Program packages method for solving closed-loop guidance problem with incomplete information for linear systems

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The object under study

Let us consider a linear dynamic control system:

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) + c(t), t \in [t_0, \vartheta] \\ x(t_0) \in X_0 \end{cases} \quad (1)$$

$A(\cdot), B(\cdot), c(\cdot)$ are continuous functions on $[t_0, \vartheta]$;
$A(t) \in \mathbb{R}^{n \times n}, B(t) \in \mathbb{R}^{n \times m}, c(t) \in \mathbb{R}^n, t \in [t_0, \vartheta]$

$u(\cdot)$ is Lebesgue measurable on $[t_0, \vartheta]$, $u(t) \in P$ – convex compact.

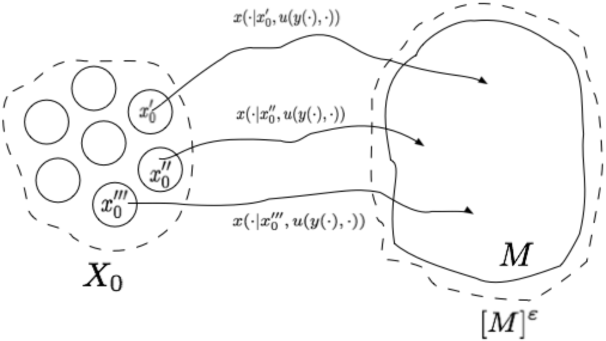
Initial states set $X_0 \subset \mathbb{R}^n$ is finite, *target set* $M \subset \mathbb{R}^n$ is closed and convex.

Signal $y(t) = Q(t)x(t)$, where $Q(\cdot) : [t_0, \vartheta] \to \mathbb{R}^{q \times n}$ - given piecewise continuous matrix-function.

Guaranteed positional guidance problem at pre-defined time

Guaranteed positional guidance problem:

$$\begin{array}{rcl} \varepsilon > 0 & & \\ x_0 \in X_0 & \xrightarrow[]{u(y(\cdot),\cdot))} & x(\vartheta | x_0, u(y(\cdot),\cdot)) \in [M]^\varepsilon \\ y(t), t \in [t_0, \vartheta] & & \end{array}$$



Homogeneous signals and initial states sets corresponding to homogeneous signals

Let us consider the homogeneous system

$$\begin{cases} \dot{x}(t) = A(t)x(t), \\ x(t_0) = x_0 \in X_0 \end{cases} \tag{2}$$

corresponding to (1).
Its solutions is given by the Cauchy formula $x(t) = F(t, t_0)x_0$,
where $F(\cdot, \cdot)$ – fundamental matrix, $F(t, t_0) = e^{\int_{t_0}^{t} A(s)ds}$.

Definition

Let us call function $g_{x_0}(\cdot)$ to be a **homogeneous signal**, corresponding to an admissible initial state x_0:

$$g_{x_0}(t) = Q(t)F(t, t_0)x_0 \ (x_0 \in X_0, t \in [t_0, \vartheta])$$

Set of all admissible initial states x_0, corresponding to the homogeneous signal $g(\cdot)$ *till time-point* $\tau \in [t_0, \vartheta]$:

$$X_0(\tau | g(\cdot)) = \{x_0 \in X_0 : g(\cdot)|_{[t_0,\tau]} = g_{x_0}(\cdot)|_{[t_0,\tau]}\};$$

Package guidance problem

Let G be the set of homogeneous signals corresponding to all admissible initial states $x_0 \in X_0$. Let us note that $|G| \leq |X_0|$.
Non-anticipatory condition:

$$\begin{array}{r} g(\cdot) \in G \\ \tau \in (t_0, \vartheta] \\ x_0', x_0'' \in X_0(\tau | g(\cdot)) \end{array} \quad \Rightarrow \quad u_{x_0'}(t) = u_{x_0''}(t), \ t \in [t_0, \tau]$$

