Working Paper

DYNAMIC LINEAR PROGRAMMING

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SUMMARY

The paper presents a survey of dynamic linear programming (DLP) models and methods, including discussion of different formalizations of dynamic linear programs, models which can be written in such a form, duality relations and optimality conditions for DLP, numerical methods -- both finite-steps and iterative.

DYNAMIC LINEAR PROGRAMMING*

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INTRODUCTION

Dynamic linear programming (DLP) can be considered as a next stage of linear programming (LP) development [1-3]. Nowadays it becomes difficult, even sometimes impossible, to make decisions in large systems and not to take into account the consequences of the decision over some period of time. Thus, almost all problems of optimization become dynamic, multistage ones. Examples are long-range energy, water and other resources supply models, problems of national settlement planning, long-range agriculture investment projects, manpower and educational planning models, resources allocation for health care, etc.

New problems require new approaches. Within DLP it is difficult to exploit only LP ideas and methods. While for the static LP the basic question consists of determining the optimal program, the implementation of this program (related to the question of the feedback control of such a program, its stability and sensitivity, etc.) is no less important for the dynamic problems. Hence, the DLP theory and methods should be both based on the methods of linear programming [1,2] and on the methods of control theory, Pontryagin's maximum principle [4] and its discrete version [5] in particular.

DLP CANONICAL FORM

In formulating DLP problems it is useful to single out [3]:

- state equations of the system with the state and control variables;
- constraints imposed on these variables;
- planning horizon T and the length of each time period;

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- objective function (performance index), which quantifies the quality of control.

State equations. State equations have the following form:

$$x(t+1) = A(t)x(t) + B(t)u(t) + s(t) \tag{1}$$

where the vector $x(t) = \{x_1(t),\ldots,x_n(t)\}$ defines the state of the system at stage t in the state space X, which is supposed to be the n-dimensional euclidean space E^n; vector $u(t) = \{u_1(t),\ldots, u_r(t)\} \in E^r$ specifies the controlling action at stage t; $s(t) = \{s_1(t),\ldots,s_n(t)\}$ is a vector defining the external effect on the system (uncontrolled, but known *a priori* in the deterministic model).

It is also assumed that the initial state of the system is given:

$$x(0) = x^0 \tag{2}$$

Planning horizon T is supposed to be fixed. Thus in (1): $t = 0,1,\ldots,T-1$. The *length* of each time period may be the same for the whole planning horizon (say, month, year, 5 years), or differs for different periods. For example, in long-range energy supply models the planning horizon might be 100 years, which forms 10-period horizon with five periods equal 6 years, three periods equal 10 years and two periods equal 20 years [6].

Constraints. In rather general form constraints imposed on the state and control variables may be written as

$$G(t)x(t) + D(t)u(t) = f(t) \tag{3}$$

$$u(t) \geq 0 \quad , \tag{4}$$

where $f(t) = \{f_1(t),\ldots,f_m(t)\} \in E^m$ $(t = 0,1,\ldots,T-1)$.

Objective function. *(Performance index)*, which is to be maximized for certainty, is

$$J_p(u) = a(T)x(T) + \sum_{t=0}^{T-1} [a(t)x(t) + b(t)u(t)] \quad , \tag{5}$$

In vector products the right vector is a column and the left vector is a row.

Definitions. The vector sequence $u = \{u(0),\ldots,u(T-1)\}$ is a *control (program)* of the system. The vector sequence $x = \{x(0),x(1),\ldots,x(T)\}$, which corresponds to control u from (1), (2), is the system's *trajectory*. The pair $\{u,x\}$, which satisfies all the constraints of the problem (e.g. (1)-(4)) is a *feasible process*. A feasible process $\{u^*,x^*\}$ which maximizes the objective function (5) is *optimal*.

The DLP problem in its canonical form is formulated as follows.

Problem 1P. Find a control u^* and a trajectory x^*, satisfying the state equations (1) with the initial state (2) and the constraints (3) and (4), which maximize the objective function (5).

