Networked Markets and Relational Contracts

Matt Elliott[1], Benjamin Golub[2] and Matt V. Leduc[3]

[1] Cambridge University, UK and Caltech, USA
matthew.l.elliott@gmail.com
[2] Harvard University, USA
ben.golub@gmail.com
[3] Aix-Marseille Univ., CNRS, EHESS, Centrale Marseille, AMSE, France and IIASA, Austria
mattvleduc@gmail.com

September 2017 (preliminary)

Abstract. Empirical studies of commercial relationships between firms reveal that (i) suppliers encounter situations in which they can gain in the short run by acting opportunistically—for example, delivering a lower quality than promised after being paid; and (ii) good conduct is sustained not exclusively by formal contracts but through informal relationships and the expectation of future business. In such relationships, the need to offer each supplier a large enough share of future business to deter cheating limits the number of supply relationships each buyer can sustain. The market thus becomes networked, with trade restricted to durable relationships. We propose and analyze a simple dynamic model to examine the structure of such overlapping relational contracts in equilibrium. Due to exogenous stochastic shocks, suppliers are not always able to make good on their promises even if they wish to, and so links are constantly dissolving and new ones are forming to take their place. This induces a Markov process on networks. We study how the stationary distribution over networks depends on the parameters—most importantly, the value of trade and the probability of shocks. When the rate at which shocks hit increases, as might happen during an economic downturn, maintaining incentive compatibility with suppliers requires promising each more future business and this necessitates maintaining fewer relationships with suppliers. This results in a destruction of social capital, and even if the rate of shocks later returns to its former level, it can take considerable time for social capital to be rebuilt because of search frictions. This creates a novel way for shocks to be persistent. It also suggests new connections between the theory of relational contracting, on the one hand, and the macroeconomic analysis of recessions, on the other.

1 Introduction

Relationships play a crucial role in the functioning of many markets. In the overnight lending market that banks rely on for liquidity, the same banks trade repeatedly with each other [5]; agricultural traders in Madagascar rely heavily on social relationships to conduct business [9]; medieval Maghribi traders utilized their social network to trade over long distances [11]; manufactures in the Japanese electronic industry rely on durable relationships with their suppliers [17]; and so on.

Building on the pioneering work of [10], an extensive literature in sociology shows that a primary function of these relationships is to help overcome short-term incentives to act opportunistically (see, for example, [22]). Common understandings of acceptable actions seem to limit opportunistic behavior and stabilize trade (e.g., [14]).[4] The relational contracting literature in economics has developed tools for modeling these kinds of informal understandings. However, with some important exceptions that we will discuss below, these models are typically limited to analyzing a single relationship or else the relationships of a single individual (say, buyer), and have stopped short or providing a theory of networked markets. This paper proposes a first model of market relationships made up of overlapping relational contracts.

[4] Measuring trust by the size of trade credit, [14] find evidence for relationships replacing formal contracts in Vietnam.

Viewing business relationships as social capital, and formalizing the role they play in facilitating trade, raises new questions. What is the dynamic impact of shocks on the stock of social capital? Does the mistrust associated with turbulent economic times destroy social capital, and if so, how does the structure of the relational contracts relate to the extent of destruction? The typical approach economists have taken in modeling the formation of networked markets does not permit such questions to be addressed. For example, [13] and [8] model relationships as partner-specific costly investments, abstracting from the roles relationships play in sustaining cooperative behavior. This limits the ability of this theory to consider the dynamic response of the network to shocks and policy interventions. Taking a more dynamic approach, based on relational contracting, complements the parsimonious static models.

We now describe the essentials of our model. Each of many principals (buyers) needs the services of one agent (supplier) every period. A buyer observes who has the capacity to fulfill her order and chooses one agent to trade with. The model features moral hazard: because contracts are incomplete and cannot cover every contingency, agents can act opportunistically ("defect") and seize surplus at the buyer's expense. In an incentive-compatible relational contract, to prevent such behavior, an agent's good conduct must yield an expectation of trading sufficiently frequently in the future. That is, the principal must promise future business after successful trade, and the expected value of that business must exceed the short-term gain of opportunism. The need to promise a large enough share of future surplus limits the number of agents with whom a principal can sustain successful trade. Thus, the market becomes networked, with each supplier having a subset of *insiders* who currently enjoy favored status. The final key ingredient of the model is that exogenous shocks may cause a trade to fail through no fault of the supplier's. When a shock hits, even if the agent acted in good faith and did not defect, the outcome observed by the principal is as if the supplier had defected. Because the principal cannot distinguish such shocks from opportunistic cheating, the shocks must also lead to a disruption in the supplier's favored status; otherwise there would be no deterrent to defection. Thus, insider relationships are sometimes destroyed. We posit a frictional process of principals' matching to new agents to as channel that can replace the lost relationships.

In general, the best relational contract for a principal in our model will be complex, rely on the fine details of the history, and involve randomizations, thus treating identical situations quite differently. Such relational contracts have two disadvantages. First, they stretch to an implausible extent the important practical constraint that contracts must be underpinned by common understandings of expected conduct. Second, they require the principal to be able to commit to intricate credible randomizing devices—something that is difficult to audit and rarely observed in practice. Instead of characterizing optimal contracts with these properties, we define a class of simple contracts that are easy for agents to understand and audit. The conditions we use to do this are intuitive and, when the principal optimizes within this class, lead to natural relational contracts. At the same time, the restrictions placed on the contracting space make the search for an optimal contract tractable, and allow us to study its properties.

The contracts we find to be optimal within our restricted class are characterized by three features. First, principals maintain relationships with a set of insiders, who are treated equally and receive priority as trading partners. Second, when no such insider is available for trade, the principal instead searches for an outsider to trade with as long as the current insider set is not too large. Finally, successful trade by an agent is always rewarded by insider status, while unsuccessful trade is always punished with outsider status. This simple structure is consistent with stylized facts. Given these contracts, new relationships are formed and existing relationships lost in a dynamic process creating a dynamic networked market, which we study. Formally, we analyze a Markov process on a set of networks that the relational contracts induce. We show how this random bipartite graph depends on the parameters of the problem.

The main questions concern how the behavior of the dynamic market depends on exogenous disruption. There are two ways in which such disruptions matter. First, when shocks are realized, there is a destruction of social capital, and restoring it takes time due to search frictions. Beyond the mechanical delay involved in meeting new agents, maintaining incentive-compatibility also implies prioritizing the remaining relationships, which delays the replacement of terminated relationships. This creates a novel source of persistence for shocks. Second, if the probability of shocks increases, the value of future trade decreases, and buyers are able to maintain fewer relationships, because they have less overall future surplus to divide. These forces suggest new connections between the theory of relational contracting on the one hand, and the macroeconomic analysis of recessions, on the other. We view relational contracts disrupted by shocks as a sort of "social" or network capital—an important input into production that is lost suddenly and replaced gradually.

There is a large related empirical literature in economics and sociology documenting the importance of relationships in markets. Beyond the examples already mentioned, [12] document that restaurateurs, particularly those from higher-end restaurants, purchase repeatedly from a small number of vendors. [18] and [19] find that the relationships between small firms and institutional creditors are crucial for gaining access to financing and that firms attempting to access many creditors face higher interest rates and end up having less access to credit. Similarly, [6] document the importance of relationships for accessing interbank lending markets and find that these relationships are especially important for distressed and smaller banks.

We now discuss the most closely related theoretical work. A few papers in the relational contracting literature have considered the relationships between a producer and her suppliers. The most closely related among these are [4] and [3]. [4] finds that an organization may prefer to maintain a limited number of strong relationships with suppliers rather than trading with the lowest-cost supplier in each period. This is the inspiration for our notion of insiders. [3] show that it can be optimal for an organization to favor suppliers that it has more recently traded with over others. In both these cases, the focus is on the interaction of a single principal (e.g., organization) with several agents (e.g., suppliers). However, the inclusion of multiple principals is crucial for developing a relational-contracting theory of networked markets and considering the effects of shocks on the market as a whole.

Our paper also related to the static mechanism design literature on competing mechanisms and to the dynamic games literature. [20] consider repeated relationships and use of transfers in sustaining them. Several recent papers also consider cooperation in networks [1, 2, 23, 16]. For a survey, see [15]. A key difference between these papers and the present one is that the networks we consider are dynamic and endogenously determined by relational contracts rather than exogenous constraints on interactions that are taken as given.

Finally, there is a literature on macroeconomics and social capital (e.g. [21, 7]). This literature does not provide microfoundations for social capital, while we do. Moreover, the microfoundations we provide are crucial for understanding the aggregate dynamic impact of shocks on social capital.

The paper is structured as follows. In Section 2, we introduce the model, focusing on the interaction setting and the timing of decisions. We analyze payoffs and incentive-compatibility in a general contracting space and then define simpler contracts and describe their basic properties. In Section 3, we provide an equilibrium characterization in this simpler contracting space and derive comparative statics and welfare results. The proofs are relegated to the Appendix.

2 Model

There is a continuum of principals and a continuum of agents, also called *suppliers*. We normalize the measure of principals to 1 and let x be the measure of agents. There are infinitely many discrete time periods $t \in \{0, 1, 2, ...\}$. The principal and all agents have a utility equal to the discounted sum (with discount rate $\delta \in (0, 1)$) of period payoffs, which we are about to define.

At each time, each principal j is associated with a set of agents $\mathcal{I}_t^j$ called *insiders*, and if we interpret this set as the neighborhood of j, the neighborhoods define a bipartite network of trading relationships.

2.1 Timing within a period and payoffs

We now present the timing of the model within each period t.

(i) Each principal is matched to one *outsider* uniformly at random.[5]

(ii) Each agent is either *available* to trade or not. The probability that any agent is available is α. The availability realizations are independent across agents and observable by the principal, all insiders, and the outsider.