Definition

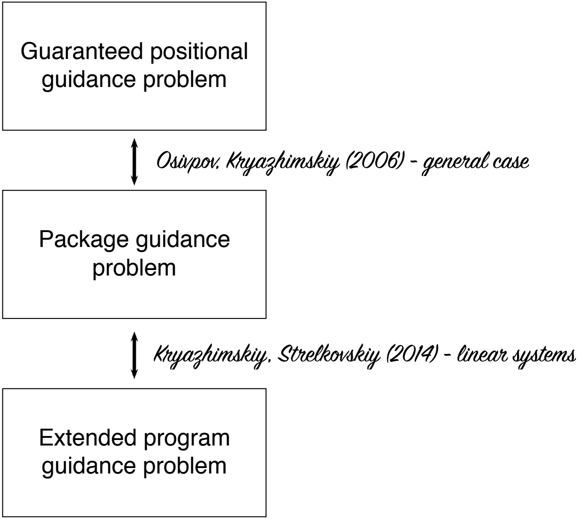
Open-loop controls family $(u_{x_0}(\cdot))_{x_0 \in X_0}$ is a **program package**, if its elements satisfy non-anticipatory condition. Program package $(u_{x_0}(\cdot))_{x_0 \in X_0}$ is **guiding**, if for every admissible initial state $x_0 \in X_0$ the condition $x(\vartheta | x_0, u_{x_0}(\cdot)) \in M$ holds.

If a guiding program package exists then the *package guidance problem* is solvable.

Theorem (Osipov, Kryazhimskiy, 2006)

The problem of positional guidance is solvable if and only if the problem of package guidance is solvable.

Guaranteed positional guidance problem at pre-defined time solution scheme



Homogeneous signals split

For an arbitrary homogeneous signal $g(\cdot) \in G$ let

$$G_0(g(\cdot)) = \left\{ \tilde{g}(\cdot) \in G : \lim_{\Delta \to +0} |\tilde{g}(t_0 + \Delta) - g(t_0 + \Delta)|_{\mathbb{R}^q} = 0 \right\}$$

be the set of all homogeneous signals equal to $g(\cdot)$ in the right-sided neighborhood of the initial time-point t_0.

Definition

Time-point $\tau_1(g(\cdot))$ such that for any homogeneous signal $\tilde{g}(\cdot) \in G_0(g(\cdot))$, certain homogeneous signal $g^*(\cdot) \in G_0(g(\cdot))$ and certain time-point $\tau_1 < \tau_1^*(g(\cdot)) \leq \vartheta$ the equalities

$$\begin{array}{rcl} |\tilde{g}(t) - g(t)|_{\mathbb{R}^q} & = & 0, \ t \in [t_0, \tau_1(g(\cdot))] \\ |g^*(t) - g(t)|_{\mathbb{R}^q} & > & 0, \ t \in (\tau_1(g(\cdot)), \tau_1^*(g(\cdot))] \end{array}$$

hold, is the first **split point** of the homogeneous signal $g(\cdot)$.

Homogeneous signals split

If $\tau_1(g(\cdot)) < \vartheta$, then for every $k = 2, 3, \ldots$ let

$$G_k(g(\cdot)) = \left\{ \tilde{g}(\cdot) \in G_{k-1}(g(\cdot)) : \lim_{\Delta \to +0} |\tilde{g}(\tau_k(g(\cdot)) + \Delta) - g(\tau_k(g(\cdot)) + \Delta)|_{\mathbb{R}^q} = 0 \right\}$$

be the set of all homogeneous signals from $G_{k-1}(g(\cdot))$, equal to $g(\cdot)$ in right-sided the neighborhood of the time-point $\tau_k(g(\cdot))$.

Definition

Time-point $\tau_{k+1}(g(\cdot))$ such that for any homogeneous signal $\tilde{g}(\cdot) \in G_k(g(\cdot))$, a certain homogeneous signal $g^*(\cdot) \in G_k(g(\cdot))$ and a certain time-point $\tau_{k+1} < \tau^*_{k+1}(g(\cdot)) \leq \vartheta$ the equalities

$$\begin{aligned} |\tilde{g}(t) - g(t)|_{\mathbb{R}^q} &= 0, \ t \in [\tau_k, \tau_{k+1}(g(\cdot))] \\ |g^*(t) - g(t)|_{\mathbb{R}^q} &> 0, \ t \in (\tau_{k+1}(g(\cdot)), \tau^*_{k+1}(g(\cdot))] \end{aligned}$$

hold, is the $(k+1)$ **split point** of the homogeneous signal $g(\cdot)$.

