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ON REACHABLE SETS FOR ONE-PULSE CONTROLS UNDER CONSTRAINTS OF ASYMPTOTIC CHARACTER

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Abstract

We study asymptotic versions of reachable sets of linear systems for two intuitive formalizations of one-pulse controls given constraints of asymptotic character. The results are presented for the simplest example of linear control systems, the double integrator, though they admit a straightforward extension to a generic linear system. We suppose that the coefficient at the control is a piecewise continuous function. To illustrate the developed theoretical framework for both formalizations, we demonstrate examples of attraction sets, asymptotic versions of reachable sets.

Key words

Finitely additive measures, attraction set, constraints of asymptotic character, ultrafilters.

1 Introduction

As the first approximation, dynamics of some objects can be described by a linear control system. Usually, it is assumed that controls comply with impulse or geometric constraints. Modeling of some real processes, e.g., rocket staging, requires not only impulse constraints, but also discontinuous coefficients at the control action. Models are studied to obtain an optimal control, but commonly models aim to explore a reachable set for a controlled object, i.e., the set of all possible terminal states that are reachable at some fixed time by using admissible controls. In general, the last task is more difficult. Note that in the presence of impulse constraints and discontinuous coefficients, estimation of reachable sets may not be robust with respect to small relaxations of phase constraints even in the case of linear control systems. This motivates us to explore robust versions of reachable sets for linear control systems with discontinuous coefficients at the control action with ‘impulsive structure’. We present results for the simplest example of linear control systems, the double integrator, but they admit a straightforward extension to a generic linear system. The double integrator is important for applications since it serves as a good approximation of spacecraft dynamics in deep space [Scharf et al., 2004].

Impulse controls may lead to the issue of the correct definition of trajectories. E.g., in [Sesekin and Fetisova, 2010] for nonlinear systems of functional differential equations with a generalized action on the right-hand side, there were proposed a notion of solution together with the corresponding sufficient conditions for the existence and uniqueness; see also [Zavalischin and Sesekin, 1997]. Note that if controls driving the system are impulsive, then the trajectories are discontinuous. In optimal control problems, this issue requires a special treatment (see [Vinter and Pereira, 1988, Miller and Rubinovich, 2003]). Impulse controls were also studied in game settings; for example, see [Goncharova and Staritsyn, 2015, Khimich and Chikrii, 2009].

Due to the presence of a discontinuous coefficient at the control action, this paper relies on the approach [Chentsov, 1996] that allows to overcome mathematical difficulties connected with the product of discontinuous and generalized functions (representing impulsive control) for linear control systems. This approach uses an extension construction in the class of finitely-additive measures. Note that geometric constraints on the control also require extension constructions to ensure ‘nice’ mathematical properties of optimal control problems; see [Warga, 1972, Young, 1980, Gamkrelidze, 1978].

We study two formalizations (models) of one-pulse controls: controls with vanishingly small duration (Model 1) and instantaneous jumps ('pushes') that used in Model 2. Controls in Model 1 naturally generate constraints of asymptotic character with respect to their duration. An additional type of the constraints of asymptotic character (similar for both models) is due to a sequential relaxation of phase constraints.

For both formalizations, the paper studies attraction sets (AS), the asymptotic versions of reachable set, which can be considered as more robust estimates of reachable sets given a potential relaxation of the phase constraints. The discontinuous coefficient at the control modifies the double integrator model to account for an effect of a sharp change of mass of the controlled object.

Let us informally describe the models.

1.1 Model 1

We consider a control problem with relaxed phase constraints and the requirement to fully utilize all available energy resources during a vanishingly small time. *Model 1* is represented by the double integrator

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = \mathbf{b}(t) f(t), \end{cases} \tag{1}$$

on a time interval $I \overset{\triangle}{=} [a, b]$ where $a, b \in \mathbb{R}$ and $a < b$. A real valued function $\mathbf{b} = \mathbf{b}(\cdot)$ is defined on I. Controls $f : I \to \mathbb{R}$ must comply with the constraints that will be introduced later. Note that $\mathbf{b}$ can be discontinuous representing a change of mass of the controlled object or as a change in an engine's 'working mode'. To derive trajectories, $\mathbf{b}$ and f are assumed to be in agreement with a measurable space (specified later) allowing integration. The measurable space restricts $\mathbf{b}$ and f to the class of piecewise continuous functions, i.e., functions that are continuous on all but a finite number of points at which their values and one-sided limits are defined. The initial conditions $x_{0,1}, x_{0,2} \in \mathbb{R}$ are given; $x_0 = (x_{0,1}, x_{0,2}) \in \mathbb{R}^2$. Then trajectory $x_f(\cdot)$ of (1) (the «scalar» double integrator) is defined for $t \in I$ as follows:

$$x_{f,1}(t) = x_{0,1} + t x_{0,2} + \int_a^t (t - \xi) \mathbf{b}(\xi) f(\xi) d\xi, \tag{2}$$

$$x_{f,2}(t) = x_{0,2} + \int_a^t \mathbf{b}(\xi) f(\xi) d\xi. \tag{3}$$

Let N be a natural number, $\mathbf{Y} \subset \mathbb{R}^N$ be a closed set, and $\rho : I \to \mathbb{R}^N$ be a piecewise continuous vector-function. Then admissible controls f comply with the following conditions: for $\varepsilon_1, \varepsilon_2 > 0$

(A) f is non-negative and $\int_a^b f \, dt = 1$, i.e., all control resources are used;
(Bε_1) $(\int_a^b \rho_i f dt)_{i=1,\dots,N}$ belongs ε_1-neighbourhood of $\mathbf{Y}$;
(Cε_2) all time instants such that f is non-zero belong to an interval with diameter that is less than ε_2.

We constrain energy resources by 1 in condition (A) without loss of generality.

The focus of this study is on ASs that are the limit representation of the reachable sets

$$\{(x_{f,1}(b), x_{f,2}(b)) : f \text{ complies with conditions (A), (B}\varepsilon_1\text{), and (C}\varepsilon_2\text{)}\}$$

as $\varepsilon_1, \varepsilon_2 \to 0$.

Using (2) and (3), we may identify the terminal positions $x_f(b)$ with the vectors

$$\Big(\int_a^b (b - \xi) \mathbf{b}(\xi) f(\xi) d\xi, \int_a^b \mathbf{b}(\xi) f(\xi) d\xi \Big) \in \mathbb{R}^2. \tag{4}$$

In what follows, we treat (4) as terminal positions. Note that $x_f(b)$ is equal to the vector in (4) if $x_0 = (0, 0)$.

