SOME PROPOSALS FOR STOCHASTIC FACILITY LOCATION MODELS

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FOREWORD

The public provision of urban facilities and services often takes the form of a few central supply points serving a large number of spatially dispersed demand points: for example, hospitals, schools, libraries, and emergency services such as fire and police. A fundamental characteristic of such systems is the spatial separation between suppliers and consumers. No market signals exist to identify efficient and inefficient geographical arrangements, thus the location problem is one that arises in both East and West, in planned and in market economies.

This problem is being studied at IIASA by the Public Facility Location Task (formerly the Normative Location Modeling Task) which started in 1979. The expected results of this Task are a comprehensive state-of-the-art survey of current theories and applications, an established network of international contacts among scholars and institutions in different countries, a framework for comparison, unification, and generalization of existing approaches, as well as the formulation of new problems and approaches in the field of optimal location theory.

Based on an earlier draft presented at the Task Force Meeting on Public Facility Location, held at IIASA in June 1980, this paper is one of the outcomes of interdisciplinary interactions between the applied area of Human Settlements and Services (HSS) and the methodological area of Systems and Decision Science (SDS). The challenging issue of applying stochastic programming to real location problems is explored, and further generalizations meaningful both for theoretical advancement and applications are proposed.

Lists of publications in the Public Facility Location Series and of related publications in the SDS Area appear at the end of this paper.

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ABSTRACT

The static facility location model with a spatial interaction-based allocation rule has been first introduced by Coelho and Wilson (1976). The main interest in introducing a spatial interaction-based allocation rule lies in the more realistic trip patterns that result from its use, which in many cases seem to fit the actual data on customer choice better than the simple nearest-facility allocation rule.

A further step towards more realistic models of customer behavior is the introduction of stochastic features, describing both the amount of total demand for facilities and the trip pattern of the customers. In this paper the usefulness of stochastic programming tools to formulate and solve such problems is explored, and some simple, but easily generalizable applied examples are given. Both numerical techniques and exact analytical methods are outlined, and some issues for further research are proposed.

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1. INTRODUCTION

It is well known that a classical "plant location" model is based on very deterministic assumptions. The main limitation of such models is the customer-choice behavior embedded within, that is, the choice of the nearest facility. The need to introduce more realistic behavioral assumptions has been recognized by many authors, among them Coelho and Wilson (1976), Hodgson (1978), Beaumont (1979), and Leonardi (1978, 1980). In all the above references the sharp distance-minimizing behavior is replaced by a smoother spatial interaction (also known as "gravity") model, thus allowing for possible substitution effects across space. Since spatial interaction models have both theoretical and empirical justifications, their use in location modeling seems a promising one. However, the classical spatial interaction models solve only part of the problem. Although they are rooted on stochastic assumptions (Wilson, 1970; McFadden, 1974; Bertuglia and Leonardi, 1979), only the expected values of the underlying stochastic processes are used. A natural further step to be undertaken is therefore to introduce the stochastic behavior explicitly, thus allowing for both uncertainty in customer choice and uncertainty in the knowledge of demand.

The aim of this paper is to explore some of the problems arising when such stochastic features are introduced, as well as to suggest some numerical tools to solve the resulting problems. Due to the exploratory nature of the paper, the examples are kept as simple as possible. However, it is felt that the suggested approach is by far more general than the applications discussed here, and can be easily extended to more complex formulations without any big change in the required theory and tools.

2. STATEMENT OF THE PROBLEM

In its most general form, the static deterministic facility location problem can be formulated as follows:

$$\max_{S,X,L} \quad B(S) - \sum_{j \in L} f_j(x_j) \tag{1}$$

s.t.

$$\sum_{j \in L} S_{ij} = P_i \quad , \qquad i \in M \tag{2}$$

$$\sum_{j \in L} S_{ij} = x_j \quad , \qquad j \in L \tag{3}$$

$$X \in \Gamma \tag{4}$$

$$L \subseteq Z \tag{5}$$

where

i — labels the demand locations, belonging to a given set M

j — labels the facility locations, belonging to a set L, to be chosen among all subsets of a given set Z

$S = (S_{ij})$ is the array of total trips made by customers between each demand-facility location pair in the unit time

$X = (x_j)$ is the array of total service capacity (in terms of customers served per unit of time) to be established in each facility location belonging to L

$P = (P_i)$ is the array of total demand (in terms of customers to be served per unit of time) in each demand location belonging to M