In Problem 1P vectors x^0, $s(t)$, $f(t)$, $a(t)$, $b(t)$ and matrices $A(t)$, $B(t)$, $G(t)$, $D(t)$ are supposed to be known.

The choice of the canonical form is to some extent arbitrary and there are various possible versions and modifications of Problem 1P. In particular, both state and control variables may be non-negative; state equations may include time lags; constraints on state and control variables may be separate, given as equalities or inequalities; the objective function may be defined only by the terminal state $x(T)$ of the system, etc. [5,7]. It should be noted, however, that such modifications can be either reduced to Problem 1P or the results stated below for Problem 1P can be easily extended for these modifications.

Note, that if $T = 1$, then Problem 1P becomes a conventional LP problem. Problem 1P itself can also be considered as a certain structured LP problem with constraints (1) - (4). In this case the optimal control Problem 1P turns out to be an LP problem with the staircase constraint matrix.

Sometimes dynamic LP problems are formulated using only LP language, as for example, Problem 2, [8] - [10]:

Problem 2. Find vectors $\{x^*(1),\ldots,x^*(T)\}$, which maximize

$$\sum_{t=1}^{T} c(t)x(t)$$

subject to

$$A(1)x(1) = d(1)$$

$$B(t-1)x(t-1) + A(t)x(t) = d(t) \quad , \qquad (t=2,\dots,T)$$

$$x(t) \geq 0 \quad , \qquad (t=1,\dots,T)$$

One can also express the state variables $x(t)$ in Problem 1P as an explicit function of control. As a result, the following LP problem with a block-triangular matrix is obtained.

Problem 3. Find vectors $\{u^*(0),\dots,u^*(T-1)\}$, which maximize

$$\sum_{t=0}^{T-1} w(t)u(t) \quad ,$$

$$D(0)u(0) = h(0)$$

$$W(t,0)u(0) + \dots + W(t,t-1)u(t-1) + D(t)u(t) = h(t) \quad ,$$

$$u(t) \geq 0 \quad , \qquad (t=0,1,\dots,T-1) \quad ,$$

where the vectors $h(t)$, $w(t)$ and the matrices $W(t,\tau)$ depend on the known parameters of Problem 1P.

Problems 2 and 3 are typical examples of structured LP and admit various modifications in the same way as Problem 1P (e.g. a block diagonal structure with coupling constraints and/or variables, etc.). They have been studied intensively (see, e.g. [1,2,8-10]. But unlike control Problem 1P, such formalizations of dynamical problems do not use explicitly the state equations of a system. Therefore this approach makes it difficult to attract the ideas and methods of the control theory. The DLP problems in the form of Problem 1P were introduced in [11,5].

DLP MODELS

There is a large and rapidly expanding field of applications of dynamic linear programming. Here only some of them are mentioned.

Dynamic multi-sector economic models. These are basically extensions of dynamic input-output models, when substitution is allowed (see for example [1,2,12-14]). Alongside with discrete time formulation continuous time economy models have also been developed (e.g. in [15]).

Energy models. For long-range planning the problem is the following: for existing initial structure of the secondary energy production capacities and under given primary energy and nonenergy resources supply constraints find a transition to such a mix of the energy production options (fossil, nuclear, etc.), which satisfied the projected energy demand and minimizes the total cost of such transition. These energy supply models have been studying intensively in the recent years [6,16-18]. The next stage in this direction is the investigation of energy-economy interaction. This problem can be analysed either as an integrated DLP model or in some man-machine iterative mode [18].

Primary resources models. These models relate to the planning of primary resources extraction and/or exploration activities. The survey of different DLP energy-resources-economy models is given in [18].

Water models. The DLP models for water systems are the water supply models (which are conceptualy close to energy supply models), the problems of alternatives evaluation in a river basin, etc. [19-21].