1.2 Model 2

Model 2 represents idealized one-pulse controls as jumps ('pushes') at moments $\tau \in I$ applied to the double integrator

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = \mathbf{b}(t) \delta(t - \tau), \end{cases} \tag{5}$$

on a time interval $I = [a, b]$; δ stands for the Dirac delta function. By choosing $\tau \in I$, we generate distinct trajectories with the discontinuity at τ. The functions $\mathbf{b}, \rho$, and the initial conditions $x_0 \in \mathbb{R}^2$ are the same as in Model 1. We assume that trajectories $x_\tau(\cdot)$ for the 'scalar' double integrator with jumps are defined for 'controls' $\tau \in I$ as follows: for $t \in [a, \tau]$

$$x_\tau(t) = \Big(x_{0,1} + t x_{0,2}, x_{0,2} \Big) \in \mathbb{R}^2 \tag{6}$$

and for $t \in]\tau, b]$

$$x_\tau(t) = \Big(x_{0,1} + t x_{0,2} + (t - \tau) \mathbf{b}(\tau), x_{0,2} + \mathbf{b}(\tau) \Big) \in \mathbb{R}^2. \tag{7}$$

Note that extension constructions and ASs for systems with jump controls in the spirit of (5) were studied in [Baklanov and Chentsov, 2010], [Berdyshev and Chentsov, 1998], [Chentsov, 2006, §21], and [Chentsov, 1997, §7.1–7.5].

We assume that admissible 'controls' $\tau \in I$ comply with the following conditions: for $\epsilon > 0$

(Dϵ) $\rho(\tau)$ belongs ϵ-neighbourhood of the closed set $\mathbf{Y}$.

For Model 2, we study ASs that are the limit versions of the reachable sets

$$\{x_\tau(b) : \tau \text{ complies with condition (D}\epsilon)\}$$

as $\epsilon \to 0$.

Similarly to Model 1, the terminal positions $x_\tau(b)$ are identified with vectors

$$\Big((b-\tau)\mathbf{b}(\tau), \mathbf{b}(\tau)\Big) \in \mathbb{R}^2 \ \ \forall \tau \in [a, b[\quad \text{and } (0,0) \text{ if } \tau = b; \quad (8)$$

clearly, $x_\tau(b)$ coincides with (8) if $x_{0,1} = x_{0,2} = 0$.

Given the way we define trajectories (see (6) and (7)), for the jump at the endpoint b, only (6) correctly defines the trajectory $x_\tau(t)$ $\forall t \in I$ since in this case $\tau = b$ makes $]\tau, b] = \emptyset$.

2 General Notation

We use the standard set-theoretic notation. We call a "family" a set in which all elements are sets. The pair set of y, z is denoted by $\{y; z\}$; $\{h\}$ is the singleton containing h; an ordered pair $z = (x, y)$ has $\mathrm{pr}_1(z) = x$ as its first element and $\mathrm{pr}_2(z) = y$ as the second one; obviously, $z = \big(\mathrm{pr}_1(z), \mathrm{pr}_2(z)\big)$.

By $\mathcal{P}(X)$ (by $\mathcal{P}'(X)$) we denote the family of all (the family of all nonempty) subsets of a set X. By definition, B^A is the set of all mappings from a set A to a set B. If $g \in B^A$ and $C \in \mathcal{P}(A)$, then $g^1(C) \triangleq \{g(x) : x \in C\} \in \mathcal{P}(B)$ is the image of C under g. Further, $\mathbb{N} \triangleq \{1; 2; \ldots\}$ and $J[k] \triangleq \{l \in \mathbb{N} |\, l \leqslant k\}$ $\forall k \in \mathbb{N}$. If T is a set and $k \in \mathbb{N}$, then, as per the common convention, we write T^k instead of $T^{J[k]}$. By $(\mathrm{top})[S]$, we denote the family of all topologies on a set S; if $\tau \in (\mathrm{top})[S]$, then (S, τ) is a topological space; if $H \in \mathcal{P}(S)$, then $\mathrm{cl}(H, \tau)$ stands for the closure of H in (S, τ). If (S, τ) is a topological space and $M \in \mathcal{P}(S)$, then $\tau\big|_M \triangleq \{M \cap G : G \in \tau\} \in (\mathrm{top})[M]$, and $(M, \tau\big|_M)$ is a subspace of (S, τ). Let $(\tau - \mathrm{comp})[S]$ stand for the family of all nonempty and compact (in (S, τ)) subsets of S.

If E is a set, then $\beta[E] \triangleq \{\mathcal{E} \in \mathcal{P}'\big(\mathcal{P}(E)\big) |\, \forall \Sigma_1 \in \mathcal{E}\ \forall \Sigma_2 \in \mathcal{E}\ \exists \Sigma_3 \in \mathcal{E} : \Sigma_3 \subset \Sigma_1 \cap \Sigma_2\}$ stands for the family of all nonempty directed subfamilies of $\mathcal{P}(E)$.

Assume that X are Y nonempty sets, $\mathcal{X} \in \mathcal{P}'\big(\mathcal{P}(X)\big)$, $\tau \in (\mathrm{top})[Y]$, and $r \in Y^X$. Then, we define AS $(\mathbf{as})[X; Y; \tau; r; \mathcal{X}]$ as in [Chentsov, 2013a, Section 3]. A sequential AS $(\mathbf{sas})[X; Y; \tau; r; \mathcal{X}]$ is defined only by the sequential limits of points in Y. Note that, if $\mathcal{X} \in \beta[X]$, then

$$(\mathbf{as})[X; Y; \tau; r; \mathcal{X}] = \bigcap_{S \in \mathcal{X}} \mathrm{cl}\big(r^1(S), \tau\big).$$

