DISCRETE APPROXIMATION OF A CONTINUOUS MODEL OF MULTISTATE DEMOGRAPHY

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ABSTRACT

This paper suggests the applicability of a method recently developed by systems engineers to the estimation of the state transition matrices involved in the construction of increment-decrement life tables. Relevant to the case of piecewise-constant force functions, this method comes out as an alternative to the usual truncation of the infinite series obtained from the exact expansion of the state transition matrices. It generates a sequence of formulas which, interestingly enough, subsumes the linear formula of Rogers and Ledent (1976) as a special case. An illustration of the method is provided with applications to the analysis of marital status, labor force participation, and interregional migration.

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INTRODUCTION

Just as in natural sciences, analytic models used in social sciences are commonly formulated in terms of <u>differential</u> equations rather than <u>difference</u> equations, i.e., relying on a continuous rather than discrete specification. For example, in mathematical demography, all the columns of the ordinary life table model originate from the simple differential equation $\dot{\ell}(x) = -\mu(x)\ell(x)$, where $\mu(x)$ is the force of mortality at age x applicable to a cohort whose number of survivors at age x is $\ell(x)$.

In all branches of applied science, including engineering as well as demography, the data necessary for the application of such continuous models usually come in discrete form. Consequently, two recurring methodological problems in applied science are (1) <u>discretization</u> of a continuous model to fit the data, and (2) <u>smoothing</u> the discrete data to fit a continuous model.

Despite the commonality of these problems, the difficulties of communication between different disciplines result in either the rediscovery of the "wheel" several times or, more unfortunately, many decades of delay for a useful method to be transmitted from one discipline to another. In this paper, we would like to avoid another manifestation of such events. Specifically, we want to introduce to other demographers a method of discrete approximation of a continuous multistate linear model, which has been discovered recently by systems engineers (Shieh et al. 1978). This method lends itself a useful application in the emerging field of multistate demography (for an enlightening review of this field, see Keyfitz 1980). We intend to make this method as immediately useable as the spline method for data smoothing presented by McNeil, Trussell, and Turner (1977).

We will first recall the continuous formulation of the demographic model of particular interest here, namely the increment-decrement life table model (see Rogers 1973; Schoen 1975), and review briefly the methods currently in existence for its discretization. Then, we will introduce the basic ideas underlying the engineering method of discretization with the help of an easily understandable example and present the sequence of approximation formulas they lead to. Finally, we will demonstrate the applicability of this method to the estimation of increment-decrement life tables with examples relating to various demographic phenomena and, in such a way, empirically evaluate the goodness of these formulas.

1. THE INCREMENT-DECREMENT LIFE TABLE MODEL: CONTINUOUS FORMULATION AND EXISTING PROCEDURES FOR ITS DISCRETE APPROXIMATION

The increment-decrement life table model is a generalization of the ordinary life table model which allows for entries into (increments) as well as withdrawals from (decrements) different states. (The state space is assumed to have n+1 states, one of which is an absorbing state of death whereas at least two of the remaining states intercommunicate.) Because of its general nature, this model is valuable in analyses of marital status, labor force participation, interregional migration, etc. In studies of marital status, the non-absorbing states may number to four: single, married, widowed, and divorced (see Krishnamoorthy 1979). In studies on labor force participation, the states may be active and inactive (see Hoem and Fong 1976 and Willekens 1980). In studies of interregional migration, they are the regions of the geographical population system under consideration (see Rogers 1975).

In general, let $\underset{\sim}{\mu}(y)$ be an n × n matrix of transition forces relating to an infinitesimal age interval (y, y+dy) such that

a) its (i,j)-th off-diagonal element is equal to minus the force of transition from state j to state i and
b) its i-th diagonal element is the sum of all the forces of transition (including death) out of state i.

Also, let ${}_z\underset{\sim}{\ell}(y)$ be an n × n matrix of transition probabilities whose (i,j)-th element is the probability for a person present at age x in state j to survive to age y in state i. Then, assuming a Markovian-generated mobility process, we have the following Kolmogorov forward differential equation (see Schoen and Land 1979; Willekens 1980)

$$ {}_z\dot{\underset{\sim}{\ell}}(y) = -\underset{\sim}{\mu}(y)\, {}_z\underset{\sim}{\ell}(y) \tag{1} $$

where ${}_z\dot{\underset{\sim}{\ell}}(y)$ is the derivative of ${}_z\underset{\sim}{\ell}(y)$ with respect to y.

Concentrating on the evolution of the cohort of people corresponding to the choice of z equal to zero and then omitting this subscript, we obtain the solution of (1) as:

$$ \underset{\sim}{\ell}(y) = \underset{\sim}{\Omega}(y)\underset{\sim}{\ell}(0) \tag{2} $$

where $\underset{\sim}{\ell}(0)$ is the diagonal matrix showing the initial state allocation of the cohort considered. As for $\underset{\sim}{\Omega}(y)$, whose (i,j)-th element represents the proportion of the individuals in the j-th radix who survive to age y in state i, it can be shown to be equal to (Krishnamoorthy 1979)

$$\underset{\sim}{\Omega}(y) = \underset{\sim}{I} - \int_0^y \underset{\sim}{\mu}(t)dt + \int_0^y \underset{\sim}{\mu}(t) \int_0^t \underset{\sim}{\mu}(s)ds\, dt \ldots \tag{3}$$

The property of this matrix or matricant (see Gantmacher 1959) is such that, when the interval (0, y) is divided into k intervals of length h, we can write

$$\underset{\sim}{\Omega}(y) = \underset{\sim}{P}_{(k-1)h}\, \underset{\sim}{P}_{(k-2)h} \cdots \underset{\sim}{P}_0 \tag{4}$$

where each $\underset{\sim}{P}_x$ is a proper transition probability matrix relating to interval (x, x+h) (see Cox and Miller 1965).

Then letting $\underset{\sim}{\ell}_x$ denote the value of $\underset{\sim}{\ell}(y)$ at the equally spaced ages x = 0,h,2h,..., we have that the exact discrete representation of the continuous model described above is

$$\underset{\sim}{\ell}_{x+h} = \underset{\sim}{P}_x\, \underset{\sim}{\ell}_x \tag{5}$$

in which all the $\underset{\sim}{\ell}_x$'s can be derived from the knowledge of $\underset{\sim}{\ell}_0$ and the set of $\underset{\sim}{P}_x$'s.

Thus, in practice, the estimation of the increment-decrement life table model reduces to the estimation of a set of transition probabilities from which all the multistate life table functions (see Rogers 1975) constituting the output of such model originate. In most situations, such estimation is performed through a linkage with the observed values of the discrete equivalents of $\underset{\sim}{\mu}(x)$, denoted as $\underset{\sim}{M}_x$. Note that the observation of $\underset{\sim}{M}_x$ is only possible when the data come in the form of counts of moves rather than transitions (for a contrast between these two notions, see Ledent 1980).