Let

$$T(g(\cdot)) = \{\tau_k(g(\cdot)) : k = 1, \ldots, K_{g(\cdot)}\}$$

be the set of all split points of the homogeneous signal $g(\cdot)$ and

$$T = \bigcup_{g(\cdot) \in G} T(g(\cdot)) = \{\tau_1, \ldots, \tau_K\}$$

be the set of all split points of all homogeneous signals. Let us note that T is finite, $|T| = K$ and $|T| \leq |X_0|$.

Initial states set clustering

Definition

For every $k = 1, \ldots, K$ let the set $\mathcal{X}_0(\tau_k) = \{X_0(\tau_k|g(\cdot)) : g(\cdot) \in G\}$ be the **cluster position** at the time-point τ_k, and each its element – a **cluster** at this time-point.

Lemma

Open-loop control family $(u_{x_0}(\cdot))_{x_0 \in X_0}$ *is a program package if and only if for every homogeneous signal* $g(\cdot) \in G$, *every* $\tau \in T(g(\cdot))$ *and all initial states* $x_0', x_0'' \in X_0(\tau|g(\cdot))$ *the equality* $u_{x_0'}(t) = u_{x_0''}(t)$ *holds for all* $t \in [t_0, \tau]$ *.*

Lemma

Programs family $(u_{x_0}(\cdot))_{x_0 \in X_0}$ *is a program package if and only if for every* $k = 1, \ldots, K$, *every cluster* $X_{0k} \in \mathcal{X}_0(\tau_k)$ *and all initial states* $x_0', x_0'' \in X_{0k}$ *equation* $u_{x_0'}(t) = u_{x_0''}(t)$ *holds for all* $t \in (\tau_{k-1}, \tau_k]$ *in case* $k > 1$ *and for all* $t \in [t_0, \tau_1]$ *in case* $k = 1$.

Extended program control

Let $\mathcal{P}$ be the set of all vector families $(u_{x_0})_{x_0 \in X_0}$ such that $u_{x_0} \in P$.

Definition

Let us call any measurable function $t \mapsto (u_{x_0}(t))_{x_0 \in X_0} : [t_0, \vartheta] \mapsto \mathcal{P}$ to be an **extended program**.

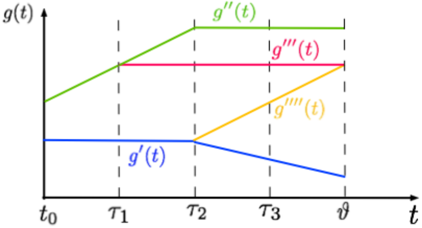
For each $k = 1, \ldots, K$ let $\mathcal{P}_k$ be the set of all families $(u_{x_0})_{x_0 \in X_0} \in \mathcal{P}$ such that for each cluster $X_{0k} \in \mathcal{X}_0(\tau_k)$ and for all $x_0', x_0'' \in X_{0k}$ holds $u_{x_0'} = u_{x_0''}$.

Definition

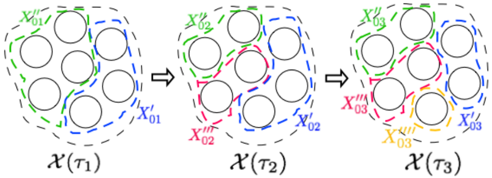
Extended program control $(u_{x_0}(\cdot))_{x_0 \in X_0}$ is **admissible**, if for every $k = 1, \ldots, K$ is held $(u_{x_0}(t))_{x_0 \in X_0} \in \mathcal{P}_k$ for all $t \in (\tau_{k-1}, \tau_k]$. We call $\mathcal{P}_k$ to be extended admissible control set on $(\tau_{k-1}, \tau_k]$.