2.1 Finitely additive measures and ultrafilters

By $\mathcal{I}$, we denote the family of sets $L \in \mathcal{P}(I)$ such that $\exists c \in I\ \ \exists d \in I : (]c, d[\subset L) \& (L \subset [c, d])$. Let $\mathcal{A}$ be the algebra of subsets of I generated by the semialgebra $\mathcal{I}$. Let $\chi_L \in \mathbb{R}^I$ be the indicator functions of sets $L \in \mathcal{P}(I)$ (see [Neveu, 1965, p. 32]) Then, $B_0(I, \mathcal{A})$ denotes the linear span of $\{\chi_A : A \in \mathcal{A}\}$. Note that $B_0(I, \mathcal{A})$ is a (linear) manifold in the Banach space $\mathbb{B}(I)$ of all bounded real-valued functions on I endowed with the standard $\sup$-norm [Dunford and Schwartz, 1958, p. 261 of the Russian translation], which we denote by $\|\cdot\|$. Let $B(I, \mathcal{A})$ stand for the closure of $B_0(I, \mathcal{A})$ in $(\mathbb{B}(I), \|\cdot\|)$. Note that $B(I, \mathcal{A})$ with the norm induced by $(\mathbb{B}(I), \|\cdot\|)$ is itself a Banach space, whose topological dual $B^*(I, \mathcal{A})$ is isometrically isomorphic to the space $\mathbb{A}(\mathcal{A})$ of all bounded finitely additive measures on $\mathcal{A}$ endowed with the (strong) norm-variation. Moreover, the isometric isomorphism between $\mathbb{A}(\mathcal{A})$ and $B^*(I, \mathcal{A})$ is defined by the rule

$$\mu \longmapsto \Big(\int_I f\, d\mu\Big)_{f \in B(I, \mathcal{A})} : \mathbb{A}(\mathcal{A}) \to B^*(I, \mathcal{A}).$$

Assume that $\mathbb{A}(\mathcal{A})$ is endowed with the $*$-weak topology $\tau_*(\mathcal{A})$ corresponding to the duality $\big(B(I, \mathcal{A}), \mathbb{A}(\mathcal{A})\big)$. Thus, $(\mathbb{A}(\mathcal{A}), \tau_*(\mathcal{A}))$ is a locally convex σ-compactum. We will also deal with the topology $\tau_0(\mathcal{A})$ of a subspace of the topological power of $\mathbb{R}$ with the discrete topology with $\mathcal{A}$ as the index set; see the definition of $\tau_0(\mathcal{A})$ in [Chentsov, 1996, (4.2.9)]. Let $(\mathrm{add})_+[\mathcal{A}]$ be the set of all real-valued non-negative finitely additive measures on $\mathcal{A}$; $(\mathrm{add})_+[\mathcal{A}] \subset \mathbb{A}(\mathcal{A})$. Further, $\mathbb{P}(\mathcal{A})$ stands for the set of all finitely additive probabilities; precisely, $\mathbb{P}(\mathcal{A}) \triangleq \{\mu \in (\mathrm{add})_+[\mathcal{A}] |\, \mu(I) = 1\} \in \big(\tau_*(\mathcal{A}) - \mathrm{comp}\big)[\mathbb{A}(\mathcal{A})]$. By definition, put

$$\mathbb{T}(\mathcal{A}) \triangleq \{\mu \in \mathbb{P}(\mathcal{A}) | \ \forall A \in \mathcal{A}\ \big(\mu(A) = 0\big) \vee \big(\mu(A) = 1\big)\} \in \big(\tau_*(\mathcal{A}) - \mathrm{comp}\big)[\mathbb{A}(\mathcal{A})].$$

Let $\mathbb{F}_0^*(\mathcal{A})$ be the set of all ultrafilters in the algebra $\mathcal{A}$ (see [Chentsov, 2011a, (3.2)]). For all $\mathcal{L} \in \mathcal{P}(\mathcal{A})$, we define $\mathbb{X}_{\mathcal{L}} \in \mathbb{R}^{\mathcal{A}}$ (the indicator of $\mathcal{L}$) by the rule $\mathbb{X}_{\mathcal{L}}(L) \triangleq 1$ if $L \in \mathcal{L}$ and $\mathbb{X}_{\mathcal{L}}(A) \triangleq 0$ if $A \in \mathcal{A} \setminus \mathcal{L}$.

Thus, $\mathbb{X}_{\mathcal{U}} \in \mathbb{T}(\mathcal{A})\ \ \forall \mathcal{U} \in \mathbb{F}_0^*(\mathcal{A})$. The mapping $\kappa \triangleq (\mathbb{X}_{\mathcal{U}})_{\mathcal{U} \in \mathbb{F}_0^*(\mathcal{A})}$ is a homeomorphism between $\mathbb{F}_0^*(\mathcal{A})$ and $\mathbb{T}(\mathcal{A})$ (see [Chentsov, 2013b, Proposition 4.2]); then, $\mathbb{F}_0^*(\mathcal{A})$ and $\mathbb{T}(\mathcal{A})$ are homeomorphic.

Let us present the structure of $\mathbb{F}_0^*(\mathcal{A})$ (see [Chentsov, 2011b] for the full exposition). First, we define the family $\beta_{\mathcal{A}}^0(I) = \{\mathcal{B} \in \beta[I] \,|\, (\emptyset \notin \mathcal{B}) \& (\mathcal{B} \subset \mathcal{A})\}$ of all bases of filters of I contained in $\mathcal{A}$. Secondly, every $\mathcal{B} \in \beta_{\mathcal{A}}^0(I)$ generates the corresponding filter $(I - \mathbf{fi})[\mathcal{B} \,|\, \mathcal{A}] \triangleq \{A \in \mathcal{A} \,|\, \exists B \in \mathcal{B} : B \subset A\}$ in $\mathcal{A}$. Thirdly, if $t \in]a,b]$, then $\mathcal{J}_t^{(-)} \triangleq \{[c,t[\,:\, c \in [a,t[\} \in \beta_{\mathcal{A}}^0(I)$ generates the ultrafilter

$$\mathcal{U}_t^{(-)} \triangleq (I - \mathbf{fi})[\mathcal{J}_t^{(-)} | \mathcal{A}] \in \mathbb{F}_0^*(\mathcal{A}).$$

Fourthly, if $t \in [a,b[$, then $\mathcal{J}_t^{(+)} \triangleq \{]t,c] \,:\, c \in]t,b]\} \in \beta_{\mathcal{A}}^0(I)$ generates the ultrafilter

$$\mathcal{U}_t^{(+)} \triangleq (I - \mathbf{fi})[\mathcal{J}_t^{(+)} | \mathcal{A}] \in \mathbb{F}_0^*(\mathcal{A}).$$

Note that all ultrafilters mentioned above are free [Engelking, 1977, Section 3.6]. Finally, $\mathbb{F}_0^*(\mathcal{A})$ coincides with the union of the set

$$\{\mathcal{U}_t^{(-)} : t \in]a,b]\} \cup \{\mathcal{U}_t^{(+)} : t \in [a,b[\}$$

and the set of all trivial ultrafilters in $\mathcal{A}$.