Γ is the set of feasible X, accounting for possible physical and economic constraints to be met by the service capacities

B(S) is a real valued function measuring the total benefit which accrues to the customers from a given trip pattern S

$f_j(x_j)$ are real valued functions measuring the cost of establishing a facility with capacity x_j in each location $j \varepsilon Z$

The objective function (1) is therefore the total net benefit, being the difference between customer benefit and establishing costs. It has to be maximized by suitably choosing the subset of locations L, the facility sizes X, the trip pattern S. This choice is subject to:

a. constraint (2), requiring the total demand to be met;
b. constraint (3), requiring the total capacity to be fully used;
c. constraint (4), requiring the facility sizes to meet the physical and economic constraints;
d. constraint (5), requiring the subset of chosen location to belong to the set of possible locations Z.

The general formulation given above can be specialized in many ways, by introducing special assumptions for the functions $B(\cdot)$ and $f_j(\cdot)$ and for the structure of the set Γ (see Leonardi, 1980, for a review).

The simplest possible form of problem (1)-(5) is obtained by introducing the following assumptions:

a. The benefit function has the form

$$B(S) = -\sum_{ij} S_{ij} \ln S_{ij} - \beta \sum_{ij} C_{ij} S_{ij} \tag{6}$$

where C_{ij} are the travel costs between each (i,j) pair, and β is a given nonnegative constant. Function (6) has been first introduced by Neuburger (1971) in transport planning evaluation and extended to location analysis by Coelho and Wilson (1976) and Coelho and Williams (1978). In the above references it is shown how this function has a sound economic interpretation, being the consumer surplus measure associated with the trip pattern (S_{ij}). Moreover, it has the useful property of embedding the spatial interaction model with an exponential discount factor, which usually has a good empirical fit on actual data.

b. The cost functions are linear and do not depend on the location

$$f_j(x_j) = ax_j \quad , \qquad \forall j.$$

c. The set Γ is

$$\Gamma = \{X : X \geq 0\}$$

that is, no physical and economic constraints must be met, except for the obvious nonnegativity requirement on the size of the facilities.

After introducing the above assumptions and dropping the constant terms, the redundant variables, and constraints, problem (1)-(5) reduces to the much simpler one

$$\min \sum_{ij} S_{ij} \ln S_{ij} + \beta \sum_{ij} C_{ij} S_{ij} \tag{7}$$

s.t.

$$\sum_{j} S_{ij} = P_i \tag{8}$$

Note that, due to the simple form of the cost functions, constraint (5) is no longer required, since an optimal solution will always have $L = Z$. The combinatorial features of (1)-(5) have thus disappeared, and the problem has been reduced to the smooth concave programming problem (7)-(8). The closed-form solution to (7)-(8) can be easily found to be

$$S_{ij} = P_i \frac{e^{-\beta C_{ij}}}{\sum_j e^{-\beta C_{ij}}} \tag{9}$$

Equation (9) states that trips from demand locations to facilities are made according to a very simple production-constrained spatial interaction model (Wilson, 1971).

Problem (7)-(8) and equation (9) can be used as a starting point to build some simple stochastic generalizations. The first one is as follows. Let it be assumed that the behavior implied by (9) is deterministic, but the demand array P is not known in advance. This assumption is sensible in many long-term planning applications, where the trip behavior is known but the total demand may fluctuate. For instance, in a high school location problem the way customers will choose facilities from each demand location can be reasonably assumed to be known and deterministic, but the total number of students living in each demand location may change over time in an unpredictable way. However, the size of the schools cannot be changed as fast as demand changes, so the planning authority is possibly faced both with unsatisfied demand and overcrowding and with unused service capacity. The above problem can be stated in mathematical terms as follows. Let

$H_i(y)$ be the distribution function of the total demand in demand location i; that is, if τ_i is the random variable giving the total demand in i, then $H_i(y) = \Pr\{\tau_i \leq y\}$

