PROOF FOR A CASE WHERE DISCOUNTING ADVANCES THE DOOMSDAY

T. C. Koopmans

January 1974 WP-74-6

PROOF FOR A CASE WHERE DISCOUNTING ADVANCES THE DOOMSDAY

by Tjalling C. Koopmans*

In a previous paper (Koopmans [1973]), I considered some problems of "optimal" consumption $\hat{r}_t$ over time of an exhaustible resource of known finite total availability R. In one of the cases studied, consumption of a minimum amount of the resource is assumed to be essential to human life, in such a way that all life ceases upon its exhaustion at time T. Assuming a constant population until that time, and denoting by $\underline{r}$ the positive minimum consumption level needed for survival of that population, the survival period T is constrained by

$$(1) \qquad 0 < T \leqq R/\underline{r} \equiv \overline{T} \quad .$$

Here equality ($T=\overline{T}$) can be attained only by consuming at the minimum level ($r_t=\underline{r}$) at all times, $0 \leqq t \leqq \overline{T}$.

However, optimality is defined in terms of maximization of the integral over time of discounted future utility levels,

$$(2) \qquad V(\rho,T,(r_t)) \equiv \int_0^T e^{-\rho t} v(r_t)dt \quad ,$$

*This research was started at the Cowles Foundation for Research in Economics at Yale University, New Haven, Conn., USA, with the support of the National Science Foundation and the Ford Foundation, and completed at the International Institute for Applied Systems Analysis in Laxenburg, Austria. I am indebted to John Casti for valuable comments.

where ρ is a discount rate, $\rho \geqq 0$, applied in continuous time to the utility flow $v(r_t)$ arising at any time t from a consumption flow r_t of the resource. The utility flow function v(r) is defined for $r \geqq \underline{r}$, is twice continuously differentiable and satisfies

$$\text{(3a,b,c,d)} \quad v'(r) > 0, \; v''(r) < 0 \quad \text{for } r > \underline{r}, \; v(\underline{r}) = 0, \quad \lim_{r \to \underline{r}} v'(r) = \infty .$$

That is, v(r) is (a) strictly increasing and (b) strictly concave. The stipulation (c) anchors the utility scale. Some such anchoring, though not necessarily the given one, is needed whenever population size is a decision variable. The last requirement (d) simplifies a step in the proof, and can be secured if needed by a distortion of v(r) in a neighborhood of $\underline{r}$ that does not affect the solution.

The paper referred to gives an intuitive argument for the following

<u>Theorem</u>: <u>For each</u> $\rho \geqq 0$ <u>there exists a unique optimal path</u> $r_t = \hat{r}_t$, $0 \leqq t \leqq \hat{T}_\rho$, <u>maximizing</u> (2) <u>subject to</u>

$$(4) \begin{cases} \text{(4a)} \quad r_t \text{ is a continuous function on } [0,T] , \\ \text{(4b)} \quad \int_0^T r_t dt \leqq R, \; r_t \geqq \underline{r}, \; 0 \leqq t \leqq T . \end{cases}$$

<u>For</u> $\rho = 0$, <u>the optimal path</u> $(\hat{r}_t | 0 \leqq t \leqq \hat{T}_0)$ <u>is defined by</u>

$$(5)\begin{cases}(5a) & \hat{r}_t = \hat{r}, \text{ a constant, for } 0 \leqq t \leqq \hat{T}_0 \ , \\ (5b) & v(\hat{r}) = \hat{r}v'(\hat{r}) \ , \\ (5c) & \hat{r}\hat{T}_0 = R \ .\end{cases}$$

<u>For</u> $\rho > 0$ <u>it is defined by</u>

$$(6)\begin{cases}(6a) & e^{-\rho t}v'(\hat{r}_t) = e^{-\rho\hat{T}_\rho}v'(\hat{r}) \ , \quad 0 \leqq t \leqq \hat{T}_\rho \ , \quad \hat{r} \text{ as in (5b)} \ , \\ (6b) & \int_0^{\hat{T}_\rho} \hat{r}_t dt = R \ .\end{cases}$$

The diagram illustrates the solution. For $\rho = 0$, (6) implies (5), and consumption of the resource is constant during survival. Its optimal level $\hat{r}$ is obtained in (5b,c) by balancing the number of years of survival against the constant level of utility flow that the total resource stock makes possible during survival. Since $\hat{r} > \underline{r}$, the optimum survival period $\hat{T}_0$ is shorter than the maximum $\overline{T}$ defined by (1).

