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ITERATIVE METHODS FOR STRUCTURED LINEAR PROGRAMMING

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SUMMARY

In this paper methods are analyzed for the solution of block-angular linear programming problems based on nondifferentiable techniques.

1. Statement of the Problem

Let us consider the LP-problem in the form

$$\sum_{i=1}^{N} c_i x_i + c_0 y \rightarrow \max \qquad (1)$$

$$B_i x_i + D_i y = b_i \qquad (i=1,\ldots,N) \qquad (2)$$

$$\sum_{i=1}^{N} A_i x_i + D_0 y = b_0 \qquad (3)$$

$$x_i \geq 0, \quad y \geq 0 \qquad (i=1,\ldots,N) \quad . \qquad (4)$$

Here vectors y, x_i, b_0, b_i have n_0, n_i, m_0, m_i components, respectively. This is the socalled block-angular LP problem with coupling constraints (3) and coupling variable y [1]. Dual to problem (1) - (4) is [1]:

$$\sum_{i=1}^{N} u_i b_i + u_0 b_0 \rightarrow \min \qquad (5)$$

$$u_i B_i + u_0 A_i \geq c_i \qquad (i=1,\ldots,N) \qquad (6)$$

$$\sum_{i=1}^{N} u_i D_i + u_0 D_0 \geq c_0 \quad . \qquad (7)$$

The finite-step methods for the solution of these problems are based on different decomposition and partition schemes or on a compact inverse technique and are analyzed, for example, in [1,2]. The purpose of this paper is to describe iterative methods, based on the nondifferentiable approach [3-5] to these problems.

2. Solution in Dual Space

To explain the idea let us consider problems (1)-(4) without coupling variable, that is, vector c_0 and matrices D_0, D_i $(i=1,\ldots,N)$ are supposed to be zeros.

Following [3] let the Lagrange function of this problem be introduced in the form:

$$L(x,u_0) = \sum_{i=1}^{N} c_i x_i + u_0 (b_0 - \sum_{i=1}^{N} A_i x_i) \quad . \tag{8}$$

Then

$$\max_{x_i \in X_i} \min_{u_0} L = \min_{u_0} \max_{x_i \in X_i} L$$

where

$$X_i = \{x_i | B_i x_i = b_i , x_i \geq 0\} \quad . \tag{9}$$

Denote

$$\Psi(u_0) = \max_{x_i \in X_i} L(x,u_0) \tag{10}$$

and

$$X_i(u_0) = \{x_i | L(x,u_0) = \Psi(u_0)\} \quad . \tag{10a}$$

The sets $X_i(u_0)$ are defined from the solution of the following LP-problems:

$$(c_i - u_0 A_i) x_i \to \max$$

$$B_i x_i = b_i$$

$$x_i \geq 0 \qquad (i=1,\ldots,N) \quad . \tag{11}$$

Lemma [3,6]. $\Psi(u_0)$ is a piece-linear concave function, low bounded for bounded X_i. The derivative of $\Psi(u_0)$ in direction δu_0 is defined by

$$\Psi'(u_0) = \max_{x_i \in X_i(u_0)} (b_0 - \sum_{i=1}^{N} A_i x_i)^T \delta u_0 \quad . \tag{12}$$

From this lemma different algorithms for the solution of dual problems (5)-(7) (c_0, D_0, D_i are zeros) can be developed (see the general approach in [4,5]). In particular, the generalized gradient method [3,5] for this case will be as follows:

1°. Let u_0^ν be the ν-th iteration.

2°. Solve N LP-problem (11) for $u_0 = u_0^\nu$. Let $\{x_i^\nu, u_i^\nu\}$ be primal and dual solutions of these problems (i=1,...,N).

3°. Compute new iteration $u_0^{\nu+1}$ from

$$u_0^{\nu+1} = u_0^\nu - \rho_\nu \frac{h^\nu}{|h^\nu|} \tag{13}$$

where

$$h^\nu = b_0 - \sum_{i=1}^{N} A_i x_i^\nu, \quad |h| \text{ is a norm of vector } h \ .$$

<u>Theorem</u> [3,5]. If

$$\rho_\nu \to 0 \ , \quad \nu \to \infty \ ; \quad \sum_{\nu=0}^{\infty} \rho_\nu = \infty \ ,$$

then $\{u_0^\nu, u_i^\nu\}$ converges to an optimal solution of dual problem (5)-(7).

It should be underlined that in this approach only dual solution $\{u_0^\nu, u_i^\nu\}$ can be directly obtained. For finding a solution to the primal problem one can apply this approach to the dual problem.

3. <u>Solution in Primal Space</u>

As we can see from above, the approach is natural if the dual problem has block-angular structure. Therefore in this section we consider problem (1) - (4) <u>without coupling constraints</u> (3), that is, all matrices D_0, A_i (i=1,...,N) and vector b_0 are zeros. In this case dual problem (5)-(7) turns out to be the block-angular problem with coupling constraints.