(iii) Simultaneously, each principal chooses at most one agent to trade with: this may be an available insider, the matched outsider, if available, or nobody. The indicator variable corresponding to principal j selecting agent i at time t is denoted by $Q_{it}^j \in \{0, 1\}$, where $\sum_i Q_{it}^j \le 1$ for each j. If j selected i, we also say that j *initiated* trade with i.

(iv) For each principal-agent pair (j, i) with $Q_i^j = 1$—i.e., where the principal has selected the agent—a *shock* is realized with probability q. In case the shock is realized, the agent cannot produce for the principal. The surplus $\tilde{v}_t$ available in that relationship in that period is thus

$$\tilde{v}_{it}^j = \begin{cases} 0 & \text{w.p. } q \\ v & \text{w.p. } 1 - q, \end{cases}$$

where $v > 0$ is a parameter of the model. The realization of $\tilde{v}_{it}^j$ is observed by the selected agent, i, at time t, but not by the principal j.

(v) The selected agent chooses whether to *defect* in period t on principal j or not. If the agent defects, he receives $\tilde{v}_{it}^j$, leaving the principal with nothing. If the agent does not defect (the agent *cooperates*), he receives

$$\tilde{p}_{it}^j = \begin{cases} 0 & \text{if } \tilde{v}_{it}^j = 0 \\ p & \text{if } \tilde{v}_{it}^j = v, \end{cases}$$

where p is a parameter of the model. We assume $p < v$.

(vi) The principal observes the amount left to him by the agent, $\tilde{v}_{it}^j - \tilde{p}_{it}^j$, and if either the shock has occurred or if the agent has defected (i.e., if $\tilde{v}_{it}^j - \tilde{p}_{it}^j \neq v - p$), the principal incurs a cost $c > 0$. The principal then selects her insider set for next period, $\mathcal{I}_{t+1}^j$. This consists of an arbitrary subset of $\mathcal{I}_t^j$ and can also include the outsider, if the principal traded with the outsider.

At the end of each period, the actions of the principal and success of trade are observed by the principal, the insiders, and the outsider. Interactions within a principal's neighborhood are not observed by anyone outside it, and there is no cheap talk technology.[6]

[5] There is no limit on how many principals an agent may be matched to through this process.

[6] Our continuum assumptions effectively rule out the prospect of interesting "communication" or "reporting" strategies of any kind, even through actions. No player in a given interaction will meet, with positive probability, anyone that his or her current partners will ever meet.

Discussion of Timing and Primitives The model features moral hazard: because contracts are incomplete—not every contingency can be described in the contract—there is an opportunity for the agent to act to benefit himself at the principal's expense. This is reflected in the model by the ability of an agent to 'hold up' the principal, i.e. he can keep the produced good of value $v > p$ for himself and the principal is not able to directly observe whether this is due to cheating or to the production shock hitting with probability q.

For the moment, the notion of insiders simply captures the idea that the principal is not able to contact every potential agent, and can trade at most with those she has met before.[7] In the class of contracts we study, the insiders correspond to the suppliers who have priority in being considered to trade at a given time. The axioms defining the restricted class of contracts we study will give additional operational content to this notion.

The key tool a principal has for incentivizing a selected agent not to defect is the choice of whether to retain that agent as an insider for the next period. This captures the idea that buyers can choose to fire a preferred supplier when trade with that supplier is unsuccessful.

The notion of availability corresponds to the idiosyncratic realization of whether a given supplier has the skills and resources to fulfill the buyer's needs. When there is no available supplier among a principal's insiders the principal can search for an outsider supplier. To capture the idea that this search is not frictionless, we assume that in a given period the principal is only able to sample one outsider. But this outsider also faces the same kind of availability draw that the insiders do: he may not be able to take the buyer's order.

The cost c incurred by the principal when either the shock or defection occurs might reflect lost upfront payments made to the agent that cannot be recovered (another aspect of incomplete contracts), revenues that the principal loses, or costs the principal incurs because her suppliers don't deliver.

2.2 Efficiency benchmark

The constrained-efficiency benchmark we will consider is one in which the agent cannot defect, and thus trade will occur if there is no shock. In this case, it is optimal for a principal never to fire any agents, to add every outsider she meets to her insider set, and thus eventually to have an arbitrarily large insider set. This makes it overwhelmingly likely, in the long run, that at least one agent is available, and the expected surplus per principal achieved per period is $v(1-q) - qc$.

2.3 General contracts and the agent's problem

Next, we formalize the incentives facing the agents and the principal. Here and at many points throughout, we will focus on a particular principal, dropping the superscript j. Recall Q_{it} is an indicator variable specifying whether the principal selects agent i in period t or not. The principal's history at the end of period t is denoted by $h^t = \{h_\tau\}_{\tau \le t}$. Because the principal's actions and the success of trade are publicly observed, $h_\tau = (\mathcal{I}_\tau, A_\tau, i, \tilde{v}_\tau - \tilde{p}_\tau)$, where $\mathcal{I}_\tau$ is the set of insiders of the agent in period τ, the set A_τ contains the agents available to the principal in that period, the index i indicates the identity of the agent selected by the principal (and may be equal to $\emptyset$, corresponding to nobody), and $\tilde{v}_\tau - \tilde{p}_\tau \in \{v-p, 0\}$ reveals the success of the trade.[8]

A *relational contract* Γ is pair (γ, η) along with a strategy for the agent. Here, γ describes the selection strategy: it is a mapping from $h^{t-1} \cup A_t$ into the space of probability distributions

[7] Note the principal may retain every agent ever encountered in her insider set, which simply acts as a "Rolodex" in this case.

[8] Here an outsider who ends up unavailable is considered unobserved; this makes no difference to any of our results.

over available agents, $\Delta(A_t)$. The other component, η, describes the retention strategy, and is a mapping from h^t into (i) distributions over subsets of $\mathcal{I}_t \cup \{O\}$, when trade in period t was with the outsider; or (ii) distributions over subsets of $\mathcal{I}_t$ otherwise. This identifies which agents will be insiders for the next period. This completes the description of the principal's strategy. A strategy for the agent, which is also part of a relational contract, specifies an action for every agent history. We will sometimes omit the description of an agent's behavior when it is clear what it should be.

Suppose an agent i is selected to trade with a particular principal in period t. To make the agent's problem separable across principals, we make our first assumption:

Assumption 1 (Infinite supplier capacity) *Each supplier (agent) can simultaneously supply any number of principals, and payoffs within a period are additively separable across principals.*

This infinite supply assumption simplifies the analysis when agents are selected by multiple principals at once. It means that principals need not care about the employment status of the agents, and whom else they are matched to as outsiders. From the supplier's perspective, the combination of unlimited capacity and additively separable payoffs means that each agent can optimize separately in every relationship with a principal[9].

Now, returning to our selected agent i, let $V_i(h^t)$ be i's continuation value, given a relational contract, after history h^t. If no shock hits the agent (which happens with probability $1-q$), then that agent's value from attempting to trade with the principal is

$$p + \delta V_i(h^t),$$

where $h^t = h^{t-1} \cup \{A_t, i, v-p\}$ and $\delta < 1$ is the discount factor representing time preferences. If agent i instead defects after being selected to trade with the principal, the agent receives

$$v + \delta V_i(h^t),$$

where $h^t = h^{t-1} \cup \{A_t, i, 0\}$.

Likewise, if a shock hits the agent (which occurs with probability q), then the agent receives

$$0 + \delta V_i(h^t),$$

where, again, $h^t = h^{t-1} \cup \{A_t, i, 0\}$.

2.4 Incentive-compatibility

We will now focus on incentive-compatible relational contracts. A contract is said to be incentive-compatible if an agent's strategy has only best responses in its support. To flesh out this condition in our application: Fixing a relational contract, and considering an interaction with a particular principal, the continuation value of agent i at the beginning of period t after history h^{t-1} is

$$V_i\left(h^{t-1}\right) = \mathbb{E}[Q_{it}(1-q)p \mid h^{t-1} \cup A_t] + \delta \mathbb{E}\left[V_i\left(h^t\right)\right], \tag{1}$$

where $h^t = h^{t-1} \cup \{A_t, i, v-p\}$ with probability $(1-q)$ and $h^t = h^{t-1} \cup \{A_t, i, 0\}$ with probability q.

Conditional on trading in period t with the principal, the relational contract with that principal induces a probability distribution over whether agent i will be selected to trade in each subsequent

[9] There are interesting implications of relaxing this assumption and they are explored in a longer version of this paper.

period. Denote the probability that agent selected to trade in period t will also be selected to trade in period s by $\rho(s \mid t, \mathit{\Gamma})$. We then have that for an agent selected to trade in period t,

$$V_i\left(h^{t-1}\right) = (1-q)p + \sum_{s=t+1}^{\infty} \delta^{s-t} \rho(s \mid t, \mathit{\Gamma})(1-q)p. \tag{2}$$

Incentive-compatibility can therefore be represented by

$$(v-p) \leq \sum_{s=t+1}^{\infty} \delta^{s-t} \rho(s \mid t, \mathit{\Gamma})(1-q)p. \tag{3}$$

Without loss of generality, we can set $p = \lambda v$ for some $\lambda \in [0, 1)$ and we get

$$\frac{(1-\lambda)}{(1-q)\lambda} \leq \sum_{s=t+1}^{\infty} \delta^{s-t} \rho(s \mid t, \mathit{\Gamma}). \tag{4}$$

The formulation emphasizes that incentive-compatibility is determined entirely by the frequency with which an agent is selected to trade in the future. Moreover, in order for an agreement to be incentive-compatible for all agents, the principal must commit to trade with all of them sufficiently often in the future. This will place a limit on the number of different agents a principal can credibly trade with.