For $\rho > 0$, the optimal path $\hat{r}_t$ follows a declining curve given by (6a), which starts from a level $\hat{r}_0$ such that, when resource exhaustion brings life to a stop at time $t = \hat{T}_\rho$, the level $\hat{r}_{\hat{T}_\rho} = \hat{r}$ is just reached. Since the decline is steeper when ρ is larger, the survival period is shorter, the larger is ρ - which explains the title of this note.

The intuitive argument already referred to gives insight into the theorem; the following proof establishes its validity.

Proof: We first consider paths optimal under the added constraint of some arbitrarily fixed value $T = T^*$ of T satisfying $0 < T^* < \overline{T}$. Assume that such a " T^* - optimal" path r_t^* exists and that

$$(7) \qquad r_t^* \geqq \underline{r} + \delta \quad \text{for } 0 \leqq t \leqq T^* \text{ and some } \delta > 0 \; .$$

Then, if s_t is a continuous function defined for $0 \leqq t \leqq T^*$ such that

$$(8) \qquad |s_t| \leqq \delta \quad , \qquad \int_0^{T^*} s_t dt = 0 \quad ,$$

the path

$$(9) \qquad r_t = r_t^* + \varepsilon s_t \quad , \qquad 0 \leqq t \leqq T^* \quad ,$$

is T^*-feasible for $|\varepsilon| \leqq 1$ and satisfies

$$(10) \begin{cases} (10a) \begin{cases} V(\rho, T^*, (r_t)) - V(\rho, T^*, (r_t^*) = \\ = \int_0^{T^*} e^{-\rho t}(v(r_t) - v(r_t^*))dt = \end{cases} \\ (10b) \begin{cases} \\ = \varepsilon \int_0^{T^*} e^{-\rho t} v'(r_t^*) s_t dt + R(\varepsilon) \quad , \end{cases} \end{cases}$$

where the remainder $R(\varepsilon)$ is of second order in ε. It is therefore a necessary condition for the T*-optimality of r_t^* that

(11) $$p_t \equiv e^{-\rho t} v'(r_t^*) = \text{constant} = e^{-\rho T^*} v'(r_{T^*}^*) \quad , \text{ say,}$$

because, if we had $p_{t'} \neq p_{t''}$, $0 \leqq t', t'' \leqq T^*$, we could by choosing s_t of one sign in a neighborhood in $[0,T^*]$ of t', s_t of the opposite sign in one of t'' and zero elsewhere while preserving (8) make the last member of (10) positive for some ε with $|\varepsilon| \leqq 1$.

In the light of (3a,b), (11) justifies our assumption that r_t^* is a continuous function of t. We now find that r_t^* is constant for $\rho = 0$, strictly decreasing for $\rho > 0$. Given $r_{T^*}^*$, say, the solution r_t^* of (11) is uniquely determined, and, for each t, r_t^* is a strictly increasing differentiable function of the given $r_{T^*}^*$. Also, by (3d),

$$\lim_{r_{T^*}^* \to \underline{r}} \int_0^T r_t^* dt = \int_0^T \underline{r} dt = T^* \underline{r} < \overline{T} \underline{r} = R \quad ,$$

whereas, for sufficiently large $r_{T^*}^*$,

$$\int_0^{T^*} r_t^* dt > R \quad .$$

Therefore there is a unique number $\alpha^* > \underline{r}$ such that the unique solution r_t^* of (11) with $r_{T^*}^* = \alpha^*$ satisfies

(12) $$\int_0^{T^*} r_t^* dt = R \quad .$$

From here on r_t^* will denote that path for the chosen T^*. Note that this path also satisfies (7).

