other hand, decision making in planning often implies the use of guaranteed strategies (minimax or maximin) which are determined only by the "worst" values of parameters and do not depend on their probability distributions.

To a great extent the topic of this paper is closely connected with such notions as stability, sensitivity, robustness, resilience and adaptivity; all of them characterize the properties of the systems under uncertainty. In essence, triplet stability, sensitivity and robustness are interchangeable notions [2,4,5]. The difference between them, if any, is in the type of systems or problems for which one or another notion is used.

Historically, the concept of stability is chiefly applied to the study of dynamic systems [6] or to analysis of the computational and analytic algorithms [7]. Anyhow, the stability is identified with the continuity of some system criterion with respect to variations in data. The concept of sensitivity is used mostly for automatic control systems [8] and the static or dynamic economic models, in particular for the planning models of water resource systems [9], when quantitatively analyzing the system response to the variations in some parameters.

As follows from [ 4,10 ] the concept of robustness originated in mathematical statistics is identical with insensitivity. Perhaps the term "robustness" is more convenient when dealing with uncertainty in the broad sense, for example, in the case of unknown structural properties of a system. None of the three notions above is quantitative and, therefore, each of them should be associated with a respective index or measure.

In a certain sense both resilience and adaptivity characterize the same property of a system under uncertainty--the ability of a system to adapt itself to some changes in data or structure and in doing so maintain the desirable characteristics [2,11,12]. The concept of resilience relates chiefly to ecological systems [13] whereas adaptivity relates chiefly to automatic control systems [14].

For the present, the discussion is confined to a study of questions related to sensitivity using this term when analyzing a system under uncertainty, and to robustness preferring this term when synthesizing a system with the desirable characteristics. Sometimes for explanatory purposes, we will refer to the Silistra Case Study [15].

## SENSITIVITY ANALYSIS

Up-to-date simulation or optimization techniques for planning complex water resource systems [9,16] are unthinkable without more-or-less detailed sensitivity analysis under uncertainty. As stated above, we are dealing with static, deterministic models for planning water resource systems having unknown parameters. By sensitivity of a system we mean the response of a system criterion to the variations in some parameters.

To be more specific, a rather general optimization model for planning a water resource system (see for example [9]) will be considered. In a very aggregated form it can be presented as follows:

$$\min_{x} \; I\,(x,\alpha) \tag{1}$$

$$x \in G(\alpha) \tag{2}$$

$$\alpha \in A \tag{3}$$

$$F_k = f_k(x,\alpha)\ , \quad k = 1,\ldots,\ell \tag{4}$$

where x is the n-dimensional vector of decisions; $\alpha$ is the m-dimensional vector of unknown parameters; $I(x,\alpha)$ is the given objective function, continuous and differentiable in x and $\alpha$; A and $G(\alpha)$ are the given sets in m- and n-dimensional spaces respectively. Moreover, the latter set depends continuously on the vector parameter $\alpha$. Finally, $F_k$ is the sensitivity criterion given by the function $f_k(x,\alpha)$, continuous and differentiable in x and $\alpha$. As a rule, the objective function $(x,\alpha)$ is one of the sensitivity criteria. The relation (2) includes physical, budgetary, institutional, water demands' constraints, etc.

Keeping in mind the relations (4) we can now quantify the sensitivity of criterion $F_k$ with respect to the variations in parameter $\alpha$. Usually, the measure for sensitivity is defined in the vicinity of the nominal or design value $\alpha^*$ of parameter $\alpha$ with respect to each its component $\alpha_i$ as follows:

$$S_k^i = \frac{\alpha_i^*}{F_k}\,\frac{\partial F_k}{\partial \alpha_i}\ , \qquad i = 1,\ldots,m \tag{5}$$

where $\partial F_k/\partial \alpha_i$ is a partial derivative of $F_k$ with respect to $\alpha_i$. Conventionally called a sensitivity coefficient, $S_k^i$ presents the fractional change in criterion $F_k$ divided by the fractional change in $\alpha_i$. However, for evaluation of sensitivity coefficients by the relations (5) we should specify the form of optimal decisions.

The point is that the relations (1)-(3), as they are, do not yet define an optimization problem, unless we choose one of the following alternatives for decision making.