Models for educational and manpower planning. The simple educational model aims to find such enrollments to different educational institutions which as well as satisfy all the constraints of the systems (e.g. teachers, buildings and other educational capacities availability) and be as close as possible to the projected manpower demand. In the manpower planning models controls are recruitments and/or promotions.

There are different modifications of these models, both for national and institutional levels [22].

National settlement planning. These DLP models relate to finding an optimal, in some sense, migration flows in a region or a country [23].

Health care systems. The problem of health care system planning may be reduced to an allocation funds, health manpower and other resources over time among different diseases treatment activities in such a way as to yield the best results in terms of reduced mortality, morbility and other loses [24].

Agriculture models. Separate DLP models concern planning problems in livestock breeding, crop production, resources utilization, etc. The integrated models relate to planning of diversified agro-industrial complexes and the problems of farm growth [25-28].

Production models. These DLP models are basically associated with the production scheduling and inventory problems in industry [8,9,29,30].

Dynamic transportation problems. Transhipment and other network problems are the important class of linear programming. Dynamical aspects of transportation problems are considered in [3,31,32].

Miscellaneous models. It is worthwhile to mention here some "individual" DLP applications: congested urban road traffic control [33], planning the acquisition and disposal of equipment in a transport fleet [34], multistage structural design problems [35].

THEORY OF DLP

The theory of DLP is connected with duality relations and optimality conditions [11,36,7,5].

Problem 1D. To find a dual control $\lambda = \{\lambda(T-1),\dots,\lambda(0)\}$ and a dual trajectory $p = \{p(T),\dots,p(0)\}$ satisfying the co-state (dual) equation

$$p(t) = p(t+1)A(t) - \lambda(t)G(t) + a(t) \quad , \tag{6}$$

with the boundary condition

$$p(T) = a(T) \quad . \tag{7}$$

subject to the constraints

$$p(t+1)B(t) - \lambda(t)D(t) \leq -b(t) \quad (8)$$

and minimize the objective function

$$J_D(\lambda) = p(0)x^0 + \sum_{t=0}^{T-1} [p(t+1)s(t) + f(t)\lambda(t)] \quad (9)$$

Here $p(t) \in E^n$ and $\lambda(t) \in E^m$ are Lagrange multipliers for constraints (1)-(4).

The dual Problem 1D is a control-type problem as is the primal one 1P. Here the variable $\lambda(t)$ is a dual control and $p(t)$ is a co-state or a dual state at stage t. We have reversed time in the dual Problem 1D: $t = T-1,\ldots,1,0$.

Theorem 1. (The DLP "Global" Duality Theorem). The solvability of either of the 1P or 1D problems implies the solvability of the other, with $J_P(u^*) = J_D(\lambda^*)$.

Let us introduce Hamilton functions

$$H_P(p(t+1),u(t)) = b(t)u(t) + p(t+1)B(t)u(t)$$

$$H_D(x(t),\lambda(t)) = \lambda(t)f(t) - \lambda(t)G(t)x(t)$$

for the primal and dual problems respectively.

Theorem 2. (The DLP "Local" Duality Theorem). The solutions of the primal $\{u^*,x^*\}$ and dual $\{\lambda^*,p^*\}$ problems are optimal if and only if the values of Hamiltonians coincide:

$$H_P(p^*(t+1),u^*(t)) = H_D(x^*(t),\lambda^*(t)) \quad .$$

Thus, the solution of the pair of dynamic dual problems can be reduced to analysis of a pair of static linear programs

$$\max H_P(p(t+1),u(t))$$
$$G(t)x(t) + D(t)u(t) = f(t) \quad ; \quad u(t) \geq 0 \quad (10)$$

$$\min H_D(x(t), \lambda(t))$$

$$p(t+1)B(T) - \lambda(t)D(t) \leq -b(t) \qquad (11)$$

linked by the state (1), (2) and co-state (6), (7) equations.

In particular, it can be shown, that *maximum principle for dual Problem 1P* and *minimum principle for dual Problem 1D* are hold [5,7]. An economic interpretation of the DLP duality relations is given in [36].