Lemma

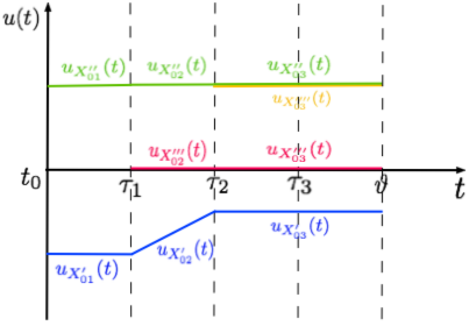
Extended program control $(u_{x_0}(\cdot))_{x_0 \in X_0}$ is a control package if and only if it is admissible.



Homogeneous signals split



Initial states set clustering



Extended program control

Extended space

Let $\mathcal{R}^j,\ j=1,2,\ldots$ be finite-dimensional Hilbert space of all vector families $(r_{x_0})_{x_0\in X_0}$ from $\mathbb{R}^j$ with scalar product $\langle\cdot,\cdot\rangle_{\mathcal{R}}$ defined as

$$\langle (r'_{x_0})_{x_0\in X_0}, (r''_{x_0})_{x_0\in X_0}\rangle_{\mathcal{R}^j} = \sum_{x_0\in X_0}\langle r'_{x_0}, r''_{x_0}\rangle \quad ((r'_{x_0})_{x_0\in X_0}, (r''_{x_0})_{x_0\in X_0}\in\mathcal{R}^j).$$

For every non-empty set $\mathcal{E}\subset\mathcal{R}^j$ let us define lower and upper support functions

$$\rho^-((l_{x_0})_{x_0\in X_0}|\mathcal{E}) = \inf_{(e_{x_0})_{x_0\in X_0}}\langle (l_{x_0})_{x_0\in X_0}, (e_{x_0})_{x_0\in X_0}\rangle_{\mathcal{R}^j}, ((l_{x_0})_{x_0\in X_0}\in\mathcal{R}^j)$$

$$\rho^+((l_{x_0})_{x_0\in X_0}|\mathcal{E}) = \sup_{(e_{x_0})_{x_0\in X_0}}\langle (l_{x_0})_{x_0\in X_0}, (e_{x_0})_{x_0\in X_0}\rangle_{\mathcal{R}^j}, ((l_{x_0})_{x_0\in X_0}\in\mathcal{R}^j)$$

Lemma

Lower support function of the admissible control set $\mathcal{P}_k$ is given by

$$\rho^-((l_{x_0})_{x_0\in X_0}|\mathcal{P}_k) = \sum_{X_{0k}\in\mathcal{X}_0(\tau_k)}\rho^-\left(\sum_{x_0\in X_{0k}} l_{x_0}\ \Big|\ P\right), \quad (l_{x_0})_{x_0\in X_0}\in\mathcal{R}^m.$$

Extended system

Let us introduce an *extended system*, composed of system (1) copies, where each copy is parametrized by elements of X_0, namely each $x_0 \in X_0$ is the initial state for the corresponding copy:

$$\dot{x}_{x_0}(t) = A(t)x_{x_0}(t) + B(t)u_{x_0}(t) + c(t), x_{x_0}(t_0) = x_0 \quad (x_0 \in X_0)$$

Let us define the phase state of the extended system (in space $\mathcal{R}^n$) as a family of phase states of the components of the extended system.

Definition

Let **extended target set** $\mathcal{M}$ be the set of all families $(z_{x_0})_{x_0 \in X_0} \in \mathcal{R}^n$ such that $z_{x_0} \in M$ for all $x_0 \in X_0$.

Definition

An admissible extended program $(u_{x_0}(\cdot))_{x_0 \in X_0}$ is **guiding the extended system**, if $(x(\vartheta | x_0, u_{x_0}(\cdot)))_{x_0 \in X_0} \in \mathcal{M}$.

The **extened problem of program guidance** is solvable, if there exists an admissible extended program which is guiding he extended system.

Extended program guidance problem solvability criterion

Theorem (Kryazhimskiy, Strelkovskiy, 2014)

1) The package guidance problem is solvable if and only if the extended program guidance problem is solvable. 2) An admissible extended program is a guiding program package if and only if it is guiding extended system.