Let η stand for the trace of the Lebesgue measure on the algebra $\mathcal{A}$; $\eta \in (\mathrm{add})_+[\mathcal{A}]$. In what follows, we deal with the compact set

$$\begin{aligned}\mathbb{P}_\eta(\mathcal{A}) \triangleq \big\{\mu \in \mathbb{P}(\mathcal{A}) \,|\, \forall A \in \mathcal{A}\ \ \big(\eta(A) = 0\big) \Rightarrow \\ \Rightarrow \big(\mu(A) = 0\big)\big\} \in \big(\tau_*(\mathcal{A}) - \mathrm{comp}\big)[\mathbb{A}(\mathcal{A})].\end{aligned} \tag{9}$$

For arbitrary $f \in B(I, \mathcal{A})$, by $f * \eta$ we denote the indefinite η-integral of f. Note that $f * \eta$ is a set function. Let $B_0^+(I, \mathcal{A})$ be the set of all non-negative functions from $B_0(I, \mathcal{A})$.

3 Rigorous Definitions of the Models

Fix $N \in \mathbb{N}$, $(\rho_i)_{i \in J[N]} \in B(I, \mathcal{A})^N, \mathbf{b} \in B(I, \mathcal{A})$, a nonempty closed set $\mathbf{Y} \in \mathcal{P}'(\mathbb{R}^N)$, and a set $M \in \mathcal{P}(J[N])$ such that $\rho_j \in B_0(I, \mathcal{A})\ \ \forall j \in M$ (the case $M = \emptyset$ is allowed).

To formally define constraints of type (B) and (D), we introduce the following neighborhoods: $\forall \varepsilon \in]0, \infty[$

$$\begin{aligned}O(\mathbf{Y}, \varepsilon) \triangleq \big\{(z_i)_{i \in J[N]} \in \mathbb{R}^N \,|\, \exists (y_i)_{i \in J[N]} \in \mathbf{Y} : \\ |y_j - z_j| < \varepsilon\ \ \forall j \in J[N]\big\},\end{aligned}$$

$$\begin{aligned}\widehat{O}(\mathbf{Y}, \varepsilon) \triangleq \big\{(z_i)_{i \in J[N]} \in \mathbb{R}^N \,|\, \exists (y_i)_{i \in J[N]} \in \mathbf{Y} : \\ (y_j = z_j\ \ \forall j \in M) \,\& \\ \& \,(|y_j - z_j| < \varepsilon\ \ \forall j \in J[N])\big\}.\end{aligned}$$

3.1 Model 1

We formally define the set of all controls complying with (A):

$$\mathbf{F} \triangleq \big\{f \in B_0^+(I, \mathcal{A}) \big| \int\limits_I f\, d\eta = 1\big\}. \tag{10}$$

For every $f \in \mathbf{F}$ we introduce the set $\mathrm{supp}(f) \triangleq \{t \in I \,|\, f(t) \neq 0\} \in \mathcal{P}'(I)$ and two values $\mathbf{t}_0(f) \triangleq \inf\big(\mathrm{supp}(f)\big) \in I$, $\mathbf{t}^0(f) \triangleq \sup\big(\mathrm{supp}(f)\big) \in I$. Given $\varepsilon \in]0, \infty[$, we define

$$\mathbf{F}_\varepsilon \triangleq \{f \in \mathbf{F} \,|\, \mathbf{t}^0(f) - \mathbf{t}_0(f) < \varepsilon\}.$$

We introduce the corresponding sets of ε-admissible controls in $\mathbf{F}$:

$$\mathbb{Y}_\varepsilon \triangleq \Big\{f \in \mathbf{F}_\varepsilon \big| \big(\int\limits_I \rho_i f\, d\eta\big)_{i \in J[N]} \in O(\mathbf{Y}, \varepsilon)\Big\},$$

$$\widehat{\mathbb{Y}}_\varepsilon \triangleq \Big\{f \in \mathbf{F}_\varepsilon \big| \big(\int\limits_I \rho_i f\, d\eta\big)_{i \in J[N]} \in \widehat{O}(\mathbf{Y}, \varepsilon)\Big\},$$

$\widehat{\mathbb{Y}}_\varepsilon \subset \mathbb{Y}_\varepsilon$. These sets lead to the following directed families of subsets of $\mathbf{F}$:

$$\mathfrak{Y} \triangleq \{\mathbb{Y}_\varepsilon : \varepsilon \in]0, \infty[\} \in \beta[\mathbf{F}],$$

$$\widehat{\mathfrak{Y}} \triangleq \{\widehat{\mathbb{Y}}_\varepsilon : \varepsilon \in]0, \infty[\} \in \beta[\mathbf{F}].$$

In a formal way, we define *Model 1* as the double integrator

$$\begin{cases}\dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = \mathbf{b}(t) f(t),\end{cases}$$

on a time interval $I = [a, b]$. Now we link informal constraints from Section 1.1 to the following formal conditions for admissible controls f : for $\varepsilon_1, \varepsilon_2 > 0$

(A) $f \in \mathbf{F}$ (see (10));
(Bε_1) $(\int_I \rho_i f\, d\eta)_{i \in J[N]} \in O(\mathbf{Y}, \varepsilon_1)$;
(Cε_2) $f \in \mathbf{F}_{\varepsilon_2}$.

We also consider the modification of (Bε_1):

($\hat{\text{B}}\varepsilon_1$) $(\int_I \rho_i f\, d\eta)_{i \in J[N]} \in \widehat{O}(\mathbf{Y}, \varepsilon_1)$.

Let us define the functions

$$\pi_1 : I \to \mathbb{R},\ \pi_2 : I \to \mathbb{R}$$

by the following rule: $\forall t \in I$

$$\pi_1(t) \triangleq (b - t)\mathbf{b}(t),\ \pi_2(t) \triangleq \mathbf{b}(t), \pi(t) \triangleq (\pi_1(t), \pi_2(t)).$$

Let the mapping Π be defined by the rule

$$f \longmapsto \Big(\int_I \pi_i f\, d\eta\Big)_{i\in J[2]} : \mathbf{F} \longrightarrow \mathbb{R}^2.$$

Values of Π generate reachable sets of Model 1 (see (4)). For Model 1, we explore the ASs $(\mathbf{as})[\mathbf{F};\mathbb{R}^2;\tau_{\mathbb{R}}^{(2)};\Pi;\mathfrak{Y}]$ and $(\mathbf{as})[\mathbf{F};\mathbb{R}^2;\tau_{\mathbb{R}}^{(2)};\Pi;\widehat{\mathfrak{Y}}]$, which are the asymptotic versions of reachable sets.