In brief, there exist three main procedures of estimating $\underset{\sim}{P}_x$ in equation (5). The first (linear) procedure introduced by Rogers and Ledent (1976) is based on the assumption that $\underset{\sim}{\ell}(y)$ is linear for $x \le y \le x+h$. The resulting approximation formula is

$$\underset{\sim}{P}_x = (\underset{\sim}{I} + \frac{h}{2}\underset{\sim}{M}_x)^{-1} (\underset{\sim}{I} - \frac{h}{2}\underset{\sim}{M}_x) \tag{6}$$

The second (exponential) procedure shown in Krishnamoorthy (1979) is based on the assumption that the forces of transition are piecewise constant. That is, $\underset{\sim}{\mu}(y)$ is constant and equal to $\underset{\sim}{M}_x$ for $x \leq y \leq x+h$. This leads to the exact formula

$$\underset{\sim}{P}_x = e^{-h\underset{\sim}{M}_x} = \underset{\sim}{I} - h\underset{\sim}{M}_x + \frac{1}{2!}[h\underset{\sim}{M}_x]^2 - \frac{1}{3!}[h\underset{\sim}{M}_x]^3 + \ldots \tag{7}$$

For computation, $\underset{\sim}{P}_x$ is generally approximated by

$$\underset{\sim}{E}_k = I - h\underset{\sim}{M}_x + \frac{1}{2!}(h\underset{\sim}{M}_x)^2 - \ldots + \frac{1}{(k-1)!}(-h\underset{\sim}{M}_x)^{(k-1)} \tag{8}$$

In other words, the tail of the Taylor series beyond the first k terms is discarded.

The third (cubic) procedure is more elaborate. Essentially, it relies on the assumption that $\underset{\sim}{\ell}(y)$ is a third degree polynomial for $x-h \leq y \leq x+2h$. The procedure, detailed in full in Ledent and Rees (1980), starts with the calculation of an initial set of matrices $\underset{\sim}{\ell}_x$ from equation (5) where $\underset{\sim}{P}_x$ is computed according to equation (6). It is continued with the calculation of $\underset{\sim}{L}_x = \int_0^h \underset{\sim}{\ell}(x+t)dt$ from

$$\underset{\sim}{L}_x = \frac{13h}{24}[\underset{\sim}{\ell}_x + \underset{\sim}{\ell}_{x+h}] - \frac{h}{24}[\underset{\sim}{\ell}_{x-h} + \underset{\sim}{\ell}_{x+2h}] \tag{9}$$

which, using the flow equation (Rogers and Ledent 1976)

$$\underset{\sim}{\ell}_x - \underset{\sim}{\ell}_{x+h} = \underset{\sim}{M}_x \underset{\sim}{L}_x \tag{10}$$

leads to a new set of matrices $\underset{\sim}{\ell}_x$. The procedure is repeated until consistency of (9) and (10) is achieved. Then $\underset{\sim}{P}_x$ is estimated from

$$\underset{\sim}{P}_x = \underset{\sim}{\ell}_{x+h} \, \underset{\sim}{\ell}_x^{-1} \qquad (11)$$

Although equation (6) is exactly of the same form as the corresponding estimation formula in the ordinary life table, the matrices $\underset{\sim}{P}_x$ obtained with the linear procedure are not necessarily proper transition probability matrices (see Ledent 1980), especially if the magnitude of the off-diagonal elements of $\underset{\sim}{M}_x$ is large. The interpretation of $\underset{\sim}{P}_x$ then becomes impossible. As for the cubic procedure, it makes the age profiles of the elements of $\underset{\sim}{\ell}_x$ less irregular than those produced by other procedures, a property which makes it particularly suitable for less reliable data base. Nevertheless, the transition probability matrices obtained with this procedure may also be improper. In contrast to the linear and cubic procedures, the exponential procedure avoids, in principle, producing improper transition matrices. In practice, this simply requires an adequate estimation of the exponential matrix in (7), i.e., the calculation of the sequence of matrices $\underset{\sim}{E}_k$, as defined by (8), until a predetermined number of digits, for all elements, remains unchanged.

Actually, an alternative sequence of formulas, based on the matrix continued fraction method recently discovered by system engineers (Shieh et al. 1978), can be used for the same purpose. In the next section, we will introduce this sequence which, interestingly enough, subsumes the estimation formula of the linear approach as a special case and moreover converges "quicker" than the sequence $\underset{\sim}{E}_k$ does. (Note that the quickness of convergence relates to the index of the sequences rather than to actual computing time.)

2. THE MATRIX CONTINUED FRACTION (MCF) METHOD OF DISCRETE APPROXIMATION

To make the basic logic of the MCF method transparent, we will start with a simple numerical example and then present the matrix results that were obtained by Shieh et al. (1978). We begin by considering the expansion of a number, say, 1.2345 into a continued fraction in the following manner:

$$1.2345 = \frac{1.2345}{1} = 1 + \frac{2345}{10000} = 1 + \cfrac{1}{\frac{10000}{2345}}$$

$$= 1 + \cfrac{1}{4 + \frac{620}{2345}} = \ldots$$

After a few divisions we get

$$1.2345 = 1 + \cfrac{1}{4 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{3 + \ldots}}}} \tag{12}$$

By letting H_i be the integer before the i-th division line, equation (12) can be written as

$$1.2345 = H_1 + \cfrac{1}{H_2 + \cfrac{1}{H_3 + \cfrac{1}{H_4 + \cfrac{1}{H_5 + \ldots}}}} \tag{13}$$

or, more compactly,

$$1.2345 = H_1 + [H_2 + [H_3 + [H_4 + [H_5 + \ldots]^{-1}]^{-1}]^{-1}]^{-1} \tag{14}$$