DLP METHODS

We shall distinguish finite and iterative methods for solving DLP problem.

Finite-steps methods. These methods are the extension of finite large-scale LP methods for the dynamic problems. Two basic approaches may be singled out here: compact inverse and decomposition methods [9,10,37,38].

Compact inverse methods. These methods are variants of the simplex method using basis factorization with various strategies as to the vector pair to enter and to leave the basis. The approach is being developed both for structured and general LP (in the latter case the sparseness of constraint matrix is exploited)(see e.g. [9,10,37,38,39,40]). For staircase structure (Problem 2) this approach was used in [37,41-44]. For DLP Problem 1P the approach was proposed in [45,46] and described in [47-49]. The method arised is a natural and straightforward extension of the simplex method. The main concept of the simplex method--the basis--is replaced by the set of local bases, introduced for the whole planning period. The idea of the method is the following. Consider the constraints (3) for t=0. Using (2), they can be written in the form

$$D(0)u(0) = \hat{f}(0); \quad \hat{f}(0) = f(0) - G(0)x^0 \qquad (12)$$

and according to the conventional simplex procedure be partitioned as

$$D(0)u_0(0) + D_1(0)u_1(0) = \hat{f}(0) \qquad (13)$$

where $D_0(0)$ is a square non-singular (mxm) matrix. If T = 1 (static problem), then one can set $u_1(0) = 0$ and find from (13) a basic feasible solution $u_0(0) \geq 0$. In the dynamic case (T > 1) not all components of $u_1(0)$ are zeros for basic solutions. These nonzero components should be transformed to the next steps t > 0. Partitioning the state equations (1) for t = 0 in the same way as in (13) and substituting $u_0(0)$ from (13) to this partitioned state equations, one can obtain the constraints for the next step in the form, similar to (12): $\hat{D}(1)\hat{u}(1) = \hat{f}(1)$, where $\hat{u}(1) = [\hat{u}_1(0); u(1)]$. The matrix $\hat{D}(1)$ depends on G(1), B(0), D(1) and can be again partitioned in the same way: $\hat{D}(1) = [\hat{D}_0(1); \hat{D}_1(1)]$, det $\hat{D}_0(1) \neq 0$ and so on, until the last step $u_1(T-1) = 0$.
The set of m linearly independent columns of the matrix $\hat{D}(t)$ is *local basis* at step t. The set of local bases $\{\hat{D}_0(t)\}$ for the whole planning period t = 0,1,...,T-1 plays the same role as basis in standard simplex-method. The introduction of local bases allows to implement all simplex procedures (selection of vectors to be removed from and to be introduced into the basis, pricing, updating) very effectively [49]. It should be also noted, that the dynamic simplex-method gives not only compact substitute for basis inverse, but uses only a part of the local bases representation at each iteration.

The extension of the dynamic simplex-method to DLP problems with time lags or non-negative state variables as well as dual versions of the method are considered in [46,49].

Decomposition methods. These methods follows from repeated application of different decomposition methods (first of all, Dantzig-Wolfe decomposition principle) to the DLP problem, and the staircase structure (Problem 2) was described in [1,50-54].

For DLP Problem 1P this approach was used in [55]. Consider the sequence of LP problems (t = T-1,...,1,0):

$$\max \quad p(t+1)x(t+1)$$

$$x(t+1) = A(t)x(t) + B(t)u(t) + s(t) \qquad (14)$$

$$G(t)x(t) + D(t)u(t) = f(t)$$
$$x(t) \in R_t(x^0); \quad u(t) \geq 0$$

where $R_t(x^0)$ is the set of all states $x(t)$ feasible from initial state x^0 for t steps [5]; $p(t+1)$ is dual state vector. Solutions of these LP problems with optimal values of $p^*(t+1)$ give an optimal solution of the original Problem 1P. Any vector $x(t)$ of $R_t(x^0)$ can be represented as a linear convex combination of its extreme points (assuming boundness of $R_t(x^0)$). Thus, problem (14) can be rewritten as

$$\begin{aligned} \max \quad & p(t+1)[\sum_i A(t)X_i(t)\mu_i(t) + B(t)u(t) + s(t)] \\ & \sum_i G(t)X_i(t)\mu_i(t) + D(t)u(t) = f(t) \\ & \sum_i \mu_i(t) = 1 \; ; \quad \mu_i(t) \geq 0 \end{aligned} \tag{15}$$

