A JOB SHOP ASSIGNMENT PROBLEM WITH QUEUING COSTS

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1. The Problem

Consider an assignment problem in which jobs are to be assigned to machines in such a way as to minimize the total cost of manufacture. In addition, there is, for each job, a queuing cost which is proportional to the time spent before completion. Each job takes a unit length of time to be completed once work is started on it by a machine.

It will be shown that this problem may be formulated as a linear program whose optimal solution will be integral.

For example, with four jobs (i = 1, 2, 3, 4) and two machines (s = 1, 2) with a fixed service cost of i times s plus a unit charge per period waited before completion, the optimal arrangement is to assign job 1 to machine 2 and the remainder to machine 1. This gives a total cost of

$$(1.2 + 1) + (2.1 + 3.1 + 4.1 + 1 + 2 + 3) = 18$$

2. The Formulation

Let r_i^s be the cost of processing job i on machine s. Let x_{is} = 1 if job i is assigned to machine s and 0 otherwise. Let y_{ks} = 1 if machine s has k jobs assigned to it and 0 otherwise.

*Carlos Winkler supplied the neat proof of the theorem. This problem was suggested by Aleksandr Butrimenko.

The following integer program models the situation

$$\min \sum_{i,s} r_i^s x_{is} + \sum_{s,k} \frac{k(k+1)}{2} y_{ks}$$

$$\sum_s x_{is} = 1 \qquad \text{for each } i$$

$$\sum_k y_{ks} = 1 \qquad \text{for each } s$$

$$\sum_k k y_{ks} = \sum_i x_{is} \qquad \text{for each } s \quad .$$

$$y_{ks} \geq 0 \quad x_{is} \geq 0$$

$$x_{is} \ \text{integer}$$

Note that it is not necessary to enforce the integrality of the y variables as y_{ks} will be integral if $\sum_i x_{is}$ is integral, because of the form of the objective function. Note too*, that if the y's are integral in the optimal solution, then so will the x's be integral because for fixed integral y's, the problem is just an assignment problem, which is known to solve in integers.

Lemma In the optimal solution to the problem

$\sum_i x_{is}$ integer => y_{ks} integer for all k

and all y_{ks} integer => whole solution is integral.

Theorem The optimal solution to the linear program (assumed to be an extreme point) is integral.

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*Observation by George Dantzig

Proof The lemma only leaves the case where at least one $\sum_i x_{is}$ is not integral. It will be shown that such an optimal solution is not extreme. Suppose that

$$0 < x^*_{is} < 1$$

in the optimal solution. Hence there exists some j for which $0 < x^*_{ij} < 1$ for the same i. Suppose first that $\sum_p x^*_{pj}$ is not integral. Then we may find an $\varepsilon > 0$ such that

$$k_1 < \sum_p x^*_{ps} \pm \varepsilon < k_1 + 1$$

$$k_2 < \sum_p x^*_{pj} \pm \varepsilon < k_2 + 1$$

Associated with the three solutions (x_{ij}, x_{is}), $(x^*_{ij} + \varepsilon, x^*_{is} - \varepsilon)$ and $(x^*_{ij} - \varepsilon, x^*_{is} + \varepsilon)$ are the solutions $(y^*_{k_1s}, y^*_{k_1+1,s}, y^*_{k_2j}, y^*_{k_2+1,j})$, $(y^*_{k_1s} + \varepsilon, y^*_{k_1+1,s} - \varepsilon, y^*_{k_2j} - \varepsilon, y^*_{k_2+1,j} + \varepsilon)$ and $(y^*_{k_1s} - \varepsilon, y^*_{k_1+1,s} + \varepsilon, y^*_{k_2j} + \varepsilon, y^*_{k_2+1,j} - \varepsilon)$.

The important point is that only these variables are affected. All three solutions are feasible and the optimal solution is a linear combination of the other two. Hence, the optimal solution is not extreme. Now the case when $\sum_p x^*_{pj}$ is integral must be considered. In this case, since x^*_{ij} is not integral, x^*_{aj} must be non integral for some $a \neq i$. Hence, x^*_{at} is not integral for some

$t \neq j$. If $t = s$, then the solution

$$(x^*_{is} + \varepsilon,\ x^*_{ij} - \varepsilon,\ x^*_{aj} + \varepsilon,\ x^*_{as} - \varepsilon)$$

is feasible without affecting the y's. The same argument about non-extremeness then applies. If $\sum_p x^*_{pt}$ is integral, the system is repeated. If it is not integral, then the first argument still applies. In summary, the argument is just that of the assignment problem proof, except that the y variables may be affected. Since these respond linearly to changes in the x variables, all is well. //

As empirical evidence of the truth of the theorem, two problems having 21 jobs and 6 machines solved in integers.