The important point is that the retention of only the first few H_j in equation (13) will result in a fairly good approximation of the original number. For example, consider the approximation

$$G_3 = H_1 + \cfrac{1}{H_2 + \cfrac{1}{H_3}} = H_1 + \frac{H_3}{H_2H_3 + 1}$$

$$= \frac{H_1H_2H_3 + H_1 + H_3}{H_2H_3 + 1} \qquad (15)$$

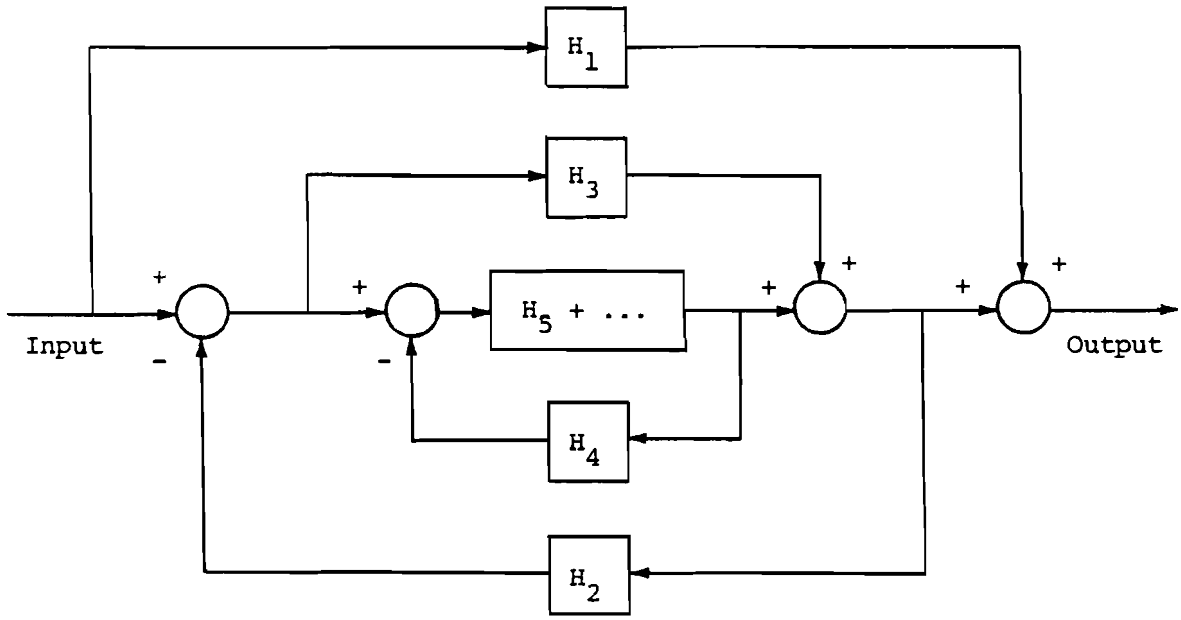
$$= [H_2H_3 + 1]^{-1} [H_1H_2H_3 + H_1 + H_3]$$

where only the first three H_j in equation (13) are preserved. Substituting the H_j by their values, equation (15) becomes

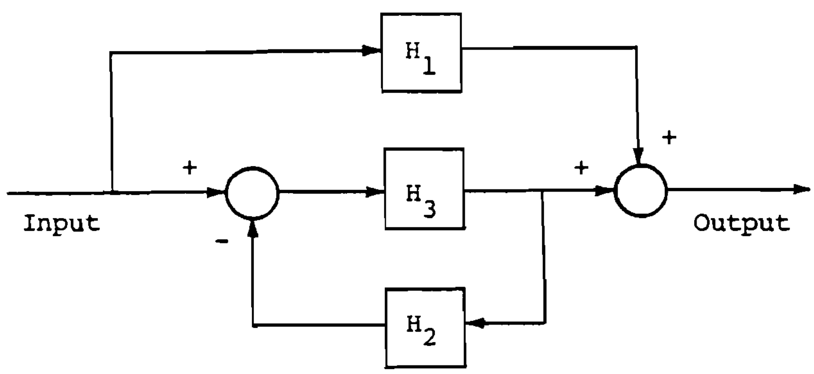
$$G_3 = [(4)(3) + 1]^{-1} [(1)(4)(3) + 1 + 3] = 16/13 \doteq 1.2308 \qquad (16)$$

The estimation error is only 0.3%.

It is useful to consider the right-hand side of equation (14) [or equivalent equation (13)] as a system which is designed for the approximation of the left-hand side. That is, when a unit input is applied (i.e., multiplied) to the system, the resulting output becomes the estimated value. The system is diagramatically shown in Figure 1a and is in the form of a multi-feedback multi-feedforward control system. It has been observed that the behavior of such a system is relatively insensitive to the changes or omissions of the <u>inner</u> paths (for an elementary explanation, see Melsa and Schultz 1969:86-91). A simplified system using only the first three outer paths is shown in Figure 1b, and we have seen it through equations (15) and (16) that it performs well for our admittedly trivial example.



(a)



(b)

Figure 1. Block diagrams for (a) $H_1 + [H_2 + [H_3 + [H_4 + [H_5 + \ldots]^{-1}]^{-1}]^{-1}]^{-1}$ and (b) $H_1 + [H_2 + H_3^{-1}]^{-1}]^{-1}$. For a good explanation of block diagrams, see Schwarz and Friedland (1965) or Melsa and Schultz (1969).

The procedure of going from equation (12) to equation (16) demonstrates the basic logic of the MCF method that was used by Shieh et al. (1978) to derive a sequence of estimation formulas for $e^{\tilde{A}h}$, where we may let $\tilde{A} = -\tilde{M}_x$ for the increment-decrement life table model. The only difference is that now we have matrices rather than just numbers. The outline of derivation is as follows.

First, we write $e^{\tilde{A}h}$ as a Taylor series divided by an identity matrix. That is,

$$e^{\tilde{A}h} = [\tilde{I} + \tilde{A}h + \frac{1}{2!}(\tilde{A}h)^2 + \ldots][\tilde{I}]^{-1} \tag{17}$$

Next, we apply the technique of continued fraction to equation (17) to obtain a sequence of matrices $\{\tilde{H}_1, \tilde{H}_2, \tilde{H}_3, \ldots\}$ such that

$$e^{\tilde{A}h} = \tilde{H}_1 + [\tilde{H}_2 + [\tilde{H}_3 + [\tilde{H}_4 + [\tilde{H}_5 + \ldots]^{-1}]^{-1}]^{-1}]^{-1} \tag{18}$$

which is exactly in the same form as equation (14). The matrices $\tilde{H}_j$ could be obtained by Routh's algorithm (see Schwarz and Friedland 1965:406-408), which is a tabular way of carrying out the type of divisions used in equation (12). It is shown in Shieh et al. (1978) that

$$\tilde{H}_1 = \tilde{I}, \quad \tilde{H}_2 = (\tilde{A}h)^{-1}, \quad \tilde{H}_3 = -2\tilde{I}, \quad \tilde{H}_4 = -3(\tilde{A}h)^{-1}$$

$$\tilde{H}_5 = 2\tilde{I}, \quad \tilde{H}_6 = 5(\tilde{A}h)^{-1}, \quad \tilde{H}_7 = -2\tilde{I}, \quad \tilde{H}_8 = -7(\tilde{A}h)^{-1}$$