α_i^+ be the unit cost to be paid for an overestimate of the demand in i

α_i^- be the unit cost to be paid for an underestimate of the demand in i

x_i be the estimate of total demand in i, given by the decision maker

Then, if $\tau_i \leq x_i$ an overestimate cost $\alpha_i^+(x_i - \tau_i)$ has to be paid, while if $\tau_i > x_i$ an underestimate cost $\alpha_i^-(\tau_i - x_i)$ has to be paid. The resulting stochastic programming problem is

$$\min_{S,X} \sum_{ij} S_{ij} \ln S_{ij} + \beta \sum_{ij} C_{ij} S_{ij} + \tag{10}$$
$$+ \sum_i \left[\alpha_i^+ \int_0^{x_i} (x_i - y)\, dH_i(y) + \alpha_i^- \int_{x_i}^{\infty} (y - x_i)\, dH_i(y) \right]$$

s.t.

$$\sum_j S_{ij} = x_i \tag{11}$$

The above generalization has been built on the assumption that the total demand is stochastic, while the trip behavior is deterministic. Let this assumption now be reversed, so that the total demand is deterministic, while the trip behavior is stochastic. This assumption can be easily introduced by suitably reinterpreting equation (9), which can be rewritten as follows:

$$S_{ij} = P_i \, q_{ij} \tag{12}$$

where

$q_{ij} = \dfrac{e^{-\beta C_{ij}}}{\sum_j e^{-\beta C_{ij}}}$ is the probability of choosing the destination j for a customer living in origin i

The interpretation of the quantities q_{ij} defined above as probabilities is rooted on the theory of probabilistic choice behavior (McFadden, 1973). It has also been shown in Bertuglia and Leonardi (1979) that these quantities can be interpreted as steady-state distribution of a suitably defined Markov process. If the customers are assumed to be mutually independent, then (12) can be interpreted as the expected value of the number of trips between i and j, whose actual values have a multinomial distribution with parameters q_{ij}. Let v_{ij} be the actual (random) number of trips from i to j, and define

$\tau_j = \sum_i v_{ij}$ the total number of customers attracted in j

$H_j(y)$ the distribution function of τ_j; that is, $H_j(y) = \Pr\{\tau_j \leq y\}$

The distribution functions $H_j(y)$ cannot be easily written in closed form, but random draws of τ_j can be computed using the probabilities q_{ij}. Let also the following costs and decision variables be introduced:

α_j^+ is the unit cost to be paid for an overestimate of the demand attracted in j

α_j^- is the unit cost to be paid for an underestimate of the demand attracted in j

x_j is the size of the facility in j

Since the planned value x_j will be usually different from the actual demand τ_j, a cost $\alpha_j^+(x_j - \tau_j)$ will have to be paid when $\tau_j \leq x_j$ and a cost $\alpha_j^-(\tau_j - x_j)$ will have to be paid when $\tau_j > x_j$.

The resulting stochastic programming problem is

$$\min_X \sum_j \left[\alpha_j^+ \int_0^{x_j} (x_j - y)\, dH_j(y) + \alpha_j^- \int_{x_j}^{\infty} (y - x_j)\, dH_j(y) \right] \qquad (13)$$

Note that the spatial interaction embedding term has been dropped in the objective function, since the customer behavior is already accounted for by the way the distribution functions $H_j(y)$ are built. If $\alpha_j = \beta_j$, $j = 1,n$, then it can be shown that the solution to problem (13) is given by the *median* of the random vector $\{\tau_j\}$, which for very large values of P_i, $i = 1,m$, is closely approximated by the expected value, i.e.:

$$x_j^* = \sum_j P_i q_{ij}$$

Although problems (10)-(11) and (13) look quite different, they belong to the same general form and can be solved with the same methods. A further generalization, allowing for a stochastic behavior of both the total demand and the trip behavior would still lead to the same problem form. The rest of this paper will be mainly concerned with problem (10)-(11) and its generalizations, but it must be kept in mind that the theory and the techniques which will be developed apply to problem (13), as well as to its generalizations.

3. THE STOCHASTIC QUASI-GRADIENT METHOD

In order to develop a computational method to solve problem (10)-(11), let it be further simplified. For given x_i by means of equations (9) the optimal values of the variables S_{ij} can be expressed in terms of the variables x_i:

$$S_{ij} = x_i \frac{e^{-\beta C_{ij}}}{\sum_j e^{-\beta C_{ij}}} \tag{14}$$

Substitution of (14) in the objective function (10) yields:

$$\min_X F(X) \tag{15}$$

where the function F(X) is defined as:

$$\begin{aligned} F(X) = {} & \sum_i x_i \ln x_i + \sum_i C_i x_i + \\ & + \sum_i \left[\alpha_i^+ \int_0^{x_i} (x_i - y)\, dH_i(y) + \alpha_i^- \int_{x_i}^{\infty} (y - x_i)\, dH_i(y) \right] \end{aligned} \tag{16}$$