Following the scheme of Section 2 one can introduce the Lagrange function of problem (5)-(7) in the form

$$L(u,y) = \sum_{i=1}^{N} u_i b_i + (c_0 - \sum_{i=1}^{N} u_i D_i) y$$

where

$$u = \{u_i\}; \quad u_i \in U_i = \{u_i | u_i B_i \geq c_i\} \quad (i=1,\ldots,N)$$

$$y \geq 0 \quad .$$

Then

$$\max_{y \geq 0} \min_{u_i \in U_i} L = \min_{u_i \in U_i} \max_{y \geq 0} L$$

Denote

$$\varphi(y) = \min_{u_i \in U_i} L(u,y)$$

$$U_i(y) = \{u_i | L(u,y) = \varphi(y)\} \quad .$$

The sets $U_i(y)$ defined from the solution of the following LP-problems:

$$u_i(b_i - D_i y) \rightarrow \min$$

$$u_i B_i \geq c_i \qquad (i=1,\ldots,N) \quad .$$

Similarly to (12) one can obtain, that

$$\varphi'(y) = \min_{u_i \in U_i(y)} (c_0 - \sum_{i=1}^{N} u_i D_i)^T \delta y \quad .$$

Therefore, the algorithm for the solution of primal problem (1)-(4) in this case will be:

1^o. Let $y^\nu \geq 0$ be the ν-th iteration.

2^o. For $y = y^\nu$ solve N LP-problems:

$$u_i(b_i - D_i y) \rightarrow \min \tag{14}$$

$$u_i B_i \geq c_i \qquad (i=1,\ldots,N)$$

Let $\{u_i^\nu, x_i^\nu\}$ be primal and dual solutions of these problems.

3^o. Compute new $y^{\nu+1} \geq 0$ from

$$y^{\nu+1} = \max \{0, y^\nu + \rho_\nu \frac{h^\nu}{|h^\nu|}\} \quad (15)$$

where

$$h^\nu = c_0 - \sum_{i=1}^{N} u_i^\nu D_i$$

and so on.

Similarly to the theorem of Section 2, $\{x_i^\nu, y^\nu\} \to \{x_i^*, y^*\}$ - optimal solution of problem (1),(2),(4), when $\nu \to \infty$ and $\rho_\nu \to 0$, $\sum_{\nu=0}^{\infty} \rho_\nu = 0$.

4. Two-stage Solution

Primal solution can be also obtained in the following way. Consider again problem (1)-(4) without coupling variable y (that is, all D_0, D_i and c_0 are zeros).

Denote

$$A_i x_i = y_i \quad ,$$

then the problem can be rewritten as (see for example [1]):

$$\sum_{i=1}^{N} c_i x_i \to \max_{\{x_i, y_i\}}$$

$$B_i x_i = b_i$$

$$A_i x_i = y_i \qquad (i=1,\ldots,N) \quad (16)$$

$$x_i \geq 0$$

$$\sum_{i=1}^{N} y_i = b_0 \quad . \quad (17)$$

This problem breaks down on N local subproblems (16) linked only by constraints (17). Therefore the following two-stage algorithm is appropriate for solution (16),(17).

Denote by $f_i(y_i)$ the optimal value of the objective function $c_i x_i$ in "local" problem (16). Then problem (16),(17) is equivalent to

$$\sum_{i=1}^{N} f_i(y_i) \to \max \tag{18}$$

$$\sum_{i=1}^{N} y_i = b_0 \quad .$$

(We define $f_i(y_i) = -\infty$, if y_i yields the empty feasible set in local problem (16).)

It can be shown that $f_i(y_i)$ $(i=1,\ldots,N)$ are piece-linear, continuous, concave functions and the generalized gradient of $f_i(y_i)$ is equal to vector v_i, where v_i is the optimal value of the dual variable associated with constraints $A_i x_i = y_i$ (cf. lemma of Section 2). Hence the algorithm for the solution of (18) will be as follows.

1°. Let $\bar{y}^\nu = \{y_i^\nu\}$ be ν-th feasible solution (that is, $\bar{y}^\nu \in Y$, $Y = \{y_i^\nu | \sum_{i=1}^{N} y_i^\nu = b_0\}$).

2°. Solve N local LP-problems (16) for $y_i = y_i^\nu$:

$$c_i x_i \to \max_{x_i}$$

$$B_i x_i = b_i$$

$$A_i x_i = y_i \qquad i = 1,\ldots,N \tag{19}$$

$$x_i \geq 0$$

Let $\{x_i^\nu, u_i^\nu, v_i^\nu\}$ be primal and dual solutions of these problems. (u_i^ν are not needed in the algorithm).

3°. Compute the new value of $\bar{y}$:

$$\bar{y}^{\nu+1} = P_Y(\bar{y}^{\nu} + \rho_\nu \frac{\bar{v}^{\nu}}{|\bar{v}^{\nu}|}) \tag{20}$$

where $\bar{v}^{\nu} = \{v_i^{\nu}\}$, P_Y is the projection operator on the set Y.

This projection operation can be reduced to a quadratic programming problem.

Then $\bar{y}^{\nu} \to \bar{y}^{*} = \{y_i^{*}\}$-optimal solution of (18) and $\{x_i^{\nu}\} \to \{x_i^{*}\}$ - optimal solution of (1)-(4), when c_0, D_0, D_i are zeros.

5. Conclusion

Above the different classes of structured LP problems were singled out for which the application of nondifferentiable technique seems to be natural.

Another class of such kind of problems are two-stage stochastic LP problems which can be handled by this approach either in deterministic [7] or stochastic way [8].

The above described algorithms can in principle be extended to the general case (1)-(7). In this general case, however, the corresponding algorithms turn out to be more complicated and not so "natural". Therefore, they may not be as effective as the other methods for the general case. However, a judgment could only be made after some numerical experiments with the algorithms.

It should also be stressed that other iterative methods (for example based on augmented Lagrange function [9,6]) are of interest for developing solution methods of large-scale LP-problems.

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