For every $(l_{x_0})_{x_0 \in X_0} \in \mathcal{R}^n$ and $a \in [0,1]$ let us define the function

$$\begin{aligned}
\gamma_a((l_{x_0})_{x_0\in X_0}) \quad &= \quad \rho^-((l_{x_0})_{x_0\in X_0}|\mathcal{A}) - \rho^+((l_{x_0})_{x_0\in X_0}|\mathcal{M}) = \sum_{x_0\in X_0} \langle l_{x_0}, F(\vartheta, t_0)x_0\rangle_{\mathbb{R}^n} + \\
&+ \quad \sum_{k=1}^{K} \int_{\tau_{k-1}}^{\tau_k} \sum_{X_{0k}\in\mathcal{X}_0(\tau_k)} \rho^-\left(\sum_{x_0\in X_{0k}} B^T(s)F^T(\vartheta, s)l_{x_0} \Big| aP\right) ds + \\
&+ \quad \int_{t_0}^{\vartheta} \left\langle \sum_{x_0\in X_0} l_{x_0}, F(\vartheta, s)c(s) \right\rangle_{\mathbb{R}^n} ds - \sum_{x_0\in X_0} \rho^+(l_{x_0}|M)
\end{aligned}$$

Theorem (Kryazhimskiy, Strelkovskiy, 2014)

Each of the three problems – (i) the extended program guidance problem, (ii) the package guidance problem and (iii) the guaranteed positional guidance problem – is solvable if and only if

$$\sup_{(l_{x_0})_{x_0\in X_0}\in\mathcal{R}^n} \gamma_1((l_{x_0})_{x_0\in X_0}) \le 0. \tag{3}$$

Construction of the guiding program package

Definition

Let the program package $(u^0_{x_0}(\cdot))_{x_0\in X_0}$ be **zero-valued**, if $u^0_{x_0}(t)=0$ for almost all $t\in[t_0,\vartheta], x_0\in X_0$.

Zero-valued program package $(u^0_{x_0}(\cdot))_{x_0\in X_0}$ is guiding if and only if

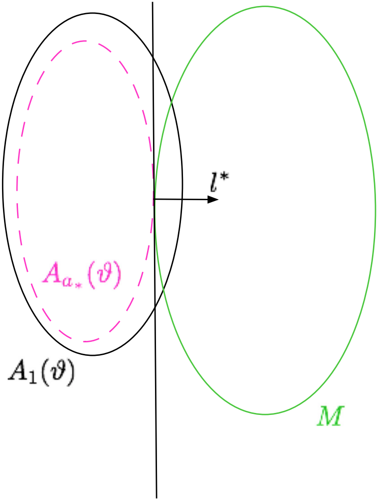
$$(x(\vartheta|x_0,u^0_{x_0}(\cdot)))_{x_0\in X_0}=(F(\vartheta,t_0)x_0+c(\vartheta))_{x_0\in X_0}\in\mathcal{M}.$$

Lemma

Let the condition (3) be satisfied, and the zero-valued program package $(u^0_{x_0}(\cdot))_{x_0\in X_0}$ is not guiding extended system. Then $a_*\in(0,1]$ exists such that

$$\max_{(l_{x_0})_{x_0\in X_0}\in\mathcal{L}}\gamma_{a_*}((l_{x_0})_{x_0\in X_0}))=0. \tag{4}$$

Separation of convex sets



Separation of two nonempty closed convex sets by hyperplane

Construction of the guiding program package

Theorem (Strelkovskiy, 2015)

Let the set P be strictly convex compact, $0 \in int\, P$, and for all $t \in [0, \vartheta]$ matrix $D(t) = B^T(t)F^T(\vartheta, t)$ is full-rank. Let the condition (3), holds and the vector family $(l^*_{x_0})_{x_0 \in X_0} \in \mathcal{L}$ is maximizing (4). Let the program package $(u^*_{x_0}(\cdot))_{x_0 \in X_0}$ satisfy $u^*_{x_0}(t) \in a_* P$ $(x_0 \in X_0,\ t \in [t_0, \vartheta])$, where a_* is the root of equation (4), and for every $k = 1, \dots, K$ and all clusters $X_{0k} \in \mathcal{X}(\tau_k)$ the equality

$$\left\langle D(t) \sum_{x_0 \in X_{0k}} l^*_{x_0}, u^*_{X_{0k}}(t) \right\rangle = \rho^- \left(D(t) \sum_{x_0 \in X_{0k}} l^*_{x_0} \middle| a_* P \right) \quad (t \in [\tau_{k-1}, \tau_k)) \tag{5}$$

holds. If for every $k = 1, \dots, K$ and $X_{0k} \in \mathcal{X}(\tau_k)$ it holds that $\sum\limits_{x_0 \in X_{0k}} l^*_{x_0} \neq 0$, then the program package $(u^*_{x_0}(\cdot))_{x_0 \in X_0}$ is guiding.