3.2 Model 2

We introduce sets of ϵ-admissible jump-controls in I :

$$\mathbb{I}_\varepsilon \triangleq \Big\{\tau \in I \big| \rho(\tau) \in O(\mathbf{Y},\varepsilon)\Big\},$$
$$\widehat{\mathbb{I}}_\varepsilon \triangleq \Big\{\tau \in I \big| \rho(\tau) \in \widehat{O}(\mathbf{Y},\varepsilon)\Big\},$$

$\widehat{\mathbb{I}}_\varepsilon \subset \mathbb{I}_\varepsilon$, and the corresponding directed families of subsets of I:

$$\mathfrak{X} \triangleq \{\mathbb{I}_\epsilon : \epsilon \in]0,\infty[\} \in \beta[I],$$
$$\widehat{\mathfrak{X}} \triangleq \{\widehat{\mathbb{I}}_\epsilon : \epsilon \in]0,\infty[\} \in \beta[I].$$

Recall that *Model 2* formalizes idealized one-pulse controls as jumps (‘pushes’) at moments $\tau \in I$. The trajectories are defined on I by (6) and (7). By choosing $\tau \in I$, we generate distinct terminal positions according to (8). We assume that admissible jump-controls $\tau \in I$ comply with the following conditions: for $\epsilon > 0$

(Dϵ) $\rho(\tau) \in O(\mathbf{Y},\epsilon)$.

We also consider the refinement of (Dϵ):

($\hat{\text{D}}\epsilon$) $\rho(\tau) \in \widehat{O}(\mathbf{Y},\epsilon)$.

Let us define the function ϕ generating terminal positions (see (8)) with components $\phi_1 : I \to \mathbb{R}$ and $\phi_2 : I \to \mathbb{R}$ by the following rule: $\forall t \in [a,b[$

$$\phi_1(t) \triangleq (b-t)\mathbf{b}(t),\ \phi_2(t) \triangleq \mathbf{b}(t),$$

$\phi_1(b) \triangleq 0$ and $\phi_2(b) \triangleq 0$. Clearly, values of π and ϕ only differ at the endpoint b.

For Model 2, we explore the ASs $(\mathbf{as})[I;\mathbb{R}^2;\tau_{\mathbb{R}}^{(2)};\phi;\mathfrak{X}]$ and $(\mathbf{as})[I;\mathbb{R}^2;\tau_{\mathbb{R}}^{(2)};\phi;\widehat{\mathfrak{X}}]$.

4 Attraction Sets for Model 1

4.1 Generalized elements

Evidently, $f * \eta \in \mathbb{P}_\eta(\mathcal{A})\ \forall f \in \mathbf{F}$. Let $\mathfrak{I}$ be defined by the rule $f \mapsto f*\eta :\ \mathbf{F} \to \mathbb{P}_\eta(\mathcal{A})$. This mapping allows us to embed $\mathbf{F}$ in the compact set (9) as a dense subset: $\mathbb{P}_\eta(\mathcal{A}) = \mathrm{cl}\big(\mathfrak{I}^1(\mathbf{F}),\tau_*(\mathcal{A})\big) = \mathrm{cl}\big(\mathfrak{I}^1(\mathbf{F}),\tau_0(\mathcal{A})\big)$; see [Chentsov, 1996, Ch. 4]. For every $t \in]a,b[$, we put $\zeta_t^0 \triangleq \inf(\{t-a; b-t\})$ and introduce the set

$$\begin{gathered}\mathbb{P}_\eta^0(\mathcal{A}|\,t) \triangleq \{\mu \in \mathbb{P}_\eta(\mathcal{A})| \\ \mu(]t-\varepsilon,t+\varepsilon[) = 1\ \ \forall \varepsilon \in]0,\zeta_t^0]\} = \\ = \{\alpha\kappa(\mathcal{U}_t^{(-)}) + (1-\alpha)\kappa(\mathcal{U}_t^{(+)}) :\ \alpha \in [0,1]\}.\end{gathered}$$

In this connection, we put by definition

$$\mathbb{P}_\eta^0[\mathcal{A}] \triangleq \Big(\bigcup_{t\in]a,b[} \mathbb{P}_\eta^0(\mathcal{A}|\,t)\Big) \cup \{\kappa(\mathcal{U}_a^{(+)});\kappa(\mathcal{U}_b^{(-)})\}.$$

The following set plays a major role in our study:

$$\widetilde{\mathbb{P}}_\eta^0(\mathcal{A}) \triangleq \Big\{\mu \in \mathbb{P}_\eta^0[\mathcal{A}] \mid \big(\int_I \rho_i\, d\mu\big)_{i\in J[N]} \in \mathbf{Y}\Big\}. \quad (11)$$

The topologies $\tau_\eta^*(\mathcal{A}) \triangleq \tau_*(\mathcal{A})\big|_{\mathbb{P}_\eta(\mathcal{A})}$ and $\tau_\eta^0(\mathcal{A}) \triangleq \tau_0(\mathcal{A})\big|_{\mathbb{P}_\eta(\mathcal{A})}$ satisfy the property $\tau_\eta^*(\mathcal{A}) \subset \tau_\eta^0(\mathcal{A})$ (see [Chentsov, 1996, Ch. 4]).