The principal's expected payoff from an incentive-compatible relational contract is simply

$$\pi(h^{t-1}) = \sum_{s=t}^{\infty} \delta^{s-t} \sum_{i} \mathbb{E}[Q_{is}][(1-q)(v-p) - qc]. \tag{5}$$

We assume that $(1-q)(v-p) > qc$ so that we are in the interesting case in which trade is at least sometimes optimal for the principal. Thus the principal's expected payoff is increasing in the probability the principal selects an available agent who is incentivized to complete the trade. Maintaining more insiders reduces the probability that there will be no agent available for trade in a given period. On the other hand, we have seen that a principal has to commit to sufficiently frequent future trade with a given agent to incentivize the agent not to defect and this places a limit on how many well-incentivized agents the principal can maintain as insiders. This is the key trade-off we study.

2.5 Simple contracts: Axioms

The need to study a simple class of contracts The purpose of this discussion is to explain why the optimal contracts within the general class we have defined so far are likely to be complex. Indeed, the arguably unrealistic features of truly optimal contracts motivate us to study our tradeoffs within a simpler and more plausible class of contracts.

First, suppose that the principal punishes failed trade with probability less than 1. If there is no public randomization device that is available, the principal is unlikely to be able to commit to actions that are unobservable and he would like to renege on. In this case the principal would like to always pretend that the realization of the randomization device does not require punishment. If the principal always punishes failed trade with probability 1, then some incentive-compatibility constraints are likely to be slack and the principal will do better by punishing with probability less than 1.

Second, there are likely to be further intricacies regarding which insider is selected for trade and when. For example, as in Andrews and Barron (2016), the principal may want to prioritize trade with insiders who recently traded (in order to strengthen incentives at the time of trade). This can lead to a complex form for γ, the probability distribution with which a principal selects an agent to trade; indeed, γ can then depend intricately on the history of play, h^t.

These observations are in tension with another feature of implicit contracts. These are often tacit agreements where common understanding is crucial (e.g. [22], [9] or [11]). To avoid misunderstanding (sincere or strategic), intuition—supported by casual empiricism—suggests relational contracts cannot be too complex.

We therefore focus on a considerably simpler, and possibly more realistic, contracting space by requiring that a number of axioms constrain the principal.

Axioms for simple contracts First we require that the contract is simple in terms of the mutual understanding of what is required from the agents—they are expected to never defect. Clearly a contract in which agents frequently defect will be suboptimal because of the cost c that defection imposes on the principal.[10] However, as there are no transfers, defection could be occasionally used as a means of transferring rents to the agents. As such defections are occasional they must depend on the history of play in a non-trivial way, creating opportunities for misunderstandings. This motivates the following Axiom.

Axiom 1 (No Defection on Path) *A contract never permits defection of an agent.*

Thus in any history reached on the equilibrium path of play, an agent's strategy will prescribe defection with probability 0.

Second, we require that insiders are treated equally. What it means to be an insider depends on the difference in the way the relational contract treats them compared with outsiders. To give insider status a simple and easily understood meaning, we require that all insiders must have the status of a principal's most favored supplier. Technically, this requirement ensures the continuation values of all insiders are the same at the start of a period.

Axiom 2 (Equal Treatment of Insiders) *After any history the treatment of all current insiders is the same and cannot depend on the insider's identity. Specifically, (i) under the measure $\gamma(h^{t-1} \cup A_t)$, any $i, j \in A_t \cap \mathcal{I}_t$ are chosen with equal probability; (ii) moreover letting i denote the selected agent, then under the measure $\eta(h^t)$ any other insiders $j, k \in \mathcal{I}_t$ have equal probabilities of being in the retained set $\mathcal{I}_{t+1}$; and (iii) the probability with which i is in $\mathcal{I}_{t+1}$ depends only on the outcome of the trade $\tilde{v}_t - \tilde{p}_t$.*

Specifically, this will imply that each insider will have to be selected with the same probability for a trade and the treatment of a selected insider after trade, in terms of being granted insider status for the next period, will not depend on the identity of the insider selected.

Next, to give insider status further meaning, we will require that insiders are prioritized over an outsider. A principal can never select an outsider if an insider is available. This requirement considerably simplifies the analysis. For a principal with a very small insider set, absent this constraint, it may be optimal to initially favor outsiders in order to build up a larger insider set more quickly, thereby increasing the chance that she will be able to trade with an available agent. While this would tighten the insiders' incentive-compatibility constraints, it may increase the principal's

[10] In practice there are often such costs representing a real loss of social surplus: for example, the failure to generate surplus with one of the principal's customers caused by the principal's lack of a key input.

expected profit. However, eventually incentive compatibility concerns will force the principal to prioritize trade with insiders. In any case, the issue considerably complicates the task of finding an optimal contract. While we expect this requirement to be restrictive in the present model, it can be motivated by robustness considerations: in a model where there is limited monitoring of the principal, she may be tempted to prioritize outsiders more often than she is supposed to. It is easier to verify compliance with simpler contracts.

Axiom 3 (Insiders' Priority) *An available outsider can only be selected if there are no available insiders. Specifically, under the measure* $\gamma(h^{t-1} \cup A_t)$*, the outsider is chosen with probability* 0 *whenever,* $\mathcal{I}_t \cap A_t \neq \emptyset$*.*

In our model, a principal can punish defection by making the selected agent an outsider (i.e., by "firing" the selected agent). Given the equal treatment of insiders Axiom, the principal cannot target punishment at a specific agent while retaining that agent as an insider. Making the agent an outsider implies that with probability 1 the agent will never be selected again by this principal, since there is a continuum of agents. Mixing when punishing would allow the agent's incentive-compatibility constraint to bind, and thus may be preferable for the principal since she would in some cases have more insiders to trade with. On the other hand, a mixed strategy is difficult to interpret in reality. Moreover, without a public randomization device, it may be hard to maintain the credibility of this mixed punishment strategy. The principal would like to convince the agents they will be punished with sufficiently high probability that they never defect. But then, upon observing a failed trade, the principal would infer that the shock must have hit and would want to renege and not punish. To better formalize this, we now state the following definition.

Definition 1 (Principal-Credible Contract). *We say that a deviation is* non-detectable *if it is on the equilibrium path, i.e. if it is consistent with the strategy prescribed by the relational contract. A relational contract is* principal credible *if all non-detectable deviations are weakly unprofitable.*

To prevent the kind of situations described previously, we require that contracts fulfill the following axiom:

Axiom 4 (Credibility) *A contract must be principal credible.*

Thus, whenever a relational contract gives the principal latitude in the actions that he can take, then the action prescribed by this contract must be weakly preferred to all others. For example, if the relational contract requires the principal to mix between actions, then the principal must be indifferent between those actions.

The next remark follows from the above axiom.

Remark 1 (Punishment of Failed Trade) *The principal cannot mix when punishing an unsuccessful trade by setting the status of the agent in question to "outsider" for the next period. That is, she punishes either with probability* 0 *or* 1*.*

Axiom 4 also limits the ability of the principal to randomize in different ways that she also would be tempted to game. While the principal generally likes to have more insiders, as having more of them increases the probability of there being an available insider to trade with each period, insiders will not typically like to share future trades with more other insiders. Thus, in order to maximize the expected number of insiders the principal has, while respecting existing insiders' incentive-compatibility constraints, the principal might want to mix over whether to take on an additional insider or not. However, if this mixing probability cannot be observed, the principal would always like to cheat and take on the extra insider for sure, while the current insiders believe the decision is random.

Moreover, to avoid intricate conditioning on the history of play that could otherwise proxy for mixing and would make the relational contract complex, we have the following axiom.

Axiom 5 *When $A_t \cap \mathcal{I}_t = \emptyset$ and $O \in A_t$, then the selection strategy γ can be conditioned only on $\mathcal{I}_t$.*

An implication of both Axioms 4 and 5 is summarized in the following remark.

Remark 2 (No Mixing over New Insiders) *If after a history h^t there are no available insiders and the outsider is available, the principal must initiate trade with the outsider with probability 1 or 0 and can condition that decision on the history h^t only through the current number of insiders. Further, if trade is successful, the principal must grant insider status to the outsider with probability 1 or 0.*

2.6 Simple contracts: Characterization

Before stating our main characterization of optimal contracts within the restricted space we have defined, we define a k-insider contract.

Definition 2 (k-Insider Contract). *A contract with the principal's strategy given by $\Gamma = (\gamma, \eta)$ is a k-*insider contract *if it has the following structure:*

(i) If $\mathcal{I}_t \cap A_t \neq \emptyset$, then each available insider is selected for trade with probability $1/|\mathcal{I}_t \cap A_t|$.

(ii) If $\mathcal{I}_t \cap A_t = \emptyset$ and the outsider is available, the outsider is selected (with probability 1) if and only if $|\mathcal{I}_t| < k$.

(iii) Successful trade is rewarded by insider status, while unsuccessful trade is punished with outsider status.

We say a relational contract is *constrained-efficient* when it is incentive-compatible and maximizes the expected payoff of a principal among the incentive-compatible contracts that satisfy Axioms 1, 2, 3, 4 and 5 (and thus Remarks 1 and 2).

Proposition 1. *A relational contract is constrained-efficient if and only if it is a k^*-insider contract with k^* given by*

$$k^* = \max\{k : V_i(k) \geq (v - p)/\delta\}$$

where $V_i(k)$ is an agent's expected payoff when there are k insiders and the principal commits to a k-insider contract.

The simple contract axioms do a lot of the work of pinning down the decisions of the principal. Proposition 1 is an exact characterization of what our simplicity conditions imply about principal behavior when combined with optimality within this class of simple contracts.

Even though the contracting space is much simplified, this further characterization requires several lemmas to be proved, which is done in the following section.