.....

$$\tilde{H}_j = 2\tilde{I}, \quad \tilde{H}_{j+1} = j(\tilde{A}h)^{-1}, \quad \tilde{H}_{j+2} = -2\tilde{I}, \quad \tilde{H}_{j+3} = -(j+2)(\tilde{A}h)^{-1}$$

for $j = 5, 9, 13, \ldots$ (19)

Next, let $\underset{\sim}{G}_j$ be the estimate of $e^{\underset{\sim}{A}h}$ by retaining only the first j H matrices in equation (18). Then, just as we did in equations (15) and (16), we get

$$\underset{\sim}{G}_2 = \underset{\sim}{H}_1 + \underset{\sim}{H}_2^{-1} = \underset{\sim}{I} + \underset{\sim}{A}h = \underset{\sim}{I} - h\underset{\sim}{M}_x \tag{20}$$

$$\underset{\sim}{G}_3 = \underset{\sim}{H}_1 + [\underset{\sim}{H}_2 + \underset{\sim}{H}_3^{-1}]^{-1} = \underset{\sim}{H}_1 + [\underset{\sim}{H}_2\underset{\sim}{H}_3 + \underset{\sim}{I}]^{-1}\underset{\sim}{H}_3$$

$$= [\underset{\sim}{H}_2\underset{\sim}{H}_3 + \underset{\sim}{I}]^{-1} \; [\underset{\sim}{H}_1\underset{\sim}{H}_2\underset{\sim}{H}_3 + \underset{\sim}{H}_1 + \underset{\sim}{H}_3]$$

$$= [(\underset{\sim}{A}h)^{-1}(-2\underset{\sim}{I}) + \underset{\sim}{I}]^{-1} \; [\underset{\sim}{I}(\underset{\sim}{A}h)^{-1}(-2\underset{\sim}{I}) + \underset{\sim}{I} - 2\underset{\sim}{I}]$$

$$= [-2(\underset{\sim}{A}h)^{-1} + \underset{\sim}{I}]^{-1} \; [-2(\underset{\sim}{A}h)^{-1} - \underset{\sim}{I}]$$

$$= [\underset{\sim}{I} - \tfrac{1}{2}(\underset{\sim}{A}h)]^{-1} \; [\underset{\sim}{I} + \tfrac{1}{2}(\underset{\sim}{A}h)]$$

$$= [\underset{\sim}{I} + \tfrac{h}{2}\underset{\sim}{M}_x]^{-1} \; [\underset{\sim}{I} - \tfrac{h}{2}\underset{\sim}{M}_x] \tag{21}$$

Clearly, equation (21) is identical to the estimation formula obtained by Rogers and Ledent (1976) using a different assumption. To avoid cluttering the text, all $\underset{\sim}{G}_j$ up to j = 13 are shown in Appendix A.

There is a basic difference between ignoring the higher $\underset{\sim}{H}_j$ in the MCF method and the chopping-off of the tail of the original Taylor series. For illustration, Shieh et al. (1978) show that

$$\underset{\sim}{G}_3 = \underset{\sim}{I} + \underset{\sim}{A}h + \frac{1}{2!}(\underset{\sim}{A}h)^2 + \sum_{j=3}^{\infty} \frac{1}{2^{j-1}}(\underset{\sim}{A}h)^j$$

and

$$\underset{\sim}{G}_4 = \underset{\sim}{I} + \underset{\sim}{A}h + \frac{1}{2!}(\underset{\sim}{A}h)^2 + \frac{1}{3!}(\underset{\sim}{A}h)^3 + \frac{1}{(1.5)(4!)} \sum_{j=4}^{\infty} \frac{1}{3^{j-4}}(\underset{\sim}{A}h)^j$$

It is argued that the retention of the first three (or four) $\underset{\sim}{H}_j$ must be preferable to discarding the tail of the Taylor series beyond the first three (or four) terms, because the MCF method (for j > 2) preserves a systematically modified tail--not a single term in the original series has been discarded. The demographic examples in the next section support this argument strongly.

3. APPLICATION OF THE MCF METHOD TO THE ESTIMATION OF INCREMENT-DECREMENT LIFE TABLES: AN EMPIRICAL EVALUATION

For evaluation purposes, the MCF method has been applied to the estimation of the age-specific transition probabilities underlying the construction of three empirical increment-decrement life tables. A comparison was then made with the more popular method based on the Taylor series expansion of formula (7).

The first application takes advantage of data originally used by Krishnamoorthy (1979) to construct a marital status life table for US females in 1970 on the basis of 18 age groups: 0-4, 5-9,..., 80-84, 85+. The second one utilizes data employed by Hoem and Fong (1976) to calculate a working life table for Danish males on the basis of 59 age groups (16, 17,..., 74). Finally, the third application makes use of interregional migration data in the Netherlands collected for 18 age groups (0-4, 5-9,..., 80-84, 85+), by Drewe (1980) in view of the calculation of a four-region life table.

Tables 1 through 3 illustrate our computational results by presenting, for young adult age groups, the elements of the following estimated matrices:

a) $\underset{\sim}{E}_3$ and $\underset{\sim}{G}_3$
b) $\underset{\sim}{E}_k$ and $\underset{\sim}{G}_k$ where k is the smallest integer for which the elements of $\underset{\sim}{G}_{k'}$, for any k' > k, has the same first five digits as in $\underset{\sim}{G}_k$
c) the "exact" solution defined as the first matrix in the series $\underset{\sim}{E}_k$ or $\underset{\sim}{G}_k$ such that all elements of $\underset{\sim}{E}_k$ and $\underset{\sim}{G}_k$ have their first five digits in common.

Table 1. Marital status life table for US females, 1970: transition probabilities between ages 20 and 25.

Transition From	To	E_3	G_3	E_{10}	G_{10}	E_{13} & G_{13}
S	S	.53175	.23001	.28593	.28593	.28593
S	M	.38646	.74260	.68218	.68213	.68213
S	W	.00678	.00325	.00380	.00380	.00380
S	D	.07137	.02040	.02435	.02440	.02440
M	S	0	0	0	0	0
M	M	1.00530	.93465	.94323	.94327	.94327
M	W	.00763	.00846	.00830	.00830	.00830
M	D	-.01667	.05314	.04473	.04468	.04468
W	S	0	0	0	0	0
W	M	.31493	.34950	.34275	.34274	.34274
W	W	.65607	.63715	.64219	.64219	.64219
W	D	.02526	.00960	.01132	.01133	.01133
D	S	0	0	0	0	0
D	M	-.31479	1.00341	.84451	.84369	.84369
D	W	.01155	.00439	.00517	.00518	.00518
D	D	1.29950	-0.01155	.14657	.14739	.14739

S = single M = married W = widowed D = divorced

SOURCE OF INPUT DATA: Krishnamoorthy (1979).