Theorem 1. *[Chentsov and Baklanov, 2015] For Model 1, (11) defines the universal AS in the space of generalized elements, i.e.,*

$$\begin{aligned}\widetilde{\mathbb{P}}_\eta^0(\mathcal{A}) &= (\mathbf{as})[\mathbf{F};\mathbb{P}_\eta(\mathcal{A});\tau_\eta^*(\mathcal{A});\mathfrak{I};\mathfrak{Y}] = \\ &= (\mathbf{as})[\mathbf{F};\mathbb{P}_\eta(\mathcal{A});\tau_\eta^0(\mathcal{A});\mathfrak{I};\mathfrak{Y}] = \\ &= (\mathbf{as})[\mathbf{F};\mathbb{P}_\eta(\mathcal{A});\tau_\eta^*(\mathcal{A});\mathfrak{I};\widehat{\mathfrak{Y}}] = \\ &= (\mathbf{as})[\mathbf{F};\mathbb{P}_\eta(\mathcal{A});\tau_\eta^0(\mathcal{A});\mathfrak{I};\widehat{\mathfrak{Y}}] = \\ &= (\mathbf{sas})[\mathbf{F};\mathbb{P}_\eta(\mathcal{A});\tau_\eta^0(\mathcal{A});\mathfrak{I};\mathfrak{Y}] = \\ &= (\mathbf{sas})[\mathbf{F};\mathbb{P}_\eta(\mathcal{A});\tau_\eta^*(\mathcal{A});\mathfrak{I};\mathfrak{Y}] = \\ &= (\mathbf{sas})[\mathbf{F};\mathbb{P}_\eta(\mathcal{A});\tau_\eta^0(\mathcal{A});\mathfrak{I};\widehat{\mathfrak{Y}}] = \\ &= (\mathbf{sas})[\mathbf{F};\mathbb{P}_\eta(\mathcal{A});\tau_\eta^*(\mathcal{A});\mathfrak{I};\widehat{\mathfrak{Y}}].\end{aligned}$$

The universality of a AS (in this case, $\widetilde{\mathbb{P}}_\eta^0(\mathcal{A})$) is understood in the sense that the AS coincides for both asymptotic constraints $\mathfrak{Y}$ and $\widehat{\mathfrak{Y}}$ (or for $\mathfrak{X}$ and $\widehat{\mathfrak{X}}$ if relevant).

We view the ASs $(\mathbf{as})[\mathbf{F};\mathbb{R}^2;\tau_{\mathbb{R}}^{(2)};\Pi;\mathfrak{Y}]$ and $(\mathbf{as})[\mathbf{F};\mathbb{R}^2;\tau_{\mathbb{R}}^{(2)};\Pi;\widehat{\mathfrak{Y}}]$ as the asymptotic versions of reachable sets. To derive the representation of these ASs, we introduce the generalized operator $\widetilde{\Pi}$ defined by

$$\mu \longmapsto \Big(\int_I \pi_i\, d\mu\Big)_{i\in J[2]} :\ \mathbb{P}_\eta(\mathcal{A}) \longrightarrow \mathbb{R}^2.$$

We stress that $\Pi = \widetilde{\Pi} \circ \mathfrak{I}$ and $\widetilde{\Pi}$ is a continuous mapping w.r.t. $\big(\mathbb{P}_\eta(\mathcal{A}), \tau^*_\eta(\mathcal{A})\big)$ and $(\mathbb{R}^2, \tau^{(2)}_{\mathbb{R}})$; here, $\tau^{(2)}_{\mathbb{R}}$ is the standard topology of coordinatewise convergence in $\mathbb{R}^2$. Combining Theorem 1 and [Chentsov, 1997, Propositions 3.3.1 and 5.2.1], we arrive at the following theorem:

Theorem 2. *[Chentsov and Baklanov, 2015] The set $\widetilde{\Pi}^1\big(\widetilde{\mathbb{P}}^0_\eta(\mathcal{A})\big)$ represents the universal AS:*

$$\begin{aligned}\widetilde{\Pi}^1\big(\widetilde{\mathbb{P}}^0_\eta(\mathcal{A})\big) &= (\mathbf{as})[\mathbf{F}; \mathbb{R}^2; \tau^{(2)}_{\mathbb{R}}; \Pi; \mathfrak{Y}] = \\ &= (\mathbf{as})[\mathbf{F}; \mathbb{R}^2; \tau^{(2)}_{\mathbb{R}}; \Pi; \widehat{\mathfrak{Y}}] = \\ &= (\mathbf{sas})[\mathbf{F}; \mathbb{R}^2; \tau^{(2)}_{\mathbb{R}}; \Pi; \mathfrak{Y}] = \\ &= (\mathbf{sas})[\mathbf{F}; \mathbb{R}^2; \tau^{(2)}_{\mathbb{R}}; \Pi; \widehat{\mathfrak{Y}}].\end{aligned}$$

4.2 The Limit Operation with Respect to Ultrafilters

In connection with a constructive representation of $\widetilde{\Pi}^1\big(\widetilde{\mathbb{P}}^0_\eta(\mathcal{A})\big)$, we highlight two properties of the limit operation with respect to an ultrafilter (see [Chentsov, 2011b]). If $t \in]a, b]$ and $g \in B(I, \mathcal{A})$, then g has a left-sided limit at t, and

$$\int_I g \, d\kappa(\mathcal{U}^{(-)}_t) = \lim_{\theta \uparrow t} g(\theta).$$

If $t \in [a, b[$ and $h \in B(I, \mathcal{A})$, then h has a right-sided limit at t, and

$$\int_I h \, d\kappa(\mathcal{U}^{(+)}_t) = \lim_{\theta \downarrow t} h(\theta).$$

We use these properties to introduce the following definitions: $\forall t \in]a, b]$

$$\begin{gathered}\Big(\hat{\rho}_\uparrow(t) \triangleq \big(\lim_{\theta\uparrow t} \rho_i(\theta)\big)_{i \in J[N]}\Big) \& \\ \Big(\overrightarrow{\pi}(t) \triangleq \big(\lim_{\theta\uparrow t} \pi_i(\theta)\big)_{i \in J[2]}\Big) \& \\ \Big(\overrightarrow{\phi}(t) \triangleq \big(\lim_{\theta\uparrow t} \phi_i(\theta)\big)_{i \in J[2]}\Big)\end{gathered}$$

and $\forall t \in [a, b[$

$$\begin{gathered}\Big(\hat{\rho}_\downarrow(t) \triangleq \big(\lim_{\theta\downarrow t} \rho_i(\theta)\big)_{i \in J[N]}\Big) \& \\ \Big(\overleftarrow{\pi}(t) \triangleq \big(\lim_{\theta\downarrow t} \pi_i(\theta)\big)_{i \in J[2]}\Big) \& \\ \Big(\overleftarrow{\phi}(t) \triangleq \big(\lim_{\theta\downarrow t} \phi_i(\theta)\big)_{i \in J[2]}\Big).\end{gathered}$$

4.3 The Asymptotic Versions of Reachable Sets

To present the final result for Model 1, we put the following definitions:

$$\begin{aligned}\Gamma \triangleq \{z \in]a, b[\times [0, 1] \mid &\operatorname{pr}_2(z)\hat{\rho}_\uparrow\big(\operatorname{pr}_1(z)\big) + \\ &+ \big(1 - \operatorname{pr}_2(z)\big)\hat{\rho}_\downarrow\big(\operatorname{pr}_1(z)\big) \in \mathbf{Y}\},\end{aligned}$$

$\Omega \triangleq \{\operatorname{pr}_2(z)\overrightarrow{\pi}\big(\operatorname{pr}_1(z)\big) + (1 - \operatorname{pr}_2(z))\overleftarrow{\pi}\big(\operatorname{pr}_1(z)\big) : z \in \Gamma\}$.