2.7 Simple contracts: Proof of characterization

We prove Proposition 1 through a series of lemmas, the proofs of which are relegated to the appendix.

First we show that an agent i is removed from the insider set only when trade is initiated with that agent and fails.

Lemma 1. *In all constrained-efficient relational contracts, if $i \in \mathcal{I}_t$ and trade with i was either not initiated in period t, or else trade with i was initiated and succeeded, then $i \in \mathcal{I}_{t+1}$.*

We next show that, given the insider priority axiom (Axiom 3), the principal will initiate trade with an insider whenever one is available.

Lemma 2. *In all constrained-efficient relational contracts, the principal initiates trade with an insider whenever there is an available insider.*

We now show that there is an upper bound on the number of insiders the principal can maintain.

Lemma 3. *There exists a $\bar{k}$ such that the principal cannot maintain more than $\bar{k}$ insiders while satisfying incentive-compatibility.*

Lemma 3 shows that if the principal always takes on available outsiders, eventually incentive-compatibility will be violated. We now use this lemma to show that the principal takes on available outsiders whenever there is no available insider and trade with the outsider is incentive-compatible. It will turn out to be incentive-compatible precisely when there are fewer than k^* insiders.

Lemma 4. *In all constrained-efficient relational contracts, there exists an integer k such that if there are no available insiders the principal trades with an available outsider if and only if there are fewer than k insiders and rewards successful trade with insider status.*

Together, Lemmas 1 and 4, along with Axiom 4 (Remark 1), establish that part (iii) of Definition 2 must hold in all constrained-efficient relational contracts: Failed trade is always punished with outsider status while successful trade is always rewarded with insider status.

We now turn to part (i) of Definition 2. Lemma 2, Axioms 2, 3 and 4 establish that all constrained-efficient relational contracts satisfy part (i) of Definition 2. When at least one insider is available, each available insider is chosen with equal probability for trade. Note that mixing here is credible since all available insiders are homogenous and thus the principal has no incentive to favor one in particular. Axiom 4 is thus not violated.

Part (ii) of Definition 2 holds in all constrained-efficient relational contracts by Lemma 4. This proves that Definition 2 establishes necessary properties of a constrained-efficient relational contract. Moreover, as under Axiom 1 there is no defection on path, Equation 5 characterizes the principal's expected profits in a constrained-efficient relational contract. This shows that the principal's expected profit is increasing in the frequency with which the principal trades and so is increasing in k, for a k-insider contract. Letting $V_i(k)$ be an agent's expected payoff when there are k insiders and the principal commits to a k-insider contract, the largest k than can be sustained without violating incentive compatibility is

$$k^* = \max\{k : V_i(k) \geq \frac{v-p}{\delta}\}.$$

Thus all constrained-efficient relational contracts are k^*-insider contracts. Moreover, the set of k^*-insider contracts is nonempty and, as this pins down all the principal's actions after every possible history, Definition 2 with $k = k^*$ also provides a sufficient condition for a constrained-efficient relational contract.

□

3 Properties of Optimal Insider Contracts

Now that we have established the optimality of insider contracts within our space of simple contracts, we can calculate a principal's expected profit and an agent's expected utility under a k-insider contract, and study in detail the conditions for incentive-compatibility.

3.1 Principal's expected profit

Even within the simple class of contracts we have described, the principal's expected discounted profits in any state involve an expectation over the stochastic process governing the size of the insider set. In this section, we describe this process and the principal's calculation.

Fix a principal j and a k-insider contract, i.e. an insider contract in which j holds a maximum of k insiders, and in which agents cooperate whenever they are available (which is justified by the incentive-compatibility of such contracts when $k \le k^*$). Then, recalling Equation 5, the principal's expected profit vector can be written as

$$\boldsymbol{\Pi}_j(k) = \mathbb{E}[\boldsymbol{Q}^j(k)]((1-q)(v-p) - qc) + \delta \boldsymbol{M}_P(k)\boldsymbol{\Pi}_j(k) \tag{6}$$

where $\boldsymbol{\Pi}_j(k) = [\Pi_j(0,k), \Pi_j(1,k), \ldots, \Pi_j(k,k)]^\top$ is the vector of expected profits when the number of insiders is $0, 1, \ldots, k$; the matrix $\boldsymbol{M}_P(k)$ describes transitions among states; and $\boldsymbol{Q}^j(k)$ is a vector we are about to define.

We let $Q^j(d,k) = \sum_{i \in \mathcal{I} \cup \{O\}} Q_i^j(d,k)$ denote the indicator variable the principal choses *some* agent to trade with at a given time, given that she currently has d agents in her insider set $\mathcal{I}$ and aims to keep a maximum of k such insiders. Given the form of the contract, $\mathbb{E}[Q^j(d,k)]$ is the probability that an agent is available in state (d,k), and so

$$\mathbb{E}[\boldsymbol{Q}^j(k)] = \begin{bmatrix} \mathbb{E}[Q^j(0,k)] \\ \mathbb{E}[Q^j(1,k)] \\ \vdots \\ \mathbb{E}[Q^j(k-2,k)] \\ \mathbb{E}[Q^j(k-1,k)] \\ \mathbb{E}[Q^j(k,k)] \end{bmatrix} = \begin{bmatrix} 1-(1-\alpha)^1 \\ 1-(1-\alpha)^2 \\ \vdots \\ 1-(1-\alpha)^{k-1} \\ 1-(1-\alpha)^k \\ 1-(1-\alpha)^k \end{bmatrix} \tag{7}$$

$\boldsymbol{M_P}(k)$ is the Markov transition matrix among those different states (d,k):

$$\boldsymbol{M_P}(k) = \begin{bmatrix} a_{0,0} & a_{0,1} & \cdots & a_{0,k} \\ a_{1,0} & a_{1,1} & \cdots & a_{1,k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k,0} & a_{k,1} & \cdots & a_{k,k} \end{bmatrix}, \tag{8}$$

where, for $0 \le d \le k$:

$$\begin{aligned} a_{d,d-1} &= (1-(1-\alpha)^d)q \\ a_{d,d} &= (1-(1-\alpha)^d)(1-q) + (1-\alpha)^d(1-\alpha) + (1-\alpha)^d q\alpha \\ a_{d,d+1} &= (1-\alpha)^d \alpha(1-q). \end{aligned}$$

with the exception that $a_{k,k} = (1-(1-\alpha)^d)(1-q) + (1-\alpha)^d$. All other entries are 0.

Thus we have the following characterization:

Lemma 5. *In any k-insider contract where agents cooperate when they are available,*

$$\boldsymbol{\Pi}_j(k) = (\boldsymbol{I} - \delta \boldsymbol{M}_P(k))^{-1}\mathbb{E}[\boldsymbol{Q}^j(k)]\big((1-q)(v-p) - qc\big). \tag{9}$$

The matrix $(\boldsymbol{I} - \delta \boldsymbol{M}_P(k))$ is a non-singular M-matrix, so its inverse is nonnegative.

We are now able to show that the principal is better off if he currently has more insiders:

Lemma 6. *For any k, it holds that $\Pi_j(0,k) < \Pi_j(1,k) < ... < \Pi_j(k,k)$.*

Proofs not in the main text are in the appendix.

3.2 Agent's expected utility

An agent i's expected utility vector from a relationship with a principal j, when this principal commits to a k-insider contract and the agent cooperates whenever able, is

$$\boldsymbol{U}_i^j(k) = (1-q)p\mathbb{E}[\boldsymbol{Q}_i^j(k)] + \delta \boldsymbol{M}_A(k)\boldsymbol{U}_i^j(k) \tag{10}$$

where $\boldsymbol{U}_i^j(k) = [U_i^j(O,k), U_i^j(I,1,k), \ldots, U_i^j(I,k,k)]^\top$ is the vector of utilities in all possible states the agent can be in and

$$\mathbb{E}[\boldsymbol{Q}_i^j(k)] = \begin{bmatrix} \mathbb{E}[Q_i^j(O,k)] \\ \mathbb{E}[Q_i^j(I,1,k)] \\ \vdots \\ \mathbb{E}[Q_i^j(I,k,k)] \end{bmatrix} \tag{11}$$

is the probability that agent i will be chosen to trade with principal j in each possible state. Note that $\mathbb{E}[Q_i^j(I,d,k)] = \alpha\mathbb{E}[\frac{1}{1+n_a^d}]$ where $n_\alpha^d \sim bin(\alpha, d-1)$. On the other hand, $\mathbb{E}[Q_i^j(O,k)] = 0$ is the probability the agent will be chosen again by principal j after being fired.

The Markov transition matrix $\boldsymbol{M}_A(k)$ is:

$$\boldsymbol{M}_{\boldsymbol{A}}(k) = \begin{bmatrix} a_{O,O} & a_{O,1} & \cdots & a_{O,k} \\ a_{1,O} & a_{1,1} & \cdots & a_{1,k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k,O} & a_{k,1} & \cdots & a_{k,k} \end{bmatrix} \tag{12}$$

where the entries are, for $1 \leq d \leq k$:

$$\begin{aligned} a_{d,d-1} &= \frac{(d-1)(1-(1-\alpha)^d)}{d}q \\ a_{d,d} &= (1-(1-\alpha)^d)(1-q) + (1-\alpha)^d q\alpha + (1-\alpha)^d(1-\alpha) \\ a_{d,d+1} &= (1-\alpha)^d\alpha(1-q), \end{aligned}$$

with the exception that $a_{k,k} = (1-(1-\alpha)^k)(1-q) + (1-\alpha)^k$. Also,

$$\begin{aligned} a_{d,O} &= \frac{(1-(1-\alpha)^d)}{d}q \\ a_{O,d} &= 0 \\ a_{O,O} &= 1 \end{aligned}$$

All other entries are 0.