Problem (15) is a master problem at step t. For its solution conventional column generator procedure can be adopted. Other methods for solution of Problem 1P based on different decomposition and partition principles were considered in [56-60].

The DLP iterative methods. The application of the LP finite methods to the dynamic problems may cause certain difficulties, especially for large planning periods T. The reason for it is that solution paths in these methods consist of a number of vertices of a feasible polyhedral set (in some space), and this number will increase exponentially with T.

The iterative methods seem to by-pass these difficulties. They also characterized by low demand to computer's core memory, simplicity of the computer implementation, low sensitivity to disturbances. Disadvantages of these methods may be low convergence rate, yeilding only approximate solution.

We shall single out the following iterative methods.

Generalized gradient methods. This group of methods is based on finding extremum of functions:

$$\Psi(\lambda,p) = \max_{x,u \geq 0} L(u,x;\lambda,p) \quad ; \quad \phi(u,x) = \min_{p,\lambda \geq 0} L(u,x;\lambda,p) \qquad (16)$$

where $L(u,x,\lambda,p)$ is the Lagrange function of Problem 1P (1D). Minimization of Ψ is equivalent to the solution of the dual Problem 1D, while the maximization of ϕ is equivalent to the solution of primal Problem 1P [7]. As these functions are non-differentiable by nature, the generalized gradient technique [61] is needed. The application of this approach to the solution of dual Problem 1D leads to the following algorithm. We consider Problem 1P with additional constraints on control variables: $u(t) \in U_t \; ; \; U_t = \{u(t) \mid R(t)u(t) \leq r(t), u(t) \geq 0\}$.

(1) Choose arbitrary dual control $\{\lambda^{\nu}(t)\}(t=0,1,\ldots,T-1)$.

(2) Compute dual state trajectory $\{p^{\nu}(t)\}$ from the dual state equations (6) with boundary condition (7).

(3) For $p^{\nu}(t+1)$ solve T LP problems: $\max H_p(p^{\nu}(t+1),u(t))$. $u(t) \in U_t$. Let $u^{\nu}(t)$ be a solution of these problems.

(4) Compute primal state trajectory $x^{\nu}(t)$ from (1) and (2) for $u^{\nu}(t)$.

(5) Compute generalized gradient of Ψ: $\partial\Psi(\lambda^{\nu}) = h^{\nu}(t)$, where $h^{\nu}(t) = f(t) - G(t)x^{\nu}(t) - D(t)u^{\nu}(t)$. (17)

(6) Compute new dual control $\lambda^{\nu+1}$:
$$\lambda^{\nu+1} = \lambda^{\nu} - \alpha_{\nu}\partial\Psi(\lambda^{\nu})/\|\partial\Psi(\lambda^{\nu})\| \quad (\nu = 0,1,2,\ldots) \qquad (18)$$

Theorem 3. Let $\alpha_{\nu} \to 0$; $\nu \to \infty$; $\sum_{\nu=0}^{\infty} \alpha_{\nu} = \infty$. Then $\Psi(\lambda^{\nu}) \to \Psi(\lambda^{*})$, $\lambda^{\nu} \to \lambda^{*}$, $\lambda^{*} = \{\lambda^{*}(t)\}$ is an optimal control of dual Problem 1D.

The algorithm (1)-(6) gives a dual solution of the problem. To obtain the primal solution one can apply the same approach to function $\phi(u)$ in (16).