Theorem 3. *[Chentsov and Baklanov, 2015] The universal AS $\widetilde{\Pi}^1\big(\widetilde{\mathbb{P}}^0_\eta(\mathcal{A})\big)$ has one of the following forms:*

1) if $\hat{\rho}_\downarrow(a) \notin \mathbf{Y}$ and $\hat{\rho}_\uparrow(b) \notin \mathbf{Y}$, then $\widetilde{\Pi}^1\big(\widetilde{\mathbb{P}}^0_\eta(\mathcal{A})\big) = \Omega$;

2) if $\hat{\rho}_\downarrow(a) \notin \mathbf{Y}$ and $\hat{\rho}_\uparrow(b) \in \mathbf{Y}$, then $\widetilde{\Pi}^1\big(\widetilde{\mathbb{P}}^0_\eta(\mathcal{A})\big) = \Omega \cup \{\overrightarrow{\pi}(b)\}$;

3) if $\hat{\rho}_\downarrow(a) \in \mathbf{Y}$ and $\hat{\rho}_\uparrow(b) \notin \mathbf{Y}$, then $\widetilde{\Pi}^1\big(\widetilde{\mathbb{P}}^0_\eta(\mathcal{A})\big) = \Omega \cup \{\overleftarrow{\pi}(a)\}$;

4) if $\hat{\rho}_\downarrow(a) \in \mathbf{Y}$ and $\hat{\rho}_\uparrow(b) \in \mathbf{Y}$, then $\widetilde{\Pi}^1\big(\widetilde{\mathbb{P}}^0_\eta(\mathcal{A})\big) = \Omega \cup \{\overleftarrow{\pi}(a); \overrightarrow{\pi}(b)\}$.

5 Attraction Sets for Model 2

5.1 Generalized Elements for Model 2

Note that the topology $\tau^*_{\mathbb{T}}(\mathcal{A}) \triangleq \tau_*(\mathcal{A})|_{\mathbb{T}(\mathcal{A})}$ provides the nonempty compactum (see [Chentsov, 1996, § 3.5])

$$(\mathbb{T}(\mathcal{A}), \tau^*_{\mathbb{T}}(\mathcal{A})). \tag{12}$$

Let $\Delta[\mathcal{A}]$ be a mapping

$$x \longmapsto (\delta_x|\mathcal{A}) : I \to \mathbb{T}(\mathcal{A}). \tag{13}$$

From [Chentsov, 1997, (7.6.20)] we get that

$$cl(\Delta[\mathcal{A}]^1(I), \tau^*_{\mathbb{T}}(\mathcal{A})) = \mathbb{T}(\mathcal{A}).$$

Hence, (13) immerses the set of 'controls' I into the compactum (12) as a dense subset.

Suppose that $\widetilde{\Phi} : \mathbb{T}(\mathcal{A}) \to \mathbb{R}^2$ is defined by the rule

$$\mu \longmapsto (\int_I \phi_i \, d\mu)_{i \in J[2]} : \mathbb{T}(\mathcal{A}) \to \mathbb{R}^2.$$

From the definitions of the *-weak topology (see [Chentsov, 1997, §3.4]) and the compactum (12), we have that $\widetilde{\Phi}$ is a continuous mapping w.r.t. $\big(\mathbb{T}(\mathcal{A}), \tau^*_{\mathbb{T}}(\mathcal{A})\big)$ and $(\mathbb{R}^2, \tau^{(2)}_{\mathbb{R}})$.

If $x \in I$, then

$$\widetilde{\Phi}((\delta_x|\mathcal{A})) = (\int_I \phi_i \, d(\delta_x|\mathcal{A}))_{i \in J[2]} = (\phi_i(x))_{i \in J[2]}.$$

Thus, $\widetilde{\Phi} \circ \Delta[\mathcal{A}] = (\phi_i)_{i \in J[2]} = \phi$.

Theorem 4. *[Baklanov and Chentsov, 2010] The universal AS for Model 2 in the class of generalized elements coincides with all admissible generalized elements:*

$$(\mathbf{as})[I;\mathbb{T}(\mathcal{A});\tau_{\mathbb{T}}^{*}(\mathcal{A});\Delta[\mathcal{A}];\mathfrak{X}] = (\mathbf{as})[I;\mathbb{T}(\mathcal{A});\tau_{\mathbb{T}}^{*}(\mathcal{A});\Delta[\mathcal{A}];\widehat{\mathfrak{X}}] = \{\mu \in \mathbb{T}(\mathcal{A}) \mid (\int_I \rho_i d\mu)_{i\in J[N]} \in \mathbf{Y}\}.$$

5.2 The Asymptotic Versions of Reachable Sets

Theorem 5. *The universal AS for Model 2 has the following form:*

$$\widetilde{\Phi}^1\Big((\mathbf{as})[I;\mathbb{T}(\mathcal{A});\tau_{\mathbb{T}}^{*}(\mathcal{A});\Delta[\mathcal{A}];\mathfrak{X}]\Big) = (\mathbf{as})[I;\mathbb{R}^2;\tau_{\mathbb{R}}^{(2)};\phi;\widehat{\mathfrak{X}}] = (\mathbf{as})[I;\mathbb{R}^2;\tau_{\mathbb{R}}^{(2)};\phi;\mathfrak{X}].$$

To present the final result for Model 2, we put the following definitions:

$$\Gamma_{\uparrow} \triangleq \{\tau \in]a,b] \mid \hat{\rho}_{\uparrow}(\tau) \in \mathbf{Y}\},$$
$$\Gamma_{\downarrow} \triangleq \{\tau \in [a,b[\mid \hat{\rho}_{\downarrow}(\tau) \in \mathbf{Y}\},$$
$$\Gamma_0 \triangleq \{\tau \in [a,b] \mid \rho(\tau) \in \mathbf{Y}\},$$
$$\Upsilon = \overrightarrow{\phi}^1(\Gamma_{\uparrow}) \cup \overleftarrow{\phi}^1(\Gamma_{\downarrow}) \cup \phi^1(\Gamma_0).$$

Theorem 6. *The universal AS for Model 2 has the following constructive form:*

$$(\mathbf{as})[I;\mathbb{R}^2;\tau_{\mathbb{R}}^{(2)};\phi;\widehat{\mathfrak{X}}] = (\mathbf{as})[I;\mathbb{R}^2;\tau_{\mathbb{R}}^{(2)};\phi;\mathfrak{X}] = \Upsilon.$$

The proof relies on the characterization of free ultrafilters [Chentsov, 2013c, Proposition 4] and the fact that trivial ultrafilters correspond to Dirac measures restricted to $\mathcal{A}$.