Thus we have

Lemma 7. *In any k-insider contract in which i cooperates when available, the agent's expected utility is*

$$\boldsymbol{U}_i^j = (\boldsymbol{I} - \delta \boldsymbol{M}_A(k))^{-1}\mathbb{E}[\boldsymbol{Q}_i^j(k)](1-q)p, \tag{13}$$

where $(\boldsymbol{I} - \delta \boldsymbol{M}_A(k))$ *is a non-singular M-matrix.*

Now we can establish:

Lemma 8. *For any k, it holds that* $U_i^j(I,1,k) > U_i^j(I,2,k) > \cdots > U_i^j(I,k,k)$.

3.3 Incentive-compatibility

Lemma 9. *Given a maximum number of insiders k, a contract is incentive-compatible if, for any* $d \leq k$,

$$p + \delta U_i^j(I,d,k) \geq v + \delta U_i^j(O,k) \tag{14}$$

It follows from Lemma 8 that if a contract is incentive-compatible with k insiders, it is also incentive-compatible with k' insiders, for any $k' < k$. By Lemma 6, a principal's expected profit is increasing in the number of insiders she keeps, k. In a principal-optimal contract, she will choose the maximum number of insiders k such that the incentive-compatibility condition holds for that k, or equivalently

$$U_i^j(I,k,k) - U_i^j(O,k) \geq \frac{v-p}{\delta} \tag{15}$$

Note also that from Eq. (10), it follows that $U_i^j(O,k) = \delta U_i^j(O,k)$ and thus that $U_i^j(O,k) = 0$. Thus this condition reduces nicely to $U_i^j(I,k,k) \geq \frac{v-p}{\delta}$.

3.4 Equilibrium, comparative statics and welfare

To analyze the behavior of the entire market, we define the insider contract game by allowing all principals to simultaneously choose, and commit to, any insider relational contracts. We look for a Nash equilibrium of this game.

Proposition 2 (Existence and Uniqueness). *There exists a unique equilibrium of the insider contract game in which all principals choose an insider contract with* k^* *insiders. To this* k^**, there corresponds a random bipartite graph with stationary principal degree distribution* $f_{k^*}(d)$*, which is the stationary distribution of the Markov transition matrix* $\boldsymbol{M}_P(k^*)$.

The steady-state distribution of the number of agents in a principal's insider set is thus given by $f_{k^*}(d)$. Therefore, in steady state, the equilibrium strategy k^* yields a random bipartite graph linking the set of principals to the set of agents and $f_{k^*}(d)$ is the distribution of the number of edges incident on a principal. Fig. 1 illustrates an equilibrium network in steady state.

3.5 Comparative statics and welfare

Comparative statics in k We first verify the intuitive statement that when each principal seeks a larger number of insiders, in equilibrium each principal has a higher degree.

Proposition 3. *Let* k *and* k' *be two symmetric strategies played by all principals, and suppose all agents cooperate when available. If* $k' > k$*, then* $f_{k'}(d) \succeq f_k(d)$*, where* $\succeq$ *indicates first-order stochastic dominance.*

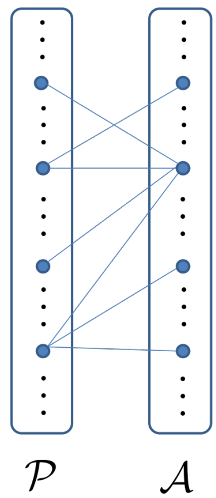

Fig. 1: A realization of the equilibrium random bipartite graph linking the set of principals $\mathcal{P}$ to the set of agents $\mathcal{A}$ in steady state. The number of edges d incident on a principal follows distribution $f_{k^*}(d)$. In this example, $k^* = 3$ and thus a principal cannot have degree higher than 3. An agent, on the other hand, can have any degree. The first-best configuration–when a principal does not have to maintain incentive-compatibility–would be the complete bipartite graph.

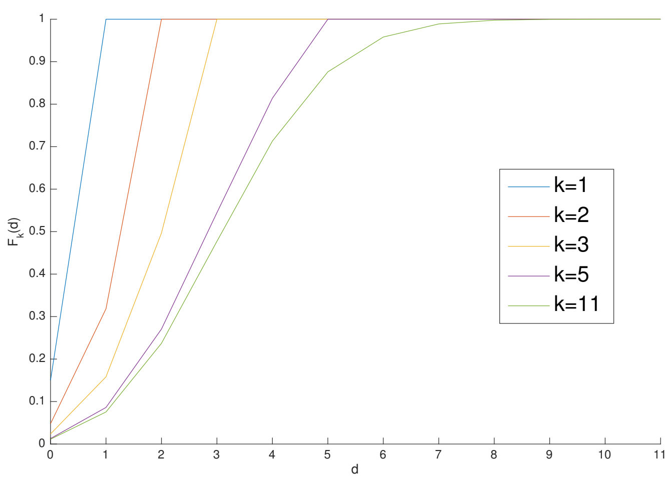

Fig. 2: Cumulative distribution functions (CDFs) of a principal's degree when the targeted number of insiders is k (i.e., k-insider contract). Model parameters are $q = 0.15$ and $\alpha = 0.2$.

Fig. 2 illustrates Proposition 3. We plot the CDF's of the degree for each k. We see that for $k' > k$, $F_{k'}(d) \leq F_k(d)$ for any d, indicating that $f_{k'} \succeq f_k$.

Given a symmetric strategy k followed by principals, and agents always cooperating, the welfare (social surplus) per period is:

$$W(k) = \big((1-q)v - qc\big)\Big(\sum_{d=0}^{k-1} f_k(d)(1-(1-\alpha)^{d+1}) + f_k(k)(1-(1-\alpha)^k)\Big). \tag{16}$$

The upper bound on welfare (first-best) is $\bar{W} = \lim_{k\to\infty} W(k) = (1-q)v - qc$, which is reached when $\alpha = 1$. This corresponds to our efficiency benchmark.

We now show that social surplus is greater when principals maintain a larger number of insiders who cooperate:

Proposition 4. *$W(k)$ is weakly increasing in k.*

Comparative statics in q Suppose that market conditions deteriorate, making it more difficult for suppliers to live up to their commitments. As a first pass, we can model this as a reduction in q. How does k^*, the principal-optimal and efficient number of insiders, depend on q?

Proposition 5. *Let $k^*(q)$ be the principal-optimal number of insiders under q. For all $\hat{q}$ such that $k^*(\hat{q}) > 0$, there exits $\underline{q} > \hat{q}$ and $\epsilon > 0$ such that $k^*(q) \leq k^*(\hat{q}) - 1$ for all $q > \underline{q}$ and $k^*(q) = k^*(\hat{q}) - 1$ for all $q \in (\underline{q}, \underline{q} + \epsilon)$*

Proposition 5 states that when q increases sufficiently, the corresponding k^* is bounded by a strictly decreasing upper bound. Moreover, this upper bound decreases by steps of size one.

Corollary 1. *Let $k'^* < k^*$ be the principal-optimal and efficient number of insiders corresponding to $q' > \underline{q}$ in Proposition 5. Then $W(k'^*) \leq W(k^*)$.*

Fig. 3 illustrates Proposition 5 and Corollary 1. We see that an increase in q leads to a rapid decrease in the number of insiders k^* a principal can keep while maintaining incentive-compatibility. These lost relationships can be viewed as a destruction of social capital and are associated with large welfare losses. Welfare is impacted by two effects: (i) the direct and continuous effect of an increase in q on $(1-q)v - qc$, seen in the slope of the $W(k^*)$ curve; (ii) the indirect effect of a decreasing $k^*(q)$ as q increases, which is seen in the downward jumps of $W(k^*)$. As a comparison, the first best—which assumes a complete bipartite graph and also that the principal does not have to maintain incentive-compatibility—only suffers a continuous decrease due to an increasing q.

Because of search frictions, the model involves an asymmetry in how it responds to unexpected positive and negative changes in q. For example, for the parameters shown in Fig. 3, if the economy enters a crisis and q suddenly rises from 0.09 to 0.12, then the principal would have to immediately adjust the relational contracts[11] in order to maintain incentive-compatibility. This would imply k^* suddenly dropping from 9 to 2 (as per Fig. 3). This implies an immediate destruction of social capital, quantified here by a sharp drop in welfare $W(k^*)$ of more than 50%. However, were the economy to unexpectedly enter a post-crisis state in which q returns to its initial value of 0.09, rebuilding the lost social capital takes a considerable amount of time. Indeed, the principal would

[11] We assume that the principal would be allowed to renegotiate contracts following the transition to an unanticipated crisis and the associated change in q.

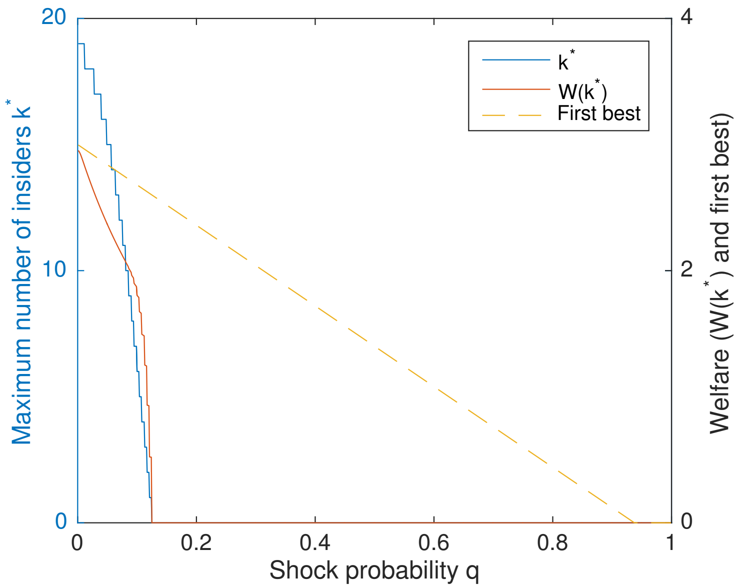

Fig. 3: Principal-optimal and efficient number of insiders k^* in equilibrium, corresponding welfare $W(k^*)$ and first best welfare (i.e. $(1-q)v - qc$) versus the shock probability q. Other model parameters are $\delta = 0.99$, $c = 0.2$, $v = 3$, $p = 0.5$, $x = 1$ and $\alpha = 0.2$.

now rewrite optimal contracts in which she aims to return to a maximum of $k^* = 9$ insiders, but search frictions would delay the rebuilding of her set of insiders. In Table 1, we show the expected first-passage times (in periods) from an insider set size of 0 to a higher insider set size (columns). We see that building up the insider set from 2 insiders (the median number of insiders in the steady state when $k^* = 2$) to 5 insiders (the median number of insiders in the steady state when $k^* = 9$) takes more 50 periods[12]. Search frictions limit the speed of the transition and are exacerbated by the relational contracts, which prioritize insiders thereby relaxing their incentive compatibility constraints.