Construction of the guiding positional strategy

Let $\mathcal{U}_t = \{u(\cdot)|_{[t_0,t)}, u(\cdot) \in \mathcal{U}, t \in (t_0, \theta]\}$ be the set of programs till the time-point t. Let us formally define $\mathcal{U}_{t_0}$ as an arbitrary one-element subset of P. Let $\mathcal{Y}_t$ be the set of all functions $y(\cdot) : [t_0, t] \mapsto \mathbb{R}^q$; let us call its elements to be *observations till the time-point* t.

Definition

Let us formally define a **positional strategy** as an arbitrary finite sequence

$$S = (\sigma_i, U_i)_{i=1}^{R},$$

where $t_0 < \sigma_1 < \ldots < \sigma_R = \vartheta$ and U_i is a map of the product $\mathcal{Y}_{\sigma_i} \times \mathcal{U}_{\sigma_i}$ to the resource set P for each $i = 1, \ldots, R$. Moments σ_i, $i = 1, \ldots, R-1$ will be called (control) **adjustment moments**, and a map U_i is called **a feedback** at the moment σ_i.

Positional strategy S is *ε-guidning*, if for all $\varepsilon > 0$ and $x_0 \in X_0$ the motion $x(\cdot)$ in the controlled process under the action of S satisfies the condition $x(\vartheta) \in [M]^\varepsilon$, $[M]^\varepsilon = M + S_\varepsilon(0)$, where $S_\varepsilon(0)$ is a ball of radius ε in the space $\mathbb{R}^n$ with the center in zero.

Construction of the guiding positional strategy

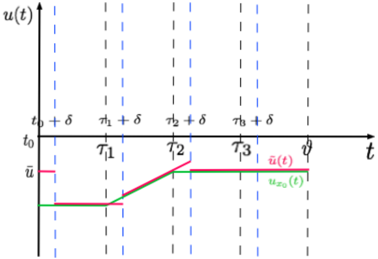
Let us constructively define the guiding positional strategy S^*. Let us define *adjustment moments* as

$$\sigma_k = \begin{cases} \delta, \ k = 0 \\ \tau_k + \delta, \ k = 1, \ldots, K \end{cases}$$

At the segment $[t_0, \sigma_1]$ let us apply an arbitrary constant control $\tilde{u}(t) = \bar{u} \in P$ to the system (1), and at the adjustment moments will adapt the control according to the following rule:

$$\tilde{u}(t) = u^*_{X_{0k}}(t), t \in (\sigma_{k-1}, \sigma_k], X_{0k} = X_0(\tau_k | g(\cdot)), \ k = 1, \ldots, K$$

Here $g(\cdot)$ is the observed homogeneous signal at the segment $[t_0, \sigma_k]$.



Construction of the guiding positional strategy

Lemma

Let the program package guidance problem be solvable for the system (1), and $\bar{x}(\cdot) = x(\cdot|x_0, \bar{u}_{x_0}(\cdot))$ is the motion from the initial admissible state $x_0 \in X_0$ under the action of program $\bar{u}_{x_0}(\cdot)$ from the guiding program package $\bar{u}_{x_0}(\cdot)_{x_0 \in X_0}$. Let $\tilde{x}(\cdot) = x(\cdot|x_0, \tilde{u}(\cdot))$ be the motion in the controlled process $(\tilde{x}(\cdot), y(\cdot), \tilde{u}(\cdot))$ with the initial state $x_0 \in X_0$ under the action of positional strategy $S = (\sigma_k, U_k)_{k=1}^{K}$, $\tilde{u}(\cdot) \in \mathcal{U}, y(t) = Q(t)\tilde{x}(t)$ $(t \in [t_0, \vartheta])$ and for each $i = 1, \ldots, R-1$ holds