Table 2. Table of working life for danish males: transition probabilities between ages 20 and 21.

Transition From	To	E_3	G_3	E_6	G_6	E_8 & G_8
I	I	.66750	.64019	.64712	.64715	.64715
I	A	.33128	.35859	.35166	.35163	.35163
A	I	.06678	.07228	.07089	.07088	.07088
A	A	.93200	.92650	.92789	.92790	.92790

I = inactive A = active

SOURCE OF INPUT DATA: Hoem and Fong (1976).

Table 3. Multiregional life table for the Netherlands: transition probabilities between ages 20 and 25.

Transition From	To	$\tilde{E}_3$	$\tilde{G}_3$	$\tilde{E}_5$	$\tilde{G}_5$	$\tilde{E}_6$ & $\tilde{G}_6$
1	1	.81193	.80886	.80977	.80976	.80976
1	2	.07186	.07472	.07391	.07392	.07392
1	3	.09096	.09180	.09153	.09153	.09153
1	4	.02151	.02086	.02104	.02103	.02103
2	1	.04061	.04218	.04174	.04175	.04175
2	2	.77242	.76554	.76750	.76747	.76747
2	3	.12614	.12994	.12885	.12887	.12887
2	4	.05706	.05856	.05813	.05814	.05814
3	1	.02412	.02438	.02430	.02430	.02430
3	2	.05553	.05722	.05674	.05674	.05674
3	3	.87103	.86852	.86925	.86925	.86925
3	4	.04631	.04689	.04671	.04671	.04671
4	1	.01068	.01036	.01045	.01045	.01045
4	2	.04660	.04774	.04741	.04741	.04741
4	3	.09382	.09510	.09472	.09472	.09472
4	4	.84505	.84295	.84357	.84357	.84357

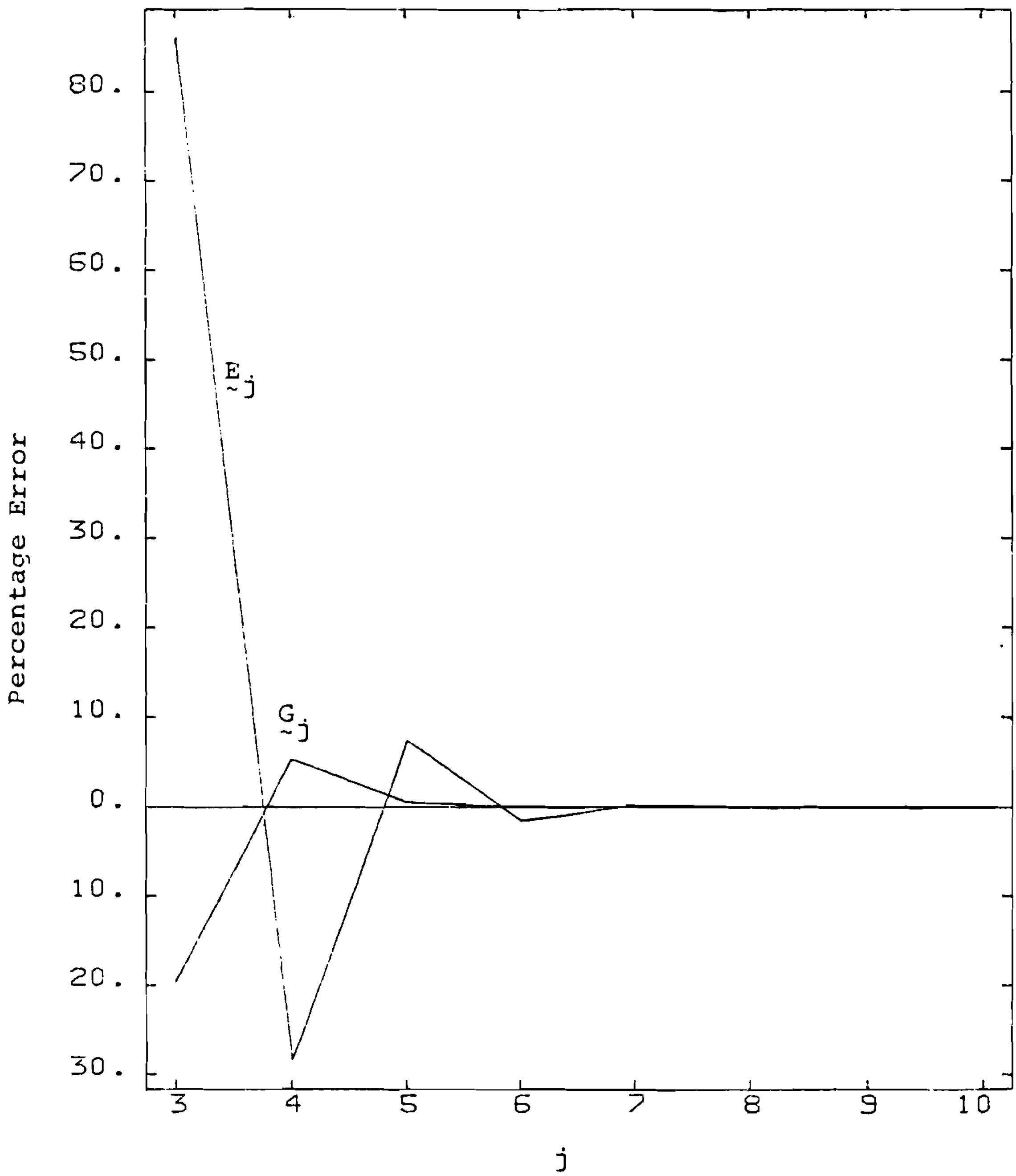
SOURCE OF INPUT DATA: Drewe (1980).