$|\mathcal{I}_t|$

0	1	2	3	4	5	6	7	8	9
0	5.49	13.04	23.73	39.49	64.15	105.93	185.04	358.66	817.15

Table 1: Expected first-passage times (in periods) from a starting insider set size $|\mathcal{I}_t|$ of 0 to various other values of $|\mathcal{I}_t|$ (columns).

[12] As the process is a Markov chain, to get from 0 insiders to 5 insiders, we must first pass by 2 insiders and thus from Table 1, the expected first-passage time from 2 insiders to 5 insiders is $64.15 - 13.04 \approx 51$.

Bibliography

[1] Ali, S. N. and Miller, D. A. 2013. Enforcing cooperation in networked societies. *Unpublished paper.[282].*

[2] Ali, S. N. and Miller, D. A. 2016. Ostracism and forgiveness. *The American Economic Review 106,* 8, 2329–2348.

[3] Andrews, I. and Barron, D. 2016. The allocation of future business: Dynamic relational contracts with multiple agents. *The American Economic Review 106,* 9, 2742–2759.

[4] Board, S. 2011. Relational contracts and the value of loyalty. *The American Economic Review 101,* 7, 3349–3367.

[5] Bräuning, F. and Fecht, F. 2012. Relationship lending and peer monitoring: Evidence from interbank payment data.

[6] Cocco, J. F., Gomes, F. J., and Martins, N. C. 2009. Lending relationships in the interbank market. *Journal of Financial Intermediation 18,* 1, 24–48.

[7] Dasgupta, P. and Serageldin, I. 2001. *Social capital: a multifaceted perspective.* World Bank Publications.

[8] Elliott, M. 2015. Inefficiencies in networked markets. *American Economic Journal: Microeconomics 7,* 4, 43–82.

[9] Fafchamps, M. and Minten, B. 1999. Relationships and traders in madagascar. *The Journal of Development Studies 35,* 6, 1–35.

[10] Granovetter, M. 1985. Economic action and social structure: The problem of embeddedness. *American journal of sociology*, 481–510.

[11] Greif, A. 1993. Contract enforceability and economic institutions in early trade: The maghribi traders' coalition. *The American economic review*, 525–548.

[12] Kirman, A. P. and Vriend, N. J. 2001. Evolving market structure: An ace model of price dispersion and loyalty. *Journal of Economic Dynamics and Control 25,* 3, 459–502.

[13] Kranton, R. and Minehart, D. F. 2001. A theory of buyer-seller networks. *American Economic Review 91,* 3, 485–508.

[14] McMillan, J. and Woodruff, C. 1999. Interfirm relationships and informal credit in vietnam. *The Quarterly Journal of Economics 114,* 4, 1285–1320.

[15] Nava, F. 2015. Efficiency in decentralized oligopolistic markets. *Journal of Economic Theory 157*, 315–348.

[16] Nava, F. and Piccione, M. 2014. Efficiency in repeated games with local interaction and uncertain local monitoring. *Theoretical Economics 9,* 1, 279–312.

[17] Nishiguchi, T. 1994. *Strategic industrial sourcing: The Japanese advantage.* Oxford University Press on Demand.

[18] Petersen, M. A. and Rajan, R. G. 1994. The benefits of lending relationships: Evidence from small business data. *The journal of finance 49,* 1, 3–37.

[19] Petersen, M. A. and Rajan, R. G. 1995. The effect of credit market competition on lending relationships. *The Quarterly Journal of Economics*, 407–443.

[20] Samuelson, L. and Stacchetti, E. 2016. Even up: Maintaining relationships. Tech. rep., working paper, Yale University.

[21] Sobel, J. 2002. Can we trust social capital? *Journal of economic literature 40,* 1, 139–154.

[22] Uzzi, B. 1997. Social structure and competition in interfirm networks: The paradox of embeddedness. *Administrative science quarterly*, 35–67.

[23] Wolitzky, A. 2013. Cooperation with network monitoring. *The Review of Economic Studies 80,* 1, 395–427.

4 Appendix

Proof (of Remark 1). Suppose the contract Γ prescribes that the principal punishes a failed trade with probability $\mu \in (0, 1)$. Then punishing with a probability smaller than μ is a strictly profitable deviation for the principal since she can keep more insiders than she would under μ and thus may increase the frequency of future trades. Moreover, such a deviation is non detectable, as defined in Definition 1. Indeed, agents cannot detect it by observing any history h^t. The contract Γ is thus not principal credible, and this violates Axiom 4. On the other hand, if a contract prescribes a pure punishment strategy $\mu = 0$ or $\mu = 1$, then any deviation is detectable and thus there are no strictly profitable non-detectable deviations. Such a contract is thus principal credible. □

Proof (of Remark 2). Suppose the contract Γ prescribes that, when there are no available insiders, the principal hires an outsider with probability $\mu \in (0, 1)$. Then hiring the outsider with a probability larger than μ is a strictly profitable deviation for the principal since she can keep more insiders than she would under μ and thus may increase the frequency of future trades. As in the previous proof, such a deviation is non-detectable. This contract is thus not principal credible, and this violates Axiom 4. On the other hand, if a contract prescribes a pure hiring strategy $\mu = 0$ or $\mu = 1$, then any deviation is detectable and thus there are no strictly profitable non-detectable deviations. Such a contract is thus principal credible.

That the decision to hire an outsider can only be conditioned on the current number of insiders follows directly from Axiom 5.

That following a successful trade, insider status must be granted to the outsider with probability 0 or 1 follows from a similar credibility argument: if the contract specified that the principal granted insider status with probability $\mu \in (0, 1)$, then the principal would always prefer deviating and granting it with a higher probability since she would then be able to trade with more insiders in future periods. Since this strictly-profitable deviation is non-detectable, then the contract is not principal credible, whereas a contract prescribing a pure strategy $\mu = 0$ or $\mu = 1$ would be credible since any deviation would be immediately detectable. □

Proof (of Lemma 1). Suppose, towards a contradiction, there is a constrained-efficient relational contract Γ in which after some history h^{t-1}, an insider is made an outsider in period t with positive probability $\mu > 0$, and in which i traded successfully. Equal treatment of insiders, Axiom 2, then implies that at the start of period $t - 1$ all insiders are equally likely to be punished following a successful trade under Γ. Consider an alternative contract Γ' that is identical to Γ, except that $\mu = 0$ and following history h^{t-1} the contract differs from Γ in the following way. Immediately after h^{t-1}, there is an additional insider under Γ' and we allow the principal to trade with an insider when one is available under Γ' but no agent would be available under Γ. In addition, trade with an outsider that would be incentive-compatible under Γ may not be incentive-compatible under Γ', in which case it is not attempted under Γ'. This continues until either: (i) a trade that would not have occurred under Γ fails, or (ii) a trade that would have been successful with an outsider under Γ and resulted in that outsider becoming an insider is not undertaken. Either way, by Remark 1, after such an event there is then the same number of insiders under Γ' as there would be under Γ, and henceforth Γ' follows Γ exactly.

The principal prefers Γ' to Γ. By construction the principal has the same sequence of trades except there are now, in expectation, additional trades. The expected value of these additional trades is positive $(1 - q)(v - p) - qc > 0$, and so the principal is better off.

The contract Γ' is incentive-compatible. It increases trade among insiders and, by Axiom 2, slackens insiders' incentive-compatibility constraints. The initial contract Γ can not, therefore, have been constrained-efficient.

□

Proof (of Lemma 2). Without loss of generality we suppose that before the start of the game nature draws an infinite sequence of binary random variables, where each random variable is drawn independently from each other and is 1 with probability $1-q$ and 0 with probability q. We then let the success of initiated trade conditional with no defection follow this sequence, where 1 represents a success and 0 represents a failure.

Suppose, towards a contradiction, there exists a constrained-efficient relational contract Γ in which, following some history h^t, the principal does not trade with an insider in period $t+1$ despite there being an available insider. Axiom 3 then implies that the principal does not trade in period $t+1$. Further, by Axiom 1 all trade under this contract is incentive-compatible. Consider the relational contract Γ' that is almost identical to Γ. We'll assume and then verify that Γ' is also incentive-compatible. Let Γ' differ from Γ in (only) the following ways. After history h^t, in period $t+1$ the principal trades with an insider. If trade is successful, the principal follows Γ henceforth. The expected value to the principal from period $t+2$ onwards is therefore the same under Γ and Γ', and as Γ was incentive-compatible from period $t+2$ onwards, so is Γ'. Also, the additional trade in period $t+1$ yields profits for the principal of $v-p$ and additional rents for agent i of p.