Modified Lagrange function method [62,63]. This approach combines the advantages of both finite and iterative methods: under minimal requirements for storage it ensures monotonic convergence in a finite number of steps. In a sense, this method can be considered as a "smooth" alternative to the nonsmooth approach, described above. Let the Lagrange function $L(u,\lambda)$ of Problem 1P (1D) be augmented by quadratic term:

$$M(u,\lambda) = L(u,\lambda) + (\theta/2) \sum_{t=0}^{T-1} h(t)h(t) \tag{19}$$

where $\theta > 0$, vector $h(t)$ is defined from (17).

Now instead of (16) the following pair of problems can be considered

$$\Psi_M(\lambda) = \max_u M(u,\lambda); \; \phi_M(u) = \min_\lambda M(u,\lambda) \tag{20}$$

Opposite to $\Psi(\lambda)$, the function $\Psi_m(\lambda)$ is a smooth concave function; derivative of $\Psi_m(\lambda)$ is a continuous function of λ and $\partial\Psi_M(\lambda)/\partial\lambda(t) = h(t)$ where $h(t)$ is computed from (17), when $u^* = u^*(\lambda)$ is a solution of the left problem in (20) [62].

Problems $\min \Psi_M(\lambda)$ and $\max \phi_M(u)$ are the pair of modified dual problems, associated with original Problem 1P. In [62] is shown, that the modification does not change the general dual relations. Thus in order to obtain a solution of Problem 1P one can solve either the dual problem $\min \Psi(\lambda)$ with the nonsmooth function Ψ or the modified dual problem $\min \Psi_M(\lambda)$ with the smooth function Ψ_M.

The latter gives the following algorithm:

(1) Choose arbitrary dual control λ^ν.

(2) Solve the left problem in (20).

This problem has linear state equations (1), quadratic objective function (19) and simple constraints on control $u(t)$ (e.g. $0 \le u(t) \le \bar{u}(t)$). Therefore for its solution optimal control technique (which reduces in this case to solution of a matrix Riccati equation) can be effectively applied [63].

(3) Let u^ν be a solution of the above problem. Compute $h^\nu(t)$ from (17) for this u^ν and new dual control $\lambda^{\nu+1}$ from

$$\lambda^{\nu+1}(t) = \lambda^\nu(t) - \theta h^\nu(t) \qquad (\nu = 0,1,2,\ldots)$$

Theorem 4. For any $\theta > 0$,

$$\Psi_M(\lambda^\nu) - \Psi_M(\lambda^{\nu+1}) \geq (\theta/2) \sum_{t=0}^{T-1} \|h^\nu(t)\|^2$$

and beginning from some finite $\nu_0 : \lambda^\nu \in \Lambda^*$, $u^\nu \in U^*$, $\nu \geq \nu_0$, where Λ^*, U^*, are solution sets of dual and primal problems 1D and 1P respectively.

Theorem 4 states that, opposite to the generalized gradient algorithm (1)-(6), the algorithm (1)-(3) converges for a finite number of steps and the values of the modified function $\Psi_M(\lambda)$ monotonously decreases with ν (in fact so does the nonmodified dual function $\Psi(\lambda)$ also [62]). Moreover, this algorithm gives simultaneously dual λ^* and primal u^* optimal solutions.

CONCLUSION

A short survey has been given above of DLP models, theory and methods. There is not much literature on computer implementation of the methods. Experiments, however, show that these methods can be much more efficient in comparison with the standard (static) approach (see, for instance, [39,53]). Therefore computer testing and evaluation of the algorithms are now an important problem. Other important directions of research are the problem of the implementation of optimal control and post-optimal analysis of the model (feedback vs program optimal control, sensitivity and stability analysis, interrelation between short and long-term problems, etc.)

It should also be noted that not all dynamical optimization problems can be kept within the framework of DLP. Therefore, the extension of DLP for stochastic, maxmin, nonlinear dynamic programming problems is of large practical interest. Some work has been already done in this direction (see, e.g. [64-66]).

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