and the constants C_i are given by

$$C_i = -\ln \sum_j e^{-\beta C_{ij}} \tag{17}$$

The solution of problems like (15) gives rise to two usually difficult problems. First, although the objective function (16) is convex, it is in general nonsmooth. The possible nonsmoothness arises from the distribution functions $H_i(y)$. First, if they are discrete distributions, then F(X) will not have continuous derivatives. Second, it is often difficult or impossible to compute the exact values of the integrals appearing in (16), unless for very special and well-behaved forms of the distribution functions $H_i(y)$. More often than not, such functions are defined not by a closed-form equation, but rather by means of a rule to generate random draws from them.

Such difficulties can be overcome by using direct stochastic programming methods, such as stochastic quasi-gradient methods (see Ermoliev, 1976, 1978 for a review). These methods are a straightforward generalization of the well-known gradient method of deterministic mathematical programming, can be used for quite arbitrary distributions $H_i(Y)$, and require very simple computations. For instance, the stochastic quasi-gradient projection method gives rise to the following rule for generating successive approximations to the optimal solution of problem (15):

$$X^{(N+1)} = \max \{0, X^{(N)} - \rho_N \xi^{(N)}\} \tag{18}$$

for

$$N = 0, 1, \ldots,$$

where

N is an iteration counter

$X^{(N)}$ is the Nth approximation to the solution vector of (15)

ρ_N is a step size, to be suitably chosen at each iteration

$\xi^{(N)} = \left\{\xi_i^{(N)}\right\}$ is a random vector, called the stochastic quasi-gradient of $F(X)$ at the point $X^{(N)}$

The stochastic quasi-gradient of $F(X)$ at $X^{(N)}$ is defined as

$$\xi_i^{(N)} = \begin{cases} \ln ex_i^{(N)} + C_i + \alpha_i^+ \ , & \text{if } x_i^{(N)} \leq \tau_i^{(N)} \\ \ln ex_i^{(N)} + C_i - \alpha_i^- \ , & \text{if } x_i^{(N)} > \tau_i^{(N)} \end{cases} \tag{19}$$

where $\{\tau_i^{(N)}\}$ is a sequence of mutually independent random draws from the distributions $H_i(y)$.

The convergence of the sequence $X^{(N)}$, as computed by (18), to the optimal solution of problem (15) is based on the fact that the random vector $\xi^{(N)}$, as defined in (19), is a stochastic estimate of a *subgradient* of the function $F(X)$. It will be briefly recalled (Rockafellar, 1970) that a subgradient $\hat{F}_X(X)$ of a convex function $F(X)$ is a vector such that the inequality

$$F(Y) - F(X) \geq \left(\hat{F}_X(X),\ y - x\right)$$

holds for all y (here the outer brakets on the right-hand side denote the inner product of two vectors). A subgradient of a differentiable function $F(X)$ is equal to the gradient

$$F_X(X) = \left(\frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n}\right) .$$

It can be shown that the conditional mathematical expectation of $\xi^{(N)} = \left(\xi_1^{(N)}, \ldots, \xi_n^{(N)}\right)$:

$$E\left(\xi^{(N)} \,|\, X^{(N)}\right)$$

where E denotes expectation, is a subgradient of the function (16) at $X = X^{(N)}$. To do this one must reformulate the problem as a minimax stochastic programming problem and apply the well-known general results (Ermoliev, 1969, 1976, 1978; Ermoliev and Nurminski, 1980). It is easily seen that

$$\alpha_i^+ \int_0^{x_i} (x_i - y)\, dH_i(y) + \alpha_i^- \int_{x_i}^{\infty} (y - x_i)\, dH_i(y) = \tag{20}$$

$$= E \max \left[\alpha_i^+ (x_i - \tau_i),\ \alpha_i^- (\tau_i - x_i)\right]$$

Substitution of (20) into (16) yields:

$$F(X) = \sum_i \left\{ x_i \ln x_i + C_i x_i + E \max \left[\alpha_i^+ (x_i - \tau_i), \alpha_i^- (\tau_i - x_i) \right] \right\} \qquad (21)$$

The requirements under which the sequence $\{X^{(N)}\}$ converge with probability 1 to the solution of (15) are very weak. For instance, a set of sufficient conditions is

$$\rho_N \geq 0 \quad , \quad \sum_{N=0}^{\infty} \rho_N = \infty \quad , \quad \sum_{N=0}^{\infty} \left[\rho_N \right]^2 < \infty \quad , \quad |\tau_i| < \text{const.}, \quad \forall i \quad ,$$

and such conditions can always be satisfied in applications.