It turns out that:

a) $\tilde{G}_k$ gives better estimates than does $\tilde{E}_k$ especially when k is small (see also Figure 2)

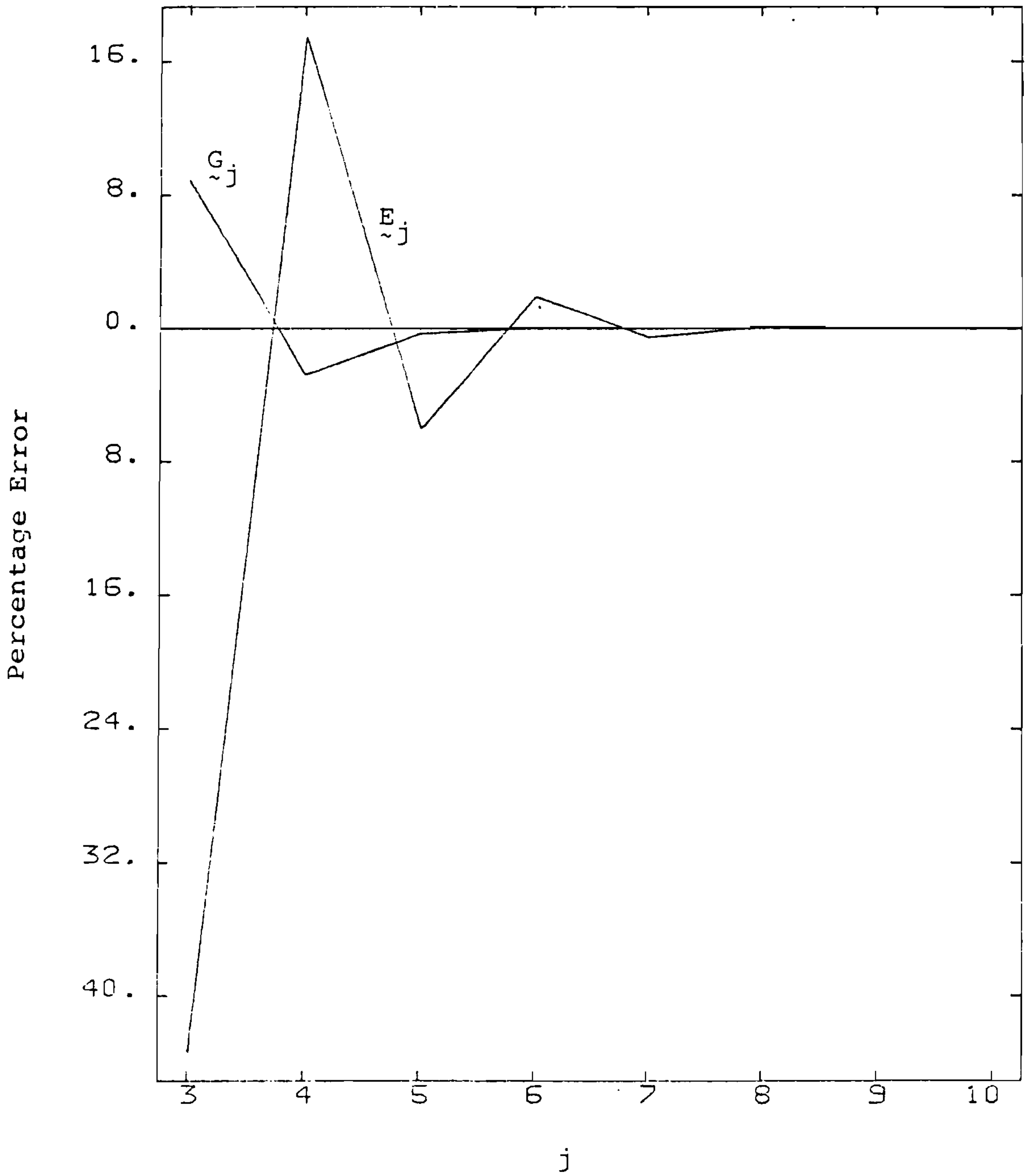
b) the higher the off-diagonal elements of the matrix $-h\tilde{M}_x$, i.e., the higher the propensity to move out of a state and the higher the width of the age groups considered, the greater the value of k necessary to reach the "exact" solution

c) even in the most favorable situations, $\tilde{G}_3$ (equivalent to the linear formula) fails to produce more than a couple of significant digits (see Table 3).