To prove the unique T^*-optimality of r_t^*, let r_t be any T^*-feasible path such that $r_{t_0} \neq r_{t_0}^*$ for some $t_0 \varepsilon [0,T]$. Then, by the continuity of r_t, r_t^*, $r_t \neq r_t^*$ for all t in some neighborhood τ of t_0 in $[0,T^*]$. By (3b), for all $t\varepsilon[0,T^*]$,

(13) $$v(r_t) - v(r_t^*) \begin{bmatrix} < \\ \leqq \end{bmatrix} (r_t - r_t^*)\, v'(r_t^*) \quad \text{for} \quad t\varepsilon \begin{bmatrix} \tau \\ \tau^* \end{bmatrix} \quad ,$$

where $\tau^* \equiv [0,T^*] - \tau$. Therefore, we have from (10a), (11), (4b) with $T = T^*$, and (12) that

$$V(\rho,T^*,(r_t)) - V(\rho,T^*,(r_t^*)) =$$

$$= \left(\int_\tau + \int_{\tau^*}\right) e^{-\rho t}(v(r_t) - v(r_t^*))dt <$$

$$< \int_0^{T^*} (r_t - r_t^*)e^{-\rho t} v'(r_t^*)dt =$$

$$= \quad e^{-\rho T^*} v'(r_{T^*}^*) \int_0^{T^*} (r_t - r_t^*)dt \leqq 0 \quad .$$

Hence r_t^* is uniquely T^*-optimal.

We now make T^* a variable, writing T instead of T^* and r_t^T instead of r_t^*. Note that, for each t, $0 \leqq t < \overline{T}$, r_t^T is a differentiable function of T for $t \leqq T < \overline{T}$. Therefore

$$V_T \equiv V(\rho, T, (r_t^T)) = \int_0^T e^{-\rho t} v(r_t^T) dt$$

is a differentiable function of T for $0 \leqq T < \overline{T}$, and

$$\frac{dV_T}{dT} = e^{-\rho T} v(r_T^T) + \int_0^T e^{-\rho t} v'(r_t^T) \frac{dr_t^T}{dT} dt =$$

$$= e^{-\rho T} v(r_T^T) + e^{-\rho T} v'(r_T^T) \int_0^T \frac{dr_t^T}{dT} dt$$

by (11). But, by (12),

$$0 = \frac{dR}{dT} = r_T^T + \int_0^T \frac{dr_t^T}{dT} dt \quad .$$

Therefore,

$$e^{\rho T} \frac{dV_T}{dT} = v(r_T^T) - r_T^T \, v'(r_T^T) \quad .$$

But then, from (5b), since $\frac{d}{dr}(v(r) - rv'(r)) = -rv''(r) > 0$ for $r > 0$, by (3b),

$$\frac{dV_T}{dT} \begin{bmatrix} < \\ = \\ > \end{bmatrix} 0 \quad \text{for} \quad r_T^T \begin{bmatrix} < \\ = \\ > \end{bmatrix} \hat{r} \quad .$$

Finally, since $0 < T < T' < \bar{T}$ implies $r_{T'}^{T'} \leqq r_T^{T'} < r_T^T$,