4. OPTIMALITY CONDITIONS

The numerical method outlined in Section 3 is quite general and can be used no matter how ill-conditioned the distributions $H_i(y)$ are. If, however, these distributions are well-behaved enough, then one may try to develop the exact optimality conditions for problem (15), and possibly find a set of simple equations for the optimal solution.

The starting point to develop necessary and sufficient optimality conditions for problem (15) is to consider it as a minimax stochastic programming problem (21). The general optimality conditions for a stochastic programming problem have been studied in Wets (1974), Ermoliev (1976), and Ermoliev and Justremski (1979). However, the special structure of problem (21) can be exploited to develop the optimality conditions in a more convenient form. Minimization of (21) is a special case of the following more general problem:

$$\min_X \quad Q(X) \qquad (22)$$

where

$$Q(X) = \sum_i \left\{ x_i \ln x_i + C_i x_i + E \max_i \left[\sum_j a_{ij}(W) x_j + b_i(W) \right] \right\} \qquad (23)$$

and

$$a_{ij}(W),\ b_j(W)$$

are random parameters.

Let, therefore, the optimalit conditions for problems (22) be analyzed. Let $\delta = (\delta_1,\dots,\delta_n)$ be a vector with nonnegative components, $Q'_\delta(X)$ the directional derivative along the direction δ. Then at an optimal solution $X = X^*$ it must be

$$\lim_{\Delta \to 0} \frac{Q(X+\Delta\delta) - Q(X)}{\Delta} = Q'_\delta(X) = \sum_{i=1}^{n}\left(\delta_i \ln ex_i + C_i\,\delta_i\right) + f'_\delta(X) \geq 0 \tag{24}$$

where $\Delta > 0$,

$$f(X) = E\,\Psi(X,W) \quad , \qquad \Psi(X,w) = \max_i \left[\sum_{j=1}^{n} a_{ij}(W)x_j + b_i(w)\right]$$

From this the following conclusion can be drawn: the components of an optimal solution are positive and (24) is satisfied for any direction δ. Under suitable hypotheses one can assert something about the equalities:

$$E\,\Psi'_\delta(X,W) = \int \Psi'_\delta(X,W)\,dH(W)$$

and

$$\lim_{\Delta \to 0} \frac{f(X+\Delta\delta) - f(X)}{\Delta} = \lim_{\Delta \to 0} \int \frac{\Psi(X+\Delta\delta,W) - \Psi(X,W)}{\Delta}\,dH(W)$$

(the integrability of the function $\Psi(X,W)$ as a function of W is automatically assumed).

For instance, it is easy to obtain the estimations

$$\left| \frac{\Psi(X + \Delta\delta, W) - \Psi(X, W)}{\Delta} \right| = \frac{1}{\Delta} \left| \max_i \left[\sum_{j=1}^{n} a_{ij}(W)(x_j + \Delta\delta_j) + b_i(W) \right] - \right.$$

$$\left. - \max_i \left[\sum_{j=1}^{n} a_{ij}(W) x_j + b_i(W) \right] \right| = \frac{1}{\Delta} \left| \sum_{j=1}^{n} \left[a_{i^*_\Delta j}(W)(x_j + \Delta\delta_j) + b^*_{i_\Delta}(W) \right. \right.$$

$$\left. \left. - a_{i^* j}(W) x_j - b_{i^*}(W) \right] \right| \leq \frac{1}{\Delta} \sum_{j=1}^{n} \left[\max\{0, a_{i^*_\Delta j}(W)(x_j + \Delta\delta_j) + \right.$$

$$+ b_{i^*_\Delta}(W) - a_{i^* j}(W) x_j - b_{i^*}(W)\} + \max\{0, a_{i^* j}(W) x_j + b_{i^*}(W) -$$

$$\left. - a_{i^* j}(W)(x_j + \Delta\delta_j) - b_{i^*_\Delta}\} \right]$$

Since

$$\max\{0, a_{i^*_\Delta j}(W) \; (x_j + \Delta\delta_j) + b_{i^*_\Delta}(W) - a_{i^* j}(W) x_j - b_{i^*}(W)\} \leq$$