I. The optimal decision $x^\circ(\alpha)$ is permitted to be a function of parameter $\alpha$. Actually, we are dealing with the problem of parametric nonlinear programming [17]. Until now this branch of mathematical programming has been relatively weakly developed. The computational algorithms here are available only for the specific problems of parametric linear programming. Using the relation (5) the sensitivity coefficient $S_k^i$ can be evaluated in the following way:

$$S_k^i = \frac{\alpha_i}{f_k(x^\circ,\alpha)} \left( \frac{\partial f_k(x^\circ,\alpha)}{\partial \alpha_i} + \sum_{j=1}^{n} \frac{\partial f_k(x^\circ,\alpha)}{\partial x_j} \frac{dx_j^\circ}{d\alpha_i} \right) \Bigg|_{\alpha^*} \qquad (6)$$

where the symbol $|_{\alpha^*}$ means that the right hand side of (6) is evaluated under $\alpha=\alpha^*$. By saying optimal sensitivity we mean the case described here.

II. The optimal decision $x^*$ does not depend on the parameter $\alpha$ and is determined by solving the usual optimization problem (1)-(3) where $I(x,\alpha) = I(x,\alpha^*)$ by definition. Here many well-elaborated methods of constrained optimization can be used [18]. The relation (5) gives the sensitivity coefficient.

$$S_k^i = \frac{\alpha_i^*}{f_k(x^*,\alpha^*)} \frac{\partial f_k(x^*,\alpha^*)}{\partial \alpha_i} \qquad (7)$$

In this case we are talking about nominal sensitivity.

To illustrate these two types of sensitivity the Silistra water supply model set out in [15] has been considered. One of the most important unknown parameters in this least-cost model is the price of lands which have been submerged by reservoirs. Figure 1 shows the variation in total cost per year associated with the establishment of a water supply system. This results from the change in price of submerged lands. It turns out that the optimal- and nominal-sensitivity coefficients are identical and equal 0.206.

## SYNTHESIS OF ROBUST SYSTEMS

We will try here to formulate a few meaningful problems of synthesizing a water resource system to be robust with respect to the uncertainty in parameters. In doing so, the objective is to reduce the uncertainty.