Suppose instead that the extra trade in period $t+1$ under Γ is unsuccessful. Following a given path of availabilities, suppose the next time the principal would initiate trade under Γ is in period $s > t+1$. The only other difference between Γ and Γ' is that in period s trade is not initiated under Γ'. Suppose the agent to fail under Γ in period s is agent j. Then under Γ' we treat agent j exactly as agent i would have been treated under Γ. With this adjustment, Γ' can again exactly mimic Γ in terms of how each insider is treated from period $s+1$ onwards. Thus an unsuccessful trade, incurring cost c, has been moved forward from period s to period $t+1 < s$.

The net effect on the principal is a gain of $(1-q)(v-p)$ at a cost of less than qc. Thus the principal strictly prefers Γ' to Γ. Moreover, the contract Γ' is still incentive-compatible. Suppose that under Γ' the trade forgone that would have occurred in period s under Γ is with an outsider. Then the insiders expected continuation values at all histories weakly increase and incentive-compatibility constraints become slacker. If instead this period s trade would have been with an insider, Axiom 2 implies that all insiders at the beginning of period $t+1$ are equally likely to trade in period $t+1$ under Γ', and equally likely to have lost the trade that would otherwise have occurred in period s. Thus relative to Γ a trade is moved forward, and because $\delta < 1$, this increases the continuation value of all the insiders.

□

Proof (of Lemma 3). By Axiom 2 (equal treatment of insiders), Remark 1 (punishment of failed trade), and Lemma 1, an insider's expected payoff is highest when when the principal never again trades with an outsider. In this case, the expected payoff of all remaining insiders in a given period, when there are $k > 1$ insiders, is:

$$V_{\text{tot}}(k) \equiv (1-(1-\alpha)^k)[(1-q)(p+\delta V_{\text{tot}}(k)) + q\delta V_{\text{tot}}(k-1)] + (1-\alpha)^k \delta V_{\text{tot}}(k).$$

When there is 1 insider

$$V_{\text{tot}}(1) = \alpha[(1-q)(p+\delta V_{\text{tot}}(1))]$$

and thus

$$V_{\text{tot}}(1) = \frac{\alpha(1-q)p}{1-\alpha(1-q)\delta}.$$

We can therefore solve this system to find an expression for $V_{\text{tot}}(k)$.

An upper bound on the continuation value of any agent will then, by the equal treatment Axiom, be $V_{\text{tot}}(k)/k$, which is decreasing in k and converges to 0 as $k \to \infty$. Denoting by $V_i(k)$ any agent's expected payoff when there are k insiders, any agent's incentive-compatibility constraint is

$$p + \delta V_i(k) \geq v$$

or

$$V_i(k) \geq \frac{v-p}{\delta}.$$

There thus exists a $\bar{k}$ such that the IC constraint of the agent is not satisfied, i.e.

$$V_i(k) \leq \frac{V_{\text{tot}}(k)}{k} \leq \frac{v-p}{\delta}.$$

This completes the proof.

□

Proof (of Lemma 4). By Remark 2 the principal cannot mix over trading with the outsider and must grant insider status following successful trade (otherwise, by Remark 2, the outsider must not be granted insider status for sure and would choose to defect violating Axiom 1). Moreover, Remark 2 implies that d insiders can be obtained only if the Principal always trades with an available outsider when there are $d-1$ insiders, all of whom are unavailable and the outsider is available. Consider two such contracts and suppose both satisfy Axiom 1 and Remark 2. Let Γ_d be such a contract in which the principal keeps at most d insiders. Let Γ_{d+1} be a similar contract, but in which the principal keeps at most $d+1$ insiders. Under both Γ_d and Γ_{d+1}, the incentive-compatibility constraint of trading agents are always satisfied by Axiom 1.

Under these two contracts, the principal trades more often under Γ_{d+1}: On all paths where there are strictly less than d insiders, the two contracts are identical. On paths where there are d insiders and none is available and the outsider is not available, the two contracts are identical. On paths where there are d unavailable insiders and the outsider is available, the principal trades and earns an expected period payoff of $(1-q)(v-p) - qc > 0$ under Γ_{d+1}, whereas she does not trade under Γ_d. On paths where there are $d+1$ insider and only one is available, the principal trades and earns a positive expected payoff under Γ_{d+1}, whereas she may or may not trade under Γ_d. Therefore Γ_{d+1} does strictly better than Γ_d on some paths and the same on other paths. It is thus strictly preferred by the principal. This is valid for all d and $d+1$ such that the trading agents' IC constrain are always satisfied. By Lemma 3 there is a maximum such d.

□

Proof (of Lemma 5).

Follows directly from Eq. (6).

□

Proof (of Lemma 6).

For any k, we may write

$$\Pi_j(d,k) = \Big(\mathbb{E}[Q^j(d,k)] + \sum_{s=t+1}^{\infty} \delta^{s-t} \boldsymbol{M}_{P,d}^{s-t}(k)\mathbb{E}[\boldsymbol{Q}^j(k)]\Big)\big((1-q)(v-p) - qc\big) \tag{17}$$

where $\boldsymbol{M}^{s-t}_{P,d}(k)$ is the d^{th} row in the $(s-t)$-period transition matrix $\boldsymbol{M}^{s-t}_{P}(k)$. Note that the dot product $\boldsymbol{M}^{s-t}_{P,d}(k)\mathbb{E}[\boldsymbol{Q}^j(k)]$ is the probability that the principal j will be able to select an agent to trade with at time $s > t$, given that she has d insiders at time t and aims to keep a maximum of k.

Let $\mathbb{P}(\tilde{d} \mid d, k)$ be the one-period transition probability to $\tilde{d}$ insiders, given that the principal currently has d insiders. $\mathbb{P}(\tilde{d} \mid d, k)$ corresponds to the d^{th} row $\boldsymbol{M}_{P,d}(k)$ of the transition matrix $\boldsymbol{M}_P(k)$. It is easy to show by inspection that $\mathbb{P}(\tilde{d} \mid d+1, k) \succeq \mathbb{P}(\tilde{d} \mid d, k)$, where $\succeq$ means first-order stochastic dominance. Equivalently, we may write $\boldsymbol{M}_{P,d+1}(k) \succeq \boldsymbol{M}_{P,d}(k)$. Since $\mathbb{E}[Q^j(d,k)]$ is an increasing function of d (see Eq. (7)), it follows that $\boldsymbol{M}_{P,d+1}(k)\mathbb{E}[\boldsymbol{Q}^j(k)] > \boldsymbol{M}_{P,d}(k)\mathbb{E}[\boldsymbol{Q}^j(k)]$.

Now suppose that for any d and for some $l > 1$, $\boldsymbol{M}^{l}_{P,d+1}(k)\mathbb{E}[\boldsymbol{Q}^j(k)] > \boldsymbol{M}^{l}_{P,d}(k)\mathbb{E}[\boldsymbol{Q}^j(k)]$. Then it follows that

$$\boldsymbol{M}^{l+1}_{P,d+1}(k)\mathbb{E}[\boldsymbol{Q}^j(k)] = \boldsymbol{M}_{P,d+1}(k)\big(\boldsymbol{M}^{l}_{P}(k)\mathbb{E}[\boldsymbol{Q}^j(k)]\big) \tag{18}$$

$$> \boldsymbol{M}_{P,d}(k)\big(\boldsymbol{M}^{l}_{P}(k)\mathbb{E}[\boldsymbol{Q}^j(k)]\big) \tag{19}$$

$$= \boldsymbol{M}^{l+1}_{P,d}(k)\mathbb{E}[\boldsymbol{Q}^j(k)] \tag{20}$$

Thus, it follows by induction that for any $s - t \geq 1$, $\boldsymbol{M}^{s-t}_{P,d+1}(k)\mathbb{E}[\boldsymbol{Q}^j(k)] > \boldsymbol{M}^{s-t}_{P,d}(k)\mathbb{E}[\boldsymbol{Q}^j(k)]$ and thus from Eq. (17), we have that $\Pi_j(d,k) < \Pi_j(d+1,k)$.

□

Proof (of Lemma 7).

Follows directly from Eq. (10).

□

Proof (of Lemma 8).

For any k, we may write

$$U(I,d,k) = \mathbb{E}[Q^j_i(I,d,k)](1-q)p + \sum_{s=t+1} \delta^{s-t}\boldsymbol{M}^{s-t}_{A,d}(k)\mathbb{E}[\boldsymbol{Q}^j_i(k)](1-q)p \tag{21}$$

where $\boldsymbol{M}^{s-t}_{A,d}(k)$ is the d^{th} row in the $(s-t)$-period transition matrix $\boldsymbol{M}^{s-t}_{A}(k)$. Note that the dot product $\boldsymbol{M}^{s-t}_{A,d}(k)\mathbb{E}[\boldsymbol{Q}^j_i(k)]$ is the probability that agent i will be selected for trading by principal j at time s, given that she is in state (I,d,k) at time t.