$$\leq \max\{0, a_{i^*_\Delta j}(W)(x_j + \Delta\delta_j) + b_{i^*_\Delta}(W) - a_{i^*_\Delta j}(W) x_j - b_{i^*_\Delta}(W)\} =$$

$$= \max\{0, a_{i^*_\Delta j}(W) \delta_j \Delta\} \leq \left| a_{i^*_\Delta j}(W) \delta_j \right| \Delta$$

and

$$\max\{0, a_{i^* j}(W) x_j + b_{i^*}(W) - a_{i^*_\Delta j}(W)(x_j + \Delta\delta_j) - b_{i^*_\Delta}(W)\} \leq$$

$$\leq \max\{0, a_{i^* j}(W) x_j + b_{i^*}(W) - a_{i^* j}(W) \; (x_j + \Delta\delta_j) - b_{i^*}(W)\} \leq$$

$$\leq \left| a_{i*j}(W)\, \delta_j \right| \Delta$$

then

$$\frac{1}{\Delta} \left| \Psi(X + \Delta\delta, W) - \Psi(X,W) \right| \leq \sum_{i,j} \left| a_{ij}(W) \right| \left| \delta_j \right|$$

and from the existence of $E\, a_{ij}$ for all i,j and the Lebesque convergence theroem one gets

$$\lim_{\Delta \to 0} \frac{f(X + \Delta\delta) - f(X)}{\Delta} = f'_\delta(X) = \int \Psi'_\delta(X,W)\, dH(W) = E\, \Psi'_\delta(X,W)$$

As is well known

$$\Psi'_\delta(X,W) = \max_{g \,\varepsilon\, G(X,W)} (\sigma, \delta)$$

where

$$G(X,W) = Co\left\{a^k(W),\ k \,\varepsilon\, K(X,W)\right\} ,$$

$$a^k(W) = \left(a_{k1}(W), \ldots, a_{kn}(W)\right) ,$$

$$K(X,W) = \left\{k : (a^k(W), X) + b_k(W) = \max_i \left[(a^i(W), X) + b_i(W)\right]\right\} .$$

Here Co denotes the set of all linear combinations of the argument vectors. Taking into account this fact, the condition (24) is replaced by

$$\sum_{j=1}^{n} (\delta_j \ln ex_j + C_j\, \delta_j) + E \max_{g \,\varepsilon\, G(X,W)} (g, \delta) \geq 0$$

or

$$\sum_{j=1}^{n} (\delta_j \ln ex_j + C_j\, \delta_j) + \max_{g(X,W) \,\varepsilon\, G(X,W)} E\left(g(X,W), \delta\right) \geq 0 ,$$

or

$$\max_{g(X,W)\,\varepsilon\, G(X,W)} E\left(\ln eX + C + g(X,W), \delta\right) \geq 0 \qquad (25)$$

where

$$\ln eX = (\ln ex_1, \ldots, \ln ex_n)$$

$$C = (C_1, \ldots, C_n)$$

Since the condition (25) is fulfilled for any δ, there exist a $g(X,W)\ \varepsilon\ G(X,W)$ such that

$$\ln eX + C + E\, g(X,W) = 0 \qquad (26)$$

Let us now return to the original problem (15) or (21). For this problem $W = (\tau_1, \ldots, \tau_n)$,

$$\Psi(X,W) = \sum_{1=1}^{n} \max\left\{\alpha_i^+ (x_i - \tau_i),\ \alpha_i^- (\tau_i - x_i)\right\}$$

$$G(X,W) = \left(G_1(X,W) \times \ldots \times G_n(X,W)\right)$$

$$G_i(X,W) = \mathrm{Co}\left\{\alpha_i^k,\ k\ \varepsilon\ K(X,W)\right\}$$

where

$$\alpha_j^1 = \alpha_i^+ \quad , \qquad \alpha_i^2 = -\ \alpha_i^-$$

$$K(X,W) = \begin{cases} \{1\} & \text{with probability } P\{x_i \geq \tau_i\} \\ \{2\} & \text{with probability } P\{x_i < \tau_i\} \\ \{1,2\} & \text{with probability } P\{x_i = \tau_i\} \end{cases}$$