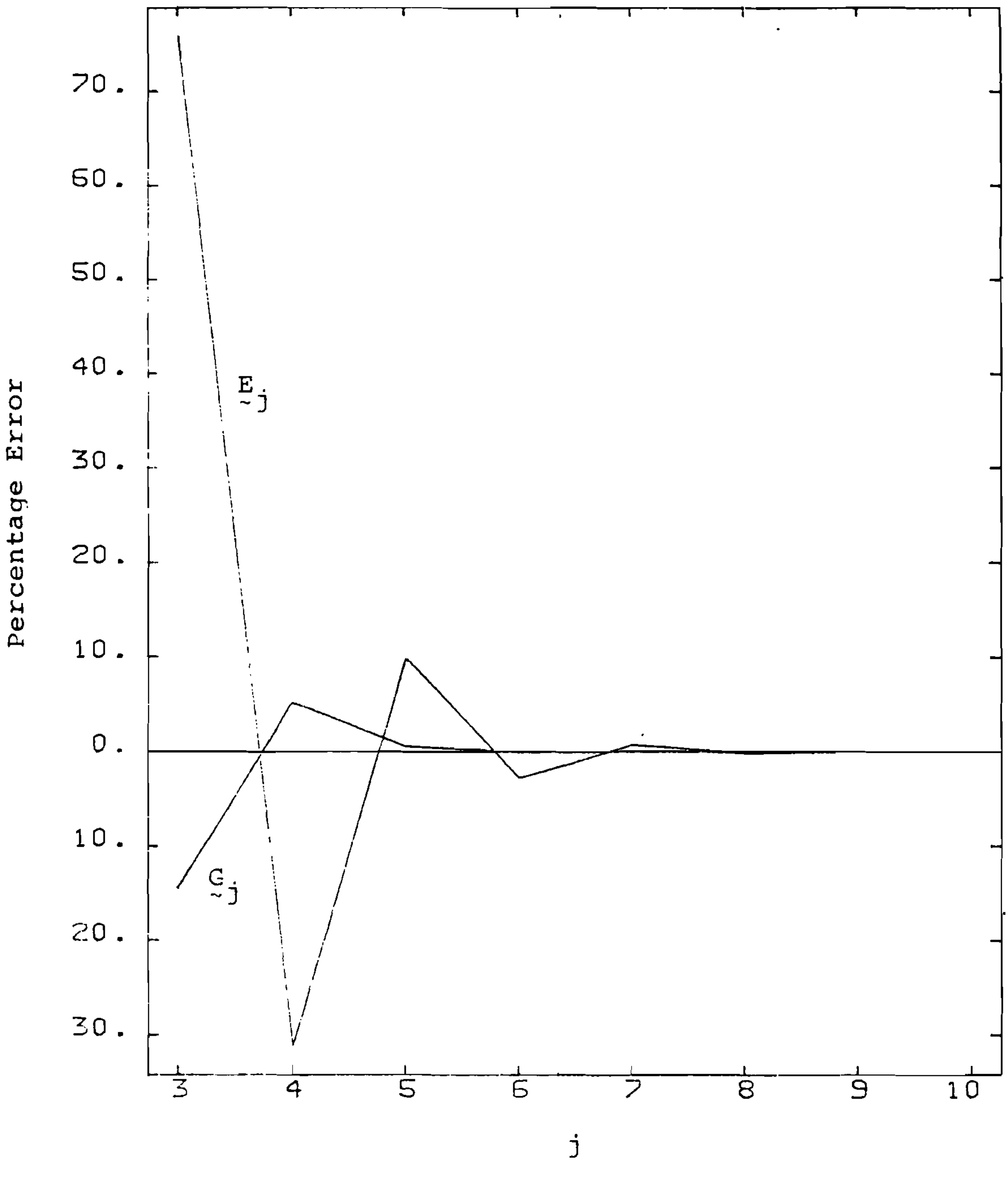
(a) From Single to Single

Figure 2. Percentage errors of the two sequences of estimation formulas: $\underset{\sim}{E}_j$ and $\underset{\sim}{G}_j$ (j = 3 to 10) for Krishnamoorthy's marriage data, ages 20 to 25.



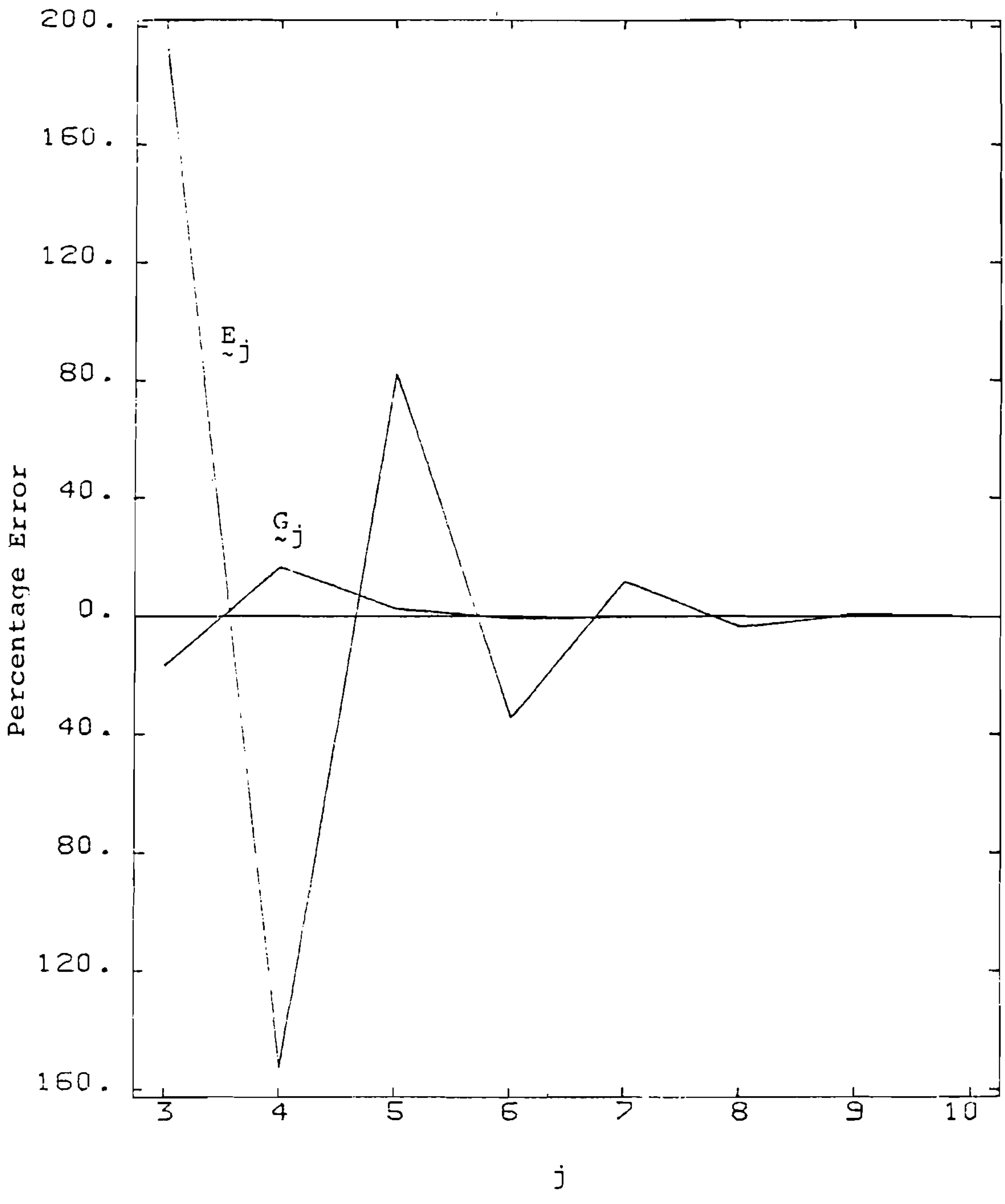
(b) From Single to Married

Figure 2. Continued.



(c) From Single to Widowed

Figure 2. Continued.



(d) From Single to Divorced

Figure 2. Continued.

DISCUSSION

Of the three main procedures currently existing for the estimation of increment-decrement life tables--the so-called linear, exponential, and cubic procedures--, only the exponential procedure (based on piecewise-constant forces of transition) ensures that the age-specific survival matrices $\underset{\sim}{P}_x$ are proper transition probability matrices.

In this paper, we have proposed for its implementation a sequence of estimation formulas, based on the method of matrix continued fraction, which a) subsumes the estimation formula of the linear procedure and b) converges "quicker" than the commonly-used sequence based on the Taylor series expansion of the exact survival matrices $\underset{\sim}{P}_x$.

Actually, the method of continued fraction and its matrix generalization have wide applicability in the analysis of linear control systems and may become more useful to mathematical demographers as we broaden our scope of investigation to control mechanisms in population systems. Besides being a good way of discretizing continuous-time linear control systems, these methods have been used to determine conveniently (i.e., without finding the roots of the characteristic equation) the stability of a system (see Schwarz and Friedland 1965:404-408) and to transform a difficult high-order linear differential equation into a relatively easy low-order one without losing the essential dynamic properties of the physical system (see Shieh and Gaudiano 1975).

Finally, let us observe that, to date, the development of multistate mathematical demography has depended heavily on extending the approach of the classical life table analysis. Recent progress in multistate analysis demostrates the fruitfulness of this research strategy. Actually, progress in other scientific disciplines also relies heavily on extending and generalizing old analytic methods. The development of the MCF method is a good example in engineering. However, to avoid wasting much time in rediscovering the methods which have already been found in other disciplines, mathematical demographers should be alert to methodological developments in mathematically advanced fields such as systems engineering.