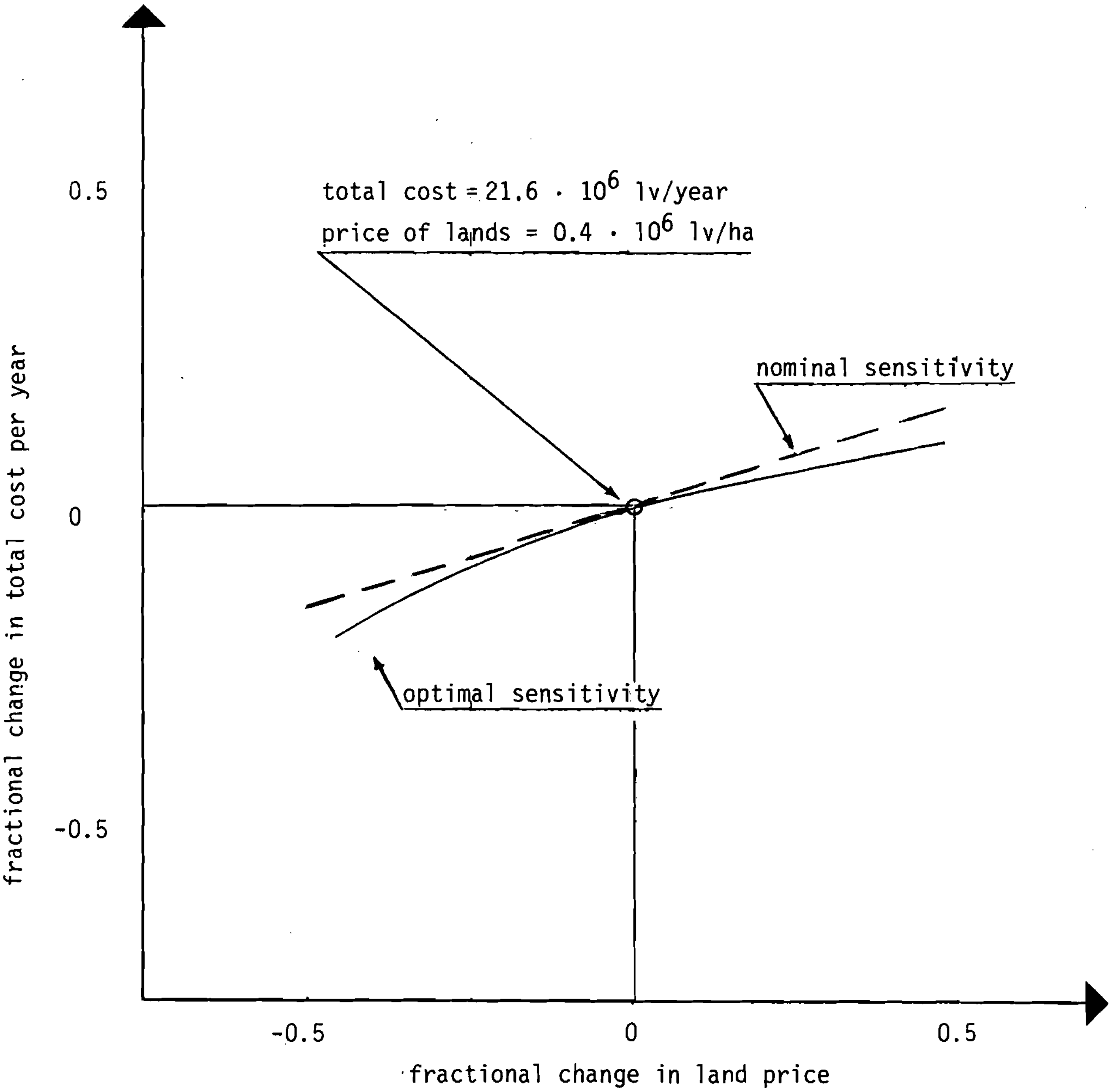


Figure 1

Sensitivity of Total Cost of Silistra's Water Supply System

In the previous section we briefly discussed the two possible alternatives for optimal decisions. Alternative I (i.e., $x^{\circ}(\alpha)$) is hardly of interest from the point of view of reducing the uncertainty. Indeed, knowing the optimal decision $x^{\circ}(\alpha)$ and, therefore, the objective function $I(\alpha) = I(x^{\circ}(\alpha),\alpha)$ and the sensitivity-analyzed criteria $F_k(\alpha) = f_k(x^{\circ}(\alpha),\alpha)$ $(k=1,...,\ell)$ is of importance only for justification of the decisions made or, in the case of high system sensitivity with respect to the variations in parameters for determining the requirements of accuracy of information about these parameters.

On the other hand, the alternative II (i.e., $x^*$) covers many practical cases of decision-making under uncertainty. Indeed, the planning decisions are made, as a rule, proceeding from the expected values of parameters before they are realized. Under these conditions the optimal decision $x^*$ must be either made in such a way that the water resource system be of low sensitivity with respect to variations in unknown parameters or determined by the "worst" combination of parameters in the domain (3). The problems of synthesis of robust systems stated below relate just to alternative II. In doing so, we have tried not only to state a problem, but to formulate it in a form allowing the use of well-known optimization techniques.

*Admissible Properties of Robustness*

Let us assume that we make decisions being based only on the nominal value $\alpha^*$ of parameter $\alpha$. Of course we would like simultaneously both to minimize the objective function and to provide the least system sensitivity. In actual fact this is nothing more than a multicriterion optimization problem. To state it correctly we should suggest some compromise settlement. In particular, one of the possible ways to reduce the parametric uncertainty is to require that the criteria (4) be of guaranteed sensitivity with respect to the variations in parameters. Following [7] we can express these requirements mathematically as follows:

$$\alpha_i^* \frac{\partial f_k(x,\alpha^*)}{\partial \alpha_i} \leq b_k f_k(x,\alpha^*) , \qquad k=1,...,\ell \quad i=1,...,m \tag{8}$$

where $b_k$ is the prespecified upper limit for the sensitivity coefficient $S_k^i$.

The relations (8) can be treated as a set of additional constraints on the decision vector x. In such a way we can formulate the following problem of synthesis of the robust systems.

Problem 1. Find the optimal decision $x^r$ minimizing the objective function $I(x,\alpha^*)$ subject to the conditions (2), (3) and (8).

Formally, this problem has been formulated in the terms of a problem of mathematical programming, because the relations (8) are ordinary constraints on the decision variables $x_1, \ldots, x_n$. Therefore, for its solving we can use methods of constrained optimization [18]. It should be stressed that the sought-for decision $x^r$ guarantees the admissible properties of robustness (in the sense of (8)) only locally, i.e., in the vicinity of the nominal parameter value $\alpha^*$.