Then from (26) one can obtain the following optimality conditions for the original problem (15): if a point X is an optimal solution, then and only then do multipliers $0 \leq \gamma_i \leq 1$ exist such that

$$\ln ex_i + C_i + \alpha_i^+ H_i(x_i) - \alpha_i^- \left[1 - H_i(x_i)\right] + \left[\gamma_i \alpha_i^+ - (1 - \gamma_i)\alpha_i^-\right] dH_i(x_i) = 0$$

Notice, that similar conditions are mentioned in Ermoliev and Justremski (1979). In particular, if $dH_i(x_i) = 0$ at an optimal solution, or if the distributions $H_i(X)$ are continuous, then one obtains

$$\ln ex_i + C_i + \alpha_i^+ H_i(x_i) - \alpha_i^- \left[1 - H_i(x_i)\right] = 0 \quad , \quad i = \overline{1,n}$$

or

$$H_i(x_i) = \frac{\ln ex_i + C_i + \alpha_i^-}{\alpha_i^+ + \alpha_i^-} \quad , \qquad i = \overline{1,n}$$

From these equations and for some kinds of distributions $H_i(X)$ it is possible to obtain a closed form for the optimal solution, or at least to compute a good approximate solution by using simple numerical techniques. In the general case with known distributions $H_i(y)$, the generalized gradient method can be used (see Ermoliev, 1976 and 1978):

$$x_i^{N+1} = x_i^N + \rho_N \Bigg[\ln ex_i^N + C_i + \alpha_i^+ H_i(x_i^N) - \alpha_i^- \left(1 - H_i(x_i^N)\right) +$$

$$+ \left(\gamma_i^N \alpha_i^+ - (1 - \gamma_i^N)\alpha_i^-\right) dH_i(x^N)\Bigg] \quad , \qquad i = \overline{1,n}$$

where ρ_N, γ_i^N satisfy the sufficient convergence conditions $\rho_N \geq 0$, $\rho_n \to 0$, $\sum_{N=0}^{\infty} \rho_N < \infty$, $0 \leq \gamma_i^N \leq 1$. The values of ρ_N and γ_i^N can be chosen in order to decrease the objective function value. This problem has some important peculiarities: there is a closed form for the set of subgradients and computing the subgradients is easier than computing the values of the objective function. This gives us the opportunity to construct descent methods of nondifferentiable optimization as well as nondescent ones.

5. CONCLUDING COMMENTS AND ISSUES FOR FURTHER RESEARCH

The examples discussed in the foregoing sections have been kept as simple as possible, in order to introduce the proposed methods in the easiest way. When some of the simplifying assumptions are dropped, some new and more realistic models are obtained.

One possible path towards generalization is the introduction of more complex cost functions and constraints. For instance, the assumption on linear homogeneous establishing costs can be generalized to linear nonhomogeneous establishing costs

$$f_j(x) = ax + b \quad \text{if} \quad x > 0 \quad , \qquad f_j(x) = 0 \quad \text{if} \quad x = 0 \quad .$$

Such cost functions introduce a fixed charge b to be paid when a facility is established, independently of its size.

The optimization problem assumes therefore combinatorial features, since in this case the decision of which locations to choose is no longer trivial. On the other hand, this generalization is realistic, since it models the economies of scale often found in real services very well. Research on this kind of problem is ongoing, and some numerical results will be produced in a forthcoming IIASA Working Paper.

Another example of possible further research would be to impose more constraints on the sizes of facilities. Some typical and usually required constraints are the limits placed on both the size of facilities and the total budget, or total capacity to be allocated. For instance, schools have usually a minimum feasible size, below which it is not reasonable to build and sometimes a maximum feasible size as well (e.g., when the available space is limited).

Another generalization is obtained by introducing many types of facilities, to be located at the same time. Using the school example again, one may be concerned with locating high schools for different specialities and trainings. All of the above constraints still hold for each type of school. Moreover, some new constraints due to interactions within different schools may be needed. For instance, total demand for each type of school may not be known in advance, and customers may be allowed to choose both the location and the type of schools. This introduces a competition among different schools. Another obvious competition arises from limited available space in each location.

When all the above generalizations are introduced, the resulting model looks much more complicated than the ones discussed in this paper. However, it still belongs to the class of stochastic programs with linear constraints discussed in Ermoliev (1976) and Wets (1974), for which theoretical results and algorithms are available. Some applications of stochastic programming to such location problems are in progress, and they will be the subject of a forthcoming IIASA Working Paper.

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