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APPENDIX A: THE MCF FORMULAS FOR ESTIMATING $\underset{\sim}{P}_x$

[Source: Shieh et al. (1978)]

$$\underset{\sim}{G}_2 = \underset{\sim}{I} - h\underset{\sim}{M}_x$$

$$\underset{\sim}{G}_3 = [\underset{\sim}{I} + \frac{1}{2} h\underset{\sim}{M}_x]^{-1} \cdot [\underset{\sim}{I} - \frac{1}{2} h\underset{\sim}{M}_x]$$

$$\underset{\sim}{G}_4 = [\underset{\sim}{I} + \frac{1}{3} h\underset{\sim}{M}_x]^{-1} \cdot [\underset{\sim}{I} - \frac{2}{3} h\underset{\sim}{M}_x + \frac{1}{6}(h\underset{\sim}{M}_x)^2]$$

$$\underset{\sim}{G}_5 = [\underset{\sim}{I} + \frac{1}{2} h\underset{\sim}{M}_x + \frac{1}{12}(h\underset{\sim}{M}_x)^2]^{-1} \cdot [\underset{\sim}{I} - \frac{1}{2} h\underset{\sim}{M}_x + \frac{1}{12}(h\underset{\sim}{M}_x)^2]$$

$$\underset{\sim}{G}_6 = [\underset{\sim}{I} + \frac{2}{5} h\underset{\sim}{M}_x + \frac{1}{20}(h\underset{\sim}{M}_x)^2]^{-1} \cdot [\underset{\sim}{I} - \frac{3}{5} h\underset{\sim}{M}_x + \frac{3}{20}(h\underset{\sim}{M}_x)^2 - \frac{1}{60}(h\underset{\sim}{M}_x)^3]$$

$$\underset{\sim}{G}_7 = [\underset{\sim}{I} + \frac{1}{2} h\underset{\sim}{M}_x + \frac{1}{10}(h\underset{\sim}{M}_x)^2 + \frac{1}{120}(h\underset{\sim}{M}_x)^3]^{-1} \cdot$$
$$[\underset{\sim}{I} - \frac{1}{2} h\underset{\sim}{M}_x + \frac{1}{10}(h\underset{\sim}{M}_x)^2 - \frac{1}{120}(h\underset{\sim}{M}_x)^3]$$

$$\underset{\sim}{G}_8 = [\underset{\sim}{I} + \frac{3}{7} h\underset{\sim}{M}_x + \frac{1}{14}(h\underset{\sim}{M}_x)^2 + \frac{1}{210}(h\underset{\sim}{M}_x)^3]^{-1} \cdot$$
$$[\underset{\sim}{I} - \frac{4}{7} h\underset{\sim}{M}_x + \frac{1}{7}(h\underset{\sim}{M}_x)^2 - \frac{2}{105}(h\underset{\sim}{M}_x)^3 + \frac{1}{840}(h\underset{\sim}{M}_x)^4]$$

$$\underset{\sim}{G}_9 = [\underset{\sim}{I} + \frac{1}{2} h\underset{\sim}{M}_x + \frac{3}{28}(h\underset{\sim}{M}_x)^2 + \frac{1}{84}(h\underset{\sim}{M}_x)^3 + \frac{1}{1680}(h\underset{\sim}{M}_x)^4]^{-1} \cdot$$
$$[\underset{\sim}{I} - \frac{1}{2} h\underset{\sim}{M}_x + \frac{3}{28}(h\underset{\sim}{M}_x)^2 - \frac{1}{84}(h\underset{\sim}{M}_x)^3 + \frac{1}{1680}(h\underset{\sim}{M}_x)^4]$$

$$\underset{\sim}{G}_{10} = [\underset{\sim}{I} + \frac{4}{9} h\underset{\sim}{M}_x + \frac{1}{12}(h\underset{\sim}{M}_x)^2 + \frac{1}{126}(h\underset{\sim}{M}_x)^3 + \frac{1}{3024}(h\underset{\sim}{M}_x)^4]^{-1} \cdot$$
$$[\underset{\sim}{I} - \frac{5}{9} h\underset{\sim}{M}_x + \frac{5}{36}(h\underset{\sim}{M}_x)^2 - \frac{5}{252}(h\underset{\sim}{M}_x)^3 + \frac{5}{3024}(h\underset{\sim}{M}_x)^4 - \frac{1}{15120}(h\underset{\sim}{M}_x)^5]$$

$$\underset{\sim}{G}^*_{11} = [\underset{\sim}{I} + \frac{1}{2} h\underset{\sim}{M}_x + \frac{1}{9}(h\underset{\sim}{M}_x)^2 + \frac{1}{72}(h\underset{\sim}{M}_x)^3 + \frac{1}{1008}(h\underset{\sim}{M}_x)^4 + \frac{1}{30240}(h\underset{\sim}{M}_x)^5]^{-1} \cdot$$
$$[\underset{\sim}{I} - \frac{1}{2} h\underset{\sim}{M}_x + \frac{1}{9}(h\underset{\sim}{M}_x)^2 - \frac{1}{72}(h\underset{\sim}{M}_x)^3 + \frac{1}{1008}(h\underset{\sim}{M}_x)^4 - \frac{1}{30240}(h\underset{\sim}{M}_x)^5]$$

* Note that the coefficient of the fifth power terms in $\underset{\sim}{G}_{11}$ as given in Shieh et al. (1978) is incorrect.

$$G_{12} = [I + \frac{5}{11} hM_x + \frac{1}{11}(hM_x)^2 + \frac{1}{99}(hM_x)^3 + \frac{1}{1584}(hM_x)^4 + \frac{1}{55440}(hM_x)^5]^{-1} \cdot$$

$$[I - \frac{6}{11} hM_x + \frac{3}{22}(hM_x)^2 - \frac{2}{99}(hM_x)^3 + \frac{1}{528}(hM_x)^4 - \frac{1}{9240}(hM_x)^5 + \frac{1}{332640}(hM_x)^6]$$

$$G_{13} = [I + \frac{1}{2} hM_x + \frac{5}{44}(hM_x)^2 + \frac{1}{66}(hM_x)^3 + \frac{1}{792}(hM_x)^4 + \frac{1}{15840}(hM_x)^5 + \frac{1}{665280}(hM_x)^6]^{-1}$$

$$[I - \frac{1}{2} hM_x + \frac{5}{44}(hM_x)^2 - \frac{1}{66}(hM_x)^3 + \frac{1}{792}(hM_x)^4 - \frac{1}{15840}(hM_x)^5 + \frac{1}{665280}(hM_x)^6]$$