*Minimax Decisions*

The principally different approach to optimization of the water resource systems with unknown parameters is based on the use of the concept of guaranteed/minimax (or maximin) strategies--the decisions determined by the "worst" combination of the parameters $\alpha_1, \ldots, \alpha_m$ in domain (3). In other words, we highlight the global variations in unknown parameters. To simplify matters, we will here consider the single sensitivity criterion--the objective function $I(x,\alpha)$. As shown in [3,19], a distinction is made between the following two principles in minimax-decision making under uncertainty.

1. Pessimistic minimax decisions. Under parametric uncertainty this principle can be expressed in the following way. Now the objective function $E(x)$ is determined by the "worst" value of parameter $\alpha$ and therefore does not depend on $\alpha$, i.e.

$$E(x) = \max_{\alpha \in A} I(x,\alpha) \tag{9}$$

If $I(x,\alpha)$ is the total cost associated with the creation of a water resource system under the decision $x$, then $E(x)$ is the highest cost of this project. If the water resource planner proceeds from the above criterion of decision making, his strategy is said to be very cautious. Sometimes this kind of decision is necessary in planning water resource systems.

2. Minimax regret decisions. The main idea of this principle formulated by Savage, consists in the following. If the true value of parameter $\alpha$ had been known a priori, we would have chosen the best decision $x^\circ(\alpha)$ and attained the least value $I(\alpha) = I(x^\circ(\alpha),\alpha)$ for objective function. In reality we would like to be as near as possible to the best decision $x^\circ(\alpha)$ in the sense of optimization criterion. The measure for our regret the best decision $x^\circ(\alpha)$, if we fix some allowable decision $x$, is the difference $I(x,\alpha) - I(\alpha)$. According to Savage's principle the criterion for decision making is the maximum regret, i.e.

$$E(x) = \max_{\alpha \in A} [I(x,\alpha) - I(\alpha)] \tag{10}$$

This principle in planning-decision making is very frequently used in practice.

In both relations (9) and (10) the maximized functions depend on x and $\alpha$ and below will be denoted by one symbol $H(x,\alpha)$. Therefore, we can formally join both minimax principles. The minimax decision $\bar{x}$ is determined by minimization of $E(x)$ over all the allowable decisions. Thus, we can rigorously formulate the following problem of synthesis of the systems being robust with respect to unknown parameters.

Problem 2. Find the minimax decision $\bar{x}$ such that

$$\max_{\alpha} H(\bar{x},\alpha) = \min_{x} \max_{\alpha} H(x,\alpha) \tag{11}$$

subject to (2) and (3).

Distinct from $x^r$, the minimax decision $\bar{x}$ provides the water resource system with the best global properties of robustness. A few words should be said about the possible approaches to solving this problem which is unconventional at first sight. First of all we could use some results of the theory of minimax problems [20].

Another way is to reduce problem 2 to a problem of mathematical programming. To simplify matters the set A is assumed to consist of a finite number of points $\alpha^1,\ldots,\alpha^p$. Introduce the auxiliary variable y. Then it is easy to see that in case of the finite set A the initial problem 2 is identical with the following problem of mathematical programming:

$$\text{minimize } y$$

$$\text{subject to} \quad x \in G(\alpha^s) \quad ,$$

$$H(x,\alpha^s) \le y \quad , \quad s = 1,\ldots,p$$

For its solution, we can use the methods of constrained optimization [18].

*Two-Stage Process in Decision Making*

In the concluding part of the paper we will study a problem of synthesis of the robust systems under uncertainty when making two-stage decisions. Being the simplest case of sequential planning, this problem is of great importance for the Silistra water supply system [15].

The Silistra irrigation system will be put into operation at least by the two stages, I and II (see Figure 2). Keeping in mind the ten year duration of the planning period, the expedience of the project's staging is almost evident. For the Silistra Case Study, the decision vector x includes such basic variables of a water supply system as capacities of reservoirs and pumping stations, and discharge capacities of canals.