Let $\mathbb{P}((I,\tilde{d}) \mid d, k)$ be the one-period transition probability to state $(I,\tilde{d})$, given that the principal currently has d insiders (i.e. $\mathbb{P}((I,\tilde{d}) \mid d, k)$ corresponds to the d^{th} row $\boldsymbol{M}_{A,d}(k)$ of the transition matrix $\boldsymbol{M}_A(k)$). It is easy to show by inspection that $\mathbb{P}((I,\tilde{d}) \mid d+1, k) \succeq \mathbb{P}((I,\tilde{d}) \mid d, k)$, where $\succeq$ indicates first-order stochastic dominance. Equivalently, we may write $\boldsymbol{M}_{A,d+1}(k) \succeq \boldsymbol{M}_{A,d}(k)$. Since $\mathbb{E}[Q^j_i(I,d,k)]$ is a decreasing function of d (from Eq. (11)), it follows that $\boldsymbol{M}_{A,d+1}(k)\mathbb{E}[\boldsymbol{Q}^j_i(k)] < \boldsymbol{M}_{A,d}(k)\mathbb{E}[\boldsymbol{Q}^j_i(k)]$,

Now suppose that for any d and for some $l > 1$, $\boldsymbol{M}^{l}_{A,d+1}(k)\mathbb{E}[\boldsymbol{Q}^j_i(k)] < \boldsymbol{M}^{l}_{A,d}(k)\mathbb{E}[\boldsymbol{Q}^j_i(k)]$. Then it follows that

$$\boldsymbol{M}^{l+1}_{A,d+1}(k)\mathbb{E}[\boldsymbol{Q}^j_i(k)] = \boldsymbol{M}_{A,d+1}(k)\big(\boldsymbol{M}^{l}_{A}(k)\mathbb{E}[\boldsymbol{Q}^j_i(k)]\big) \tag{22}$$

$$< \boldsymbol{M}_{A,d}(k)\big(\boldsymbol{M}^{l}_{A}(k)\mathbb{E}[\boldsymbol{Q}^j_i(k)]\big) \tag{23}$$

$$= \boldsymbol{M}^{l+1}_{A,d}(k)\mathbb{E}[\boldsymbol{Q}^j_i(k)] \tag{24}$$

It thus follows by induction that for any $s-t>1$, $\boldsymbol{M}^{s-t}_{A,d+1}(k)\mathbb{E}[\boldsymbol{Q}^j_i(k)] < \boldsymbol{M}^{s-t}_{A,d}(k)\mathbb{E}[\boldsymbol{Q}^j_i(k)]$ and thus from Eq. (21), we have that $U^j_i(I,d,k) > U^j_i(I,d+1,k)$.

□

Proof (of Lemma 9). Whenever the agent is selected, his payoff from not deviating in state (I,d,k) is $p+\delta U^j_i(I,d,k)$, i.e. the price he obtains from the transaction plus his continuation value after the trade. His payoff from deviating and then reverting back to on-path play is $v+\delta U^j_i(O,k)$, i.e. the value of the good plus his continuation value when fired by the principal. Thus i will not have a profitable one-period deviation if and only if $p+\delta U^j_i(I,d,k) \geq v+\delta U^j_i(O,k)$. Invoking the one-shot deviation principle completes the proof.

□

Proof (of Proposition 2).

We first show that any equilibrium must be symmetric. Let $\Pi_j(d,k_j)$ denote principal j's expected profit in state (d,k_j). It is easy to show, by an argument similar to that of Lemma 6, that $\Pi_j(d,k_j)$ is increasing in k_j. Principal j's optimal strategy is thus necessarily the largest k such that $U^j_i(I,k,k) \geq \frac{v-p}{\delta}$. From Lemma 8, it then follows that $U^j_i(I,d,k) \geq \frac{v-p}{\delta}$, $\forall d \leq k$ and thus the incentive-compatibility constraints of all insiders are satisfied.

Any k^* such that $U^j_i(I,k^*,k^*) \geq \frac{v-p}{\delta}$ and $U^j_i(I,k^*+1,k^*+1) < \frac{v-p}{\delta}$ is an equilibrium. Note that all principals face the same decision problem. It follows that any equilibrium must be symmetric.

Any k^* that just precedes the crossing of $U^j_i(I,k^*,k^*)$ and $\frac{v-p}{\delta}$ is thus an equilibrium. There exists an equilibrium since $U^j_i(I,1,1) > \frac{v-p}{\delta}$ and $\lim_{k\to\infty} U^j_i(I,k,k) = 0 < \frac{v-p}{\delta}$ and thus there exists a crossing. Moreover, the equilibrium is unique since $U^j_i(I,k,k)$ is strictly decreasing in k and thus there is a single crossing of $U^j_i(I,k,k)$ and $\frac{v-p}{\delta}$. In this case, the unique, symmetric equilibrium strategy followed by all principals is $k^* = \max\{k \mid U^j_i(I,k,k) \geq \frac{v-p}{\delta}\}$.

When all principals follow strategy k^*, then in steady state, any principal's degree (the number of insiders she can potentially trade with) follows a distribution $f_{k^*}(d)$, which is the stationary distribution of the Markov transition matrix $\boldsymbol{M}_P(k^*)$. Since the Markov chain described by $\boldsymbol{M}_P(k^*)$ is time-homogenous, irreducible and all its states are positive recurrent, then this stationary distribution necessarily exists and is unique.

□

Proof (of Proposition 3). Let x' and x be distribution vectors over the support $\{0,1,2,...,k\}$. We will first show that for any $x' \succeq x$, then $x'^{\top}\boldsymbol{M}_P(k) \succeq x^{\top}\boldsymbol{M}_P(k-1)$. Here $\boldsymbol{M}_P(k)$ is the $(k+1)\times(k+1)$ Markov transition matrix in Eq. (8) when the principal keeps a maximum of k insiders. $\boldsymbol{M}_P(k-1)$ is the Markov transition matrix when the principal keeps a maximum of $k-1$ insiders with an additional last row and last column consisting of zeros so that $\boldsymbol{M}_P(k-1)$ is a $(k+1)\times(k+1)$ matrix.

First note that $x'^{\top}\boldsymbol{M}_P(k)$ and $x^{\top}\boldsymbol{M}_P(k-1)$ can be expressed as mixture distributions. Indeed, let $\boldsymbol{M}_{P,d}(k)$ be the d^{th} row of the transition matrix $\boldsymbol{M}_P(k)$. This row corresponds to the transition probability distribution $\mathbb{P}(\tilde{d} \mid d,k)$. We may then write $x'^{\top}\boldsymbol{M}_P(k) = \sum_{d=0}^{k} x'(d)\boldsymbol{M}_{P,d}(k)$ and $x^{\top}\boldsymbol{M}_P(k-1) = \sum_{d=0}^{k} x(d)\boldsymbol{M}_{P,d}(k-1)$. Moreover, $\boldsymbol{M}_{P,d}(k) \succeq \boldsymbol{M}_{P,d}(k-1)$ for all d.

If $x' \succeq x$, then it follows that $\sum_{d=0}^{k} x'(d)\boldsymbol{M}_{P,d}(k) \succeq \sum_{d=0}^{k} x(d)\boldsymbol{M}_{P,d}(k-1)$ and thus that $x'^{\top}\boldsymbol{M}_P(k) \succeq x^{\top}\boldsymbol{M}_P(k-1)$. Thus, if for some $l>0$, $x'^{\top}\boldsymbol{M}^l_P(k) \succeq x^{\top}\boldsymbol{M}^l_P(k-1)$, then it follows that

$$\begin{aligned} x'^{\top}\boldsymbol{M}_P^{l+1}(k) &= (x'^{\top}\boldsymbol{M}_P^{l}(k))\boldsymbol{M}_P(k) \\ &\succeq (x^{\top}\boldsymbol{M}_P^{l}(k-1))\boldsymbol{M}_P(k-1) \\ &= x^{\top}\boldsymbol{M}_P^{l+1}(k-1) \end{aligned}$$

Thus, by induction we conclude that $x'^{\top}\boldsymbol{M}_P^{l}(k) \succeq x^{\top}\boldsymbol{M}_P^{l}(k-1)$ for any $l \geq 1$ and $x' \succeq x$. It thus follows that the stationary distributions satisfy

$$f_k(d) = \lim_{l\to\infty} x'^{\top}\boldsymbol{M}_P^{l}(k) \succeq \lim_{l\to\infty} x^{\top}\boldsymbol{M}_P^{l}(k-1) = f_{k-1}(d)$$

By transitivity, $f_{k'}(d) \succeq f_k(d)$ for any $k' > k$. □

Proof (of Proposition 4).
Eq. (16) can be rewritten as

$$W(k) = \big((1-q)v - qc\big)\mathbb{E}[g(d)] \tag{25}$$

where $d \sim f_k(d)$ and

$$g(d) = \begin{cases} 1-(1-\alpha)^{d+1} & \text{for } \ d \leq k-1 \\ 1-(1-\alpha)^{d} & \text{for } \ d = k \end{cases}$$

is a non-decreasing function of d. When $k' > k$, then it follows from Proposition 3 that $f_{k'} \succeq f_k$. It thus follows in turn that $W(k') \geq W(k)$. □

Proof (of Proposition 5). Let $\underline{q} = \max\{\tilde{q} \in [q,1] : U^j_{\tilde{q},i}(I,k^*,k^*) = \frac{v-p}{\delta}\}$, where k^* is the principal-optimal number of insiders under q. Such a $\underline{q}$ exists since $U^j_{\tilde{q},i}(I,k,k)$ is continuous in $\tilde{q}$, since $U^j_{\tilde{q}=q,i}(I,k^*,k^*) \geq \frac{v-p}{\delta}$ and since $U^j_{\tilde{q}=1,i}(I,k^*,k^*) = 0$.

Then $\forall q' > \underline{q}$, we have that $U^j_{q',i}(I,k^*,k^*) < \frac{v-p}{\delta}$ and thus it must be that $k'^* \leq k^*-1$, since this is the only way to potentially satisfy the IC constraint. Moreover, $U^j_{\underline{q},i}(I,k^*-1,k^*-1) > \frac{v-p}{\delta}$ and continuity in $\tilde{q}$ implies that $U^j_{q',i}(I,k^*-1,k^*-1) > \frac{v-p}{\delta}$ for all $q' \in (\underline{q}, \underline{q}+\epsilon)$ for some $\epsilon > 0$. Thus, $k'^* = k^*-1$ for all $q' \in (\underline{q}, \underline{q}+\epsilon)$, since this is the greatest number of insiders for which the incentive compatibility constraint is satisfied. □

Proof (of Corollary 1).
The result follows directly from Proposition 4. □