$$\tilde{u}(t) = \tilde{u}(\sigma_k) = U_i(y_{\sigma_k}(\cdot), u_{\sigma_k}(\cdot)) \quad (t \in [\sigma_k, \sigma_{k+1}]),$$

where $u_{\sigma_i}(\cdot)$ is restriction of $u(\cdot)$ to $[t_0, \sigma_k)$ and $y_{\sigma_k}(\cdot)$ is restriction of $y(\cdot)$ to $[t_0, \sigma_i]$. Then

$$|\tilde{x}(t) - \bar{x}(t)| \leq KC\delta \quad (t \in [t_0, \vartheta]),$$

where C is a positive constant.

Theorem (Strelkovskiy, 2015)

Let the program package guidance problem be solvable for the system (1), and condition $KC\delta < \varepsilon$ holds, where $\delta > 0$, and C is a certain positive constant. Then the positional strategy $S = (\sigma_k, U_k)_{k=1}^{K}$, corresponding to the guiding program package, is ε-guiding.

Example

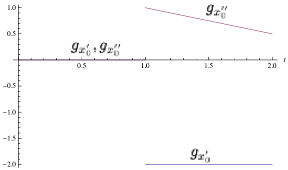
Let us consider a dynamic controlled system

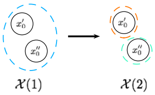
$$\begin{cases} \dot{x}_1 = x_2, x_1(0) = x_{01} \\ \dot{x}_2 = u, x_2(0) = x_{02} \end{cases} \qquad (6)$$

on the time segment $[0, 2]$.

$$X_0 = \{\overset{(-2,0)}{x_0'}, \overset{(3/2,-1/2)}{x_0''}\}; \qquad M = \left\{ \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \in \mathbb{R}^2 : |x_1| \leq 1, x_2 \in \mathbb{R} \right\}$$

$$u(t) \in P = \{u : |u| \leq 1\}, \; t \in [0, 2]; \qquad Q(t) = \begin{cases} (0, 0), t \in [0, 1] \\ (1, 0), t \in (1, 2]. \end{cases}$$





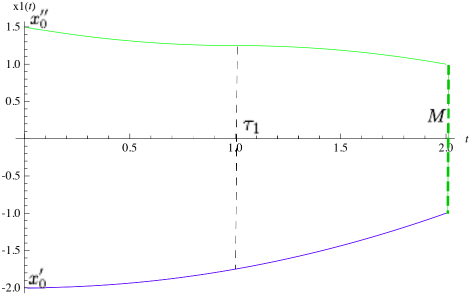
Example

Solving maximization problem (3) we get

$$\begin{aligned}\gamma_1 \approx -0.35 &< 0 \Rightarrow \text{The package guidance problem is solvable}\\ \gamma_{a_*} &= 0,\ a_* = \frac{1}{2}\end{aligned}$$

From the theorem (5) it is clear that the guiding program package is

$$\begin{cases} u^*_{x_0' x_0''}(t) = \frac{1}{2},\ t \in [0,1],\\ u^*_{x_0'}(t) = \frac{1}{2},\ t \in (1,2],\\ u^*_{x_0''}(t) = -\frac{1}{2},\ t \in (1,2]. \end{cases}$$



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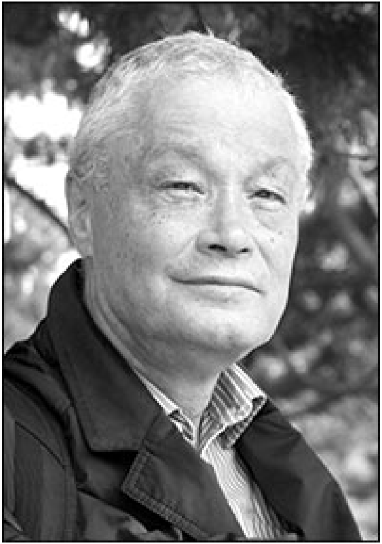
Acknowledgements

Yu. S. Osipov



A.V.Kryazhimskiy (1949 – 2014)