Let $x^1$ and $x^2$ be the $n_1$ - and $n_2$ - dimensional subvectors of decisions made at the stages I and II respectively, so that the union of them is the aggregate vector of decisions x, i.e.

$$(x^1, x^2) = x \quad , \quad n_1 + n_2 = n$$

An essential assumption is that once decisions $x^1$ and $x^2$ are made, they can not be changed during the planning period. We should now specify the very concept of uncertainty in systems' parameters for a two-stage process of decision making. Let $\alpha$ be some typical parameter of a water resource system which takes on values in domain (3). The two following points completely characterize the kind of parametric uncertainty with which we are dealing here:

a) at stage I of decision making $x^1$, the true value of parameter $\alpha \in A$, is unknown;
b) at stage II of decision making $x^2$, the true value of parameter $\alpha$ is known.

Let us interpret these points in reference to the example of the Silistra water supply model. The price of lands submerged by reservoirs is a rather subjective value which can be unexpectedly changed in time very much. At the same time the costs associated with losses of submerged lands constitute 60%, 45% and 30% of the total costs of reservoirs 1,2, and 3. Therefore, the total cost of the Silistra water supply system is very sensitive with respect to the price of submerged lands. When making the first-stage decision $x^1$, the price of lands submerged by reservoirs 1 and 2 is known, but unknown for reservoir 3. The second-stage of decision making $x^2$ is ordinary because we deal with the complete certainty--the price of lands submerged by reservoir 3 is known.

The crucial point in this problem is stage I. We have here the only unknown parameter, which becomes known at stage II. In other words, two types of information about unknown parameter $\alpha$ are available at the first stage:

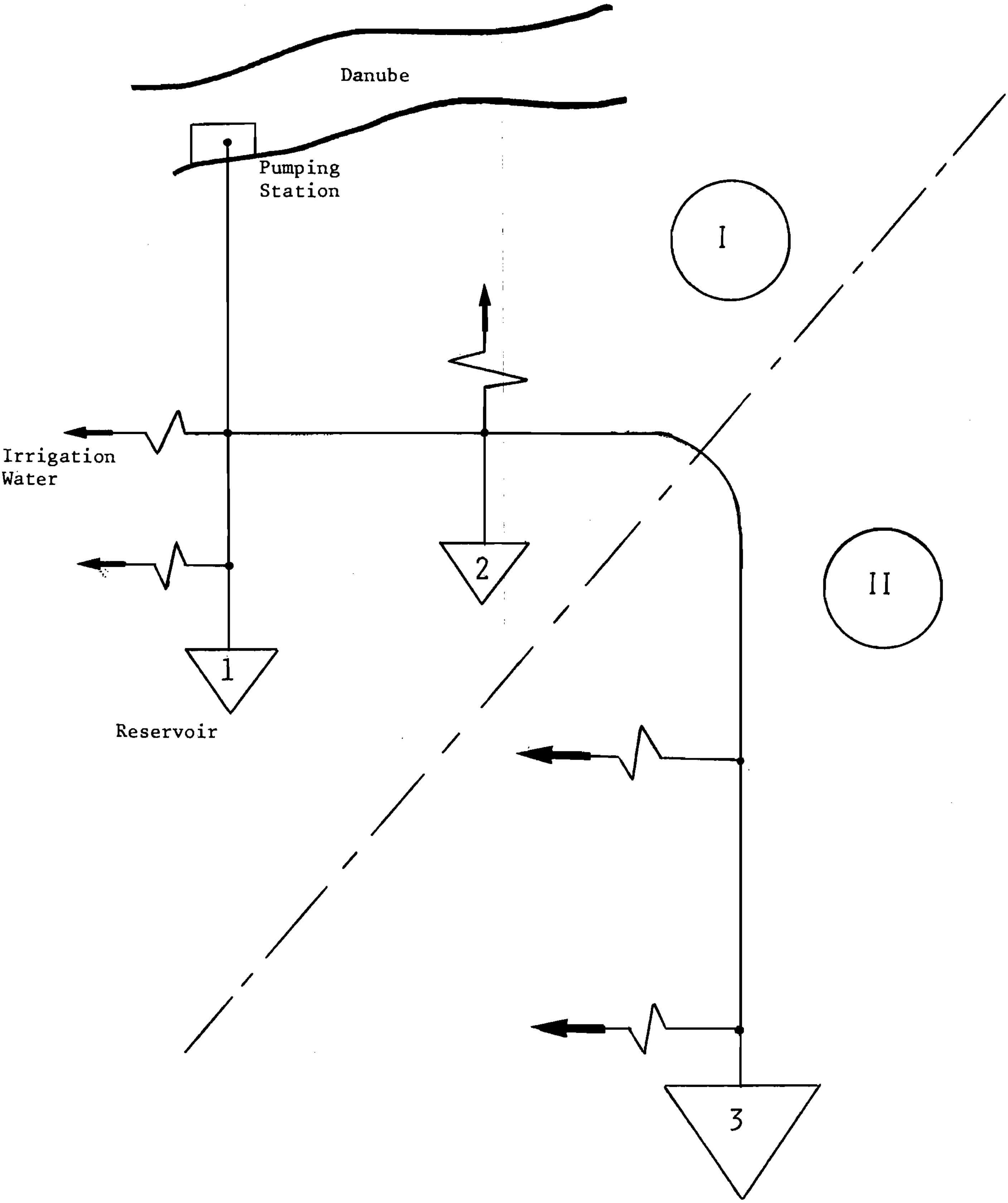


Figure 2
Staging Silistra's Irrigation System

A) $\alpha \in \Lambda$, and
B) that at stage II parameter $\alpha$ will be known.

In comparison with the above mentioned problems 1 and 2 we now have some additional information B which is rather scant at first sight. The question is whether it is helpful in reducing the uncertainty and, if so, how to use it. If only the information A) about parameter $\alpha$ had been available, the decision $x^1$ would be obtained from the appropriate analogies of problem 1 or 2. In this case the two-stage process in decision making would have divided into two unconnected problems. It turns out that the availability of information B influences the decision $x^1$ because we now have at our disposal the broader choice of the decisions $x^2$ which can be adapted to the parameter $\alpha$ becoming known at the stage II. Mathematically, this means the decision $x^2$ is a function of $\alpha$.

Holding to one of the minimax principles above we can rigorously formulate the following problem for the two-stage process in decision making.

<u>Problem 3.</u> Find the minimax decisions $\hat{x}^1$ and $\hat{x}^2(\alpha)$ such that

$$\max_{\alpha} H(\hat{x}^1, \hat{x}^2, \alpha) = \min_{x^1, x^2(\alpha)} \max_{\alpha} H(x^1, x^2, \alpha) \tag{12}$$

subject to (2) and (3).

This form of the problem is rather inconvenient for numerical or analytic solution. In comparison with (11), the sought-for decision $\hat{x}^2(\alpha)$ is a vector function of $\alpha$ rather than a finite vector. Formally, that means we deal here with a problem of calculus of variations. Therefore, we will try to reformulate problem 3 by reducing (12) to an optimality condition such as (11).

For this purpose some allowable decision $x^1$ will be fixed. It then follows from (12) that the second-stage decision $x^2$ must be determined by conventional problem

$$\min_{x^2} H(x^1, x^2, \alpha) \tag{13}$$

subject to (2) for all $\alpha \in A$. Thus, the optimal strategy at the stage II is a function of $x^1$ and $\alpha$, i.e. $x^2 = f(x^1, \alpha)$. Then the condition (12) can be rewritten as follows:

$$\max_{\alpha} H(\hat{x}^1, f(\hat{x}^1, \alpha), \alpha) = \min_{x^1} \max_{\alpha} H(x^1, f(x^1, \alpha), \alpha) , \tag{14}$$

where $f(x^1, \alpha)$ is the known function. In principle, the relations (11) and (14) are identical. It follows that the problem 3 can be solved by the same methods as problem 2. Finally, it should be stressed that such a reduction of problem 3 is attained at high cost, i.e. by the solution of the optimization problem (13) with the two vector parameters $x^i$ and $\alpha$.

## CONCLUSION

Thus, the problems 1,2 and 3 discussed above introduce us to the problem of the synthesis of water resource systems to be robust with respect to the deterministic, unknown parameters. Each of these problems is characterized by its own approach to reducing the uncertainty in a system's parameters.

1. Problem 1 is mainly concerned with guaranteeing the admissible, local properties of robustness of systems in the vicinity of the nominal values of unknown parameters.
2. The solution of problem 2 provides a water resource system with the extreme, global properties of robustness with respect to variations in systems' parameters.
3. Problem 3 introduces us to the synthesis of robust systems under the multistage process of decision making.

Each of these problems, especially 2 and 3, needs the development of rigorous computational or analytic algorithms for its solution.

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