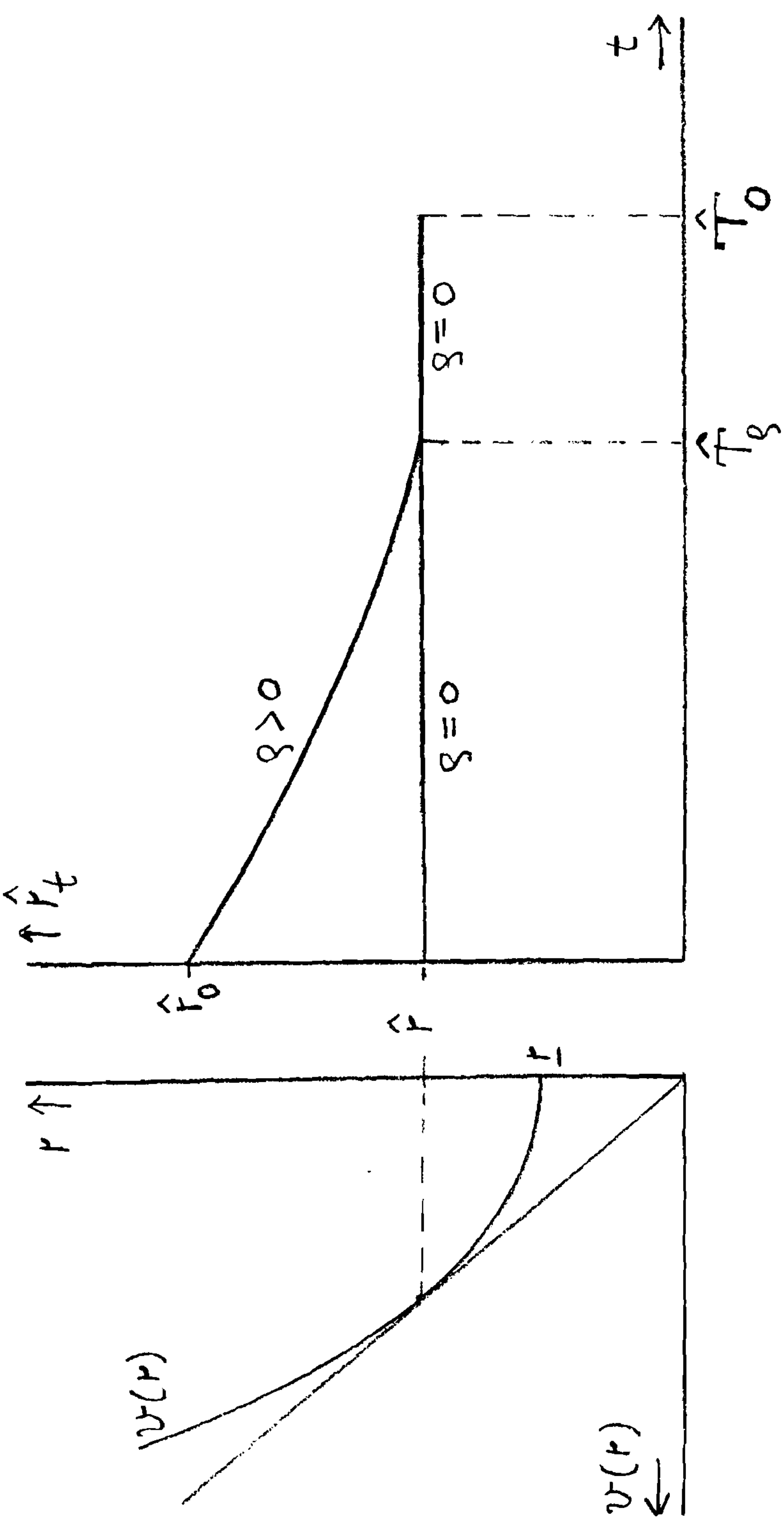
$$\frac{dV_T}{dT} \begin{bmatrix} < \\ = \\ > \end{bmatrix} 0 \quad \text{for} \quad T \begin{bmatrix} > \\ = \\ < \end{bmatrix} \hat{T}_\rho \quad .$$

Thus, V_T reaches its unique maximum for that value $\hat{T}_\rho$ of T for which $r_T^T = \hat{r}$.

This establishes the second part of the theorem. The first part follows by specialization when $\rho = 0$.

REFERENCE

Koopmans, T.C., "Some observations on 'optimal' economic growth and exhaustible resources", in Bos, Linnemann and de Wolff, Ed$^{\underline{s}}$, <u>Economic Structure and Development</u>, essays in honour of Jan Tinbergen, Holland Publishing Co., 1973, pp. 239-55.


r
v(r)
v(r)
r_1
$\hat{r}$
$\hat{r}_0$
$\hat{r}_t$
g>0
g=0
g=0
$\hat{T}_g$
$\hat{T}_0$
t