6 Interpretation and examples

Let us elaborate an informal interpretation of Theorem 3 and Theorem 6. If ρ and π are continuous, then both models have the same AS. According to the theorems, two models treat discontinuities differently. Idealized controls 'attached' to a discontinuity of π in Model 1 allocate energy resources in two parts and utilize the resources in two time instants: just before the discontinuity and right after it (i.e., convex combinations of left- and right-sided limits are used). In contrast, idealized controls 'attached' to a discontinuity of π in Model 2 apply all energy resources in one of the three possible time instants: the exact instant of the discontinuity, just before the discontinuity or right after it. Note that idealized *admissible* controls change the velocity of the controlled object instantaneously and comply with the original constraint in terms of ρ and $\mathbf{Y}$ without ϵ-relaxations.

Let us illustrate the developed theoretical framework by presenting some examples. Without loss of generality, we consider both models on the time interval $[0,1]$; thus, $a = 0, b = 1$, and $I = [0,1]$. Assume that $N = 1$ and the function $\rho(t)$ is specified as follows: $\rho(t) = \mathbf{b}(t)\ \forall t \in I$.

To employ Theorem 3 and Theorem 6, we specify the functions $\overrightarrow{\pi}, \overrightarrow{\phi} :]0,1] \to \mathbb{R}^2$ and $\overleftarrow{\pi}, \overleftarrow{\phi} : [0,1[\to \mathbb{R}^2$. It is easy to see that

$$\overrightarrow{\pi}(t) = \Big((1-t)\lim_{\theta\uparrow t}\mathbf{b}(\theta), \lim_{\theta\uparrow t}\mathbf{b}(\theta)\Big)\ \ \forall t \in]0,1];$$
$$\overleftarrow{\pi}(t) = \Big((1-t)\lim_{\theta\downarrow t}\mathbf{b}(\theta), \lim_{\theta\downarrow t}\mathbf{b}(\theta)\Big)\ \ \forall t \in [0,1[;$$
$$\overrightarrow{\phi}(t) = \overrightarrow{\pi}(t)\ \ \forall t \in]0,1];\ \overleftarrow{\phi}(t) = \overleftarrow{\pi}(t)\ \ \forall t \in [0,1[.$$

Combining the last formulas, Theorem 3, and Theorem 6, we see that the construction of ASs for both models is essentially the matter of finding one-sided limits of $\mathbf{b}$. Note that ASs of Model 1 were originally studied in [Chentsov and Baklanov, 2015].

6.1 Example 1

Assume that $\mathbf{b} = \chi_{[0,1[} + 2\chi_{\{1\}}, \mathbf{Y} = \{2\}$. It is easy to see that $(\Gamma = \emptyset) \Rightarrow (\Omega = \emptyset)$. Hence, the AS of Model 1 is empty. In contrast, the AS of Model 2 equals $\{(0,0)\}$ since $\Gamma_{\uparrow} = \Gamma_{\downarrow} = \emptyset$ and $\Gamma_0 = \{1\}$. There is a subtlety here. The AS of Model 2 is nonempty, but contains only the initial point. The unique admissible control is the jump at the endpoint, which doesn't change the coordinates.

6.2 Example 2

Suppose that $\mathbf{b} = \chi_{[0,0.5[} + 2\chi_{\{0.5\}} + 3\chi_{]0.5,1]}$ and $\mathbf{Y} = [1.1, 1.5]$. Clearly, $(\Gamma \neq \emptyset) \Rightarrow (\Omega \neq \emptyset)$. Thus, the AS of Model 1 is not empty; moreover, it is infinite. The AS of Model 2 is empty since $\Gamma_{\uparrow} = \Gamma_{\downarrow} = \Gamma_0 = \emptyset$.

If one modifies $\mathbf{Y}$ such that $\mathbf{Y} = \{2\}$, then the ASs of both models coincide with $\{(1,2)\}$.

6.3 Example 3

Assume that

$$\mathbf{b}(t) = \begin{cases} 4 - 2t & \text{if } t \in [0, 0.5[, \\ 2 & \text{if } t = 0.5, \\ 2 - 2t & \text{if } t \in]0.5, 1], \end{cases}$$

and $\mathbf{Y} = [1,3]$. It is easy to see that the AS of Model 1 is infinite and equal to $\{(0.5x_2, x_2) : x_2 \in [1,3]\}$. The AS of Model 2 is finite and equal to $\{(1.5,3);(1,2);(0.5,1)\}$; $\Gamma_{\uparrow} = \Gamma_{\downarrow} = \Gamma_0 = \{0.5\}$.

7 Conclusion

In this paper we obtained the full constructive characterization of ASs — asymptotic versions of reachable sets — for two formalizations of one-pulse controls given constraints of asymptotic character. The novelty is in the combination of the rather 'rich' measurable space, which admits piecewise continuous functions as the coefficient at the control, and the requirement to fully consume available control resources. The developed extension scheme relies on the results [Chentsov, 2011b] and uses ultrafilters and finitely-additive measures as generalized elements (controls). The results being presented for the double integrator, a basic second-order (linear) control system, admit a straightforward extension to a generic linear system. Though the formalizations are very intuitive and simple, the examples show that the actual realizations of the ASs vary significantly due to the 'richness' of the measurable space. This calls for future research investigating the connection between ASs of the models. Note that the main objects of study can be also understood as expected values of random variables. Thus, the abstract version of the setting of this paper may also be applied in robust statistics [Huber, 1981] and in the theory of statistical solutions [Blackwell and Girshick, 1954].

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