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FACILITY LOCATION WITH SPATIALLY INTERACTIVE TRAVEL BEHAVIOR

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FOREWORD

The public provision of urban facilities and services often takes the form of a few central supply points serving a large number of spatially dispersed demand points: for example, hospitals, schools, libraries, and emergency services such as fire and police. A fundamental characteristic of such systems is the spatial separation between suppliers and consumers. No market signals exist to identify efficient and inefficient geographical arrangements, thus the location problem is one that arises in both East and West, in planned and in market economies.

This problem is being studied at IIASA by the Public Facility Location Task, which started in 1979. The expected results of this Task are a comprehensive state-of-the-art survey of current theories and applications, an established network of international contacts among scholars and institutions in different countries, a framework for comparison, unification, and generalization of existing approaches, as well as the formulation of new problems and approaches in the field of optimal location theory.

This paper develops a spatial interaction model of facility location that adopts a gravity model specification of customer travel patterns. A model that can be solved by a nonlinear branch-and-bound algorithm is set out and several computational results arising out of applications of the model to shopping center location in Leeds, England, high school location in Turin, Italy, and hospital location in London, England are reported.

Related publications in the Public Facility Location Task are listed at the end of this report.

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ABSTRACT

This paper sets out a spatially interactive facility location model that specifies client travel behavior according to a "gravity" formula. The well-known uncapacitated facility location model is a limiting case of this model. Analytical partial optimization yields a condensed formulation that can be solved by a nonlinear branch-and-bound approach. Computational results are presented for several problems having as many as 69 potential facility locations.

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1. INTRODUCTION

Most of the literature on facility location models is concerned with systems in which spatial flows are assigned entirely to the nearest, or more generally minimum-cost, facility. This assignment rule clearly is optimal for typical plant and warehouse location problems where goods are delivered from the facility to the demand location and the costs for both establishing the facilities and transporting the goods are paid by the producer. The simplest such problem is the uncapacitated facility location problem, for which solution methods have progressed from the early branch-and-bound approach of Efroymson and Ray (1966) to much more effective dual-based approaches (Erlenkotter, 1978).

For many service facility location problems, clients are free to make their own choice of facility, and empirical evidence demonstrates that not all clients select the nearest facility. Actual travel from a given client location tends to be distributed among a number of facilities, and the relative location of facilities affects the distribution of travel. This form of spatially interactive travel behavior, often described through a spatial "gravity" model, has been discussed by Wilson (1971) and has led to locational criteria interpreted variously as "consumers' surplus" (see Neuberger, 1971; Coelho and Williams, 1978), "accessibility" (see Leonardi, 1978), and "entropy" (see Wilson, 1970). A plausible modeling foundation for this behavior has been provided through random utility theory, as discussed recently with reference specifically to location modeling by Leonardi (1981).

Spatially interactive travel behavior has been incorporated into several location models, including those of Coelho and Wilson (1976), Leonardi (1978), and Beaumont (1980). Recently Leonardi (1980) has proposed spatially interactive location models in which establishment of facilities is restricted to a subset of available locations either by a policy constraint or by the presence of fixed charges for opening facilities. These models are closely related to the uncapacitated facility location problem (UFLP). In this paper we develop a solution algorithm for such a model, and present the results of computational tests conducted to evaluate its effectiveness.

2. A FACILITY LOCATION MODEL WITH SPATIALLY INTERACTIVE TRAVEL BEHAVIOR

We begin with the following budget-constrained spatially interactive location problem:

$$\underset{\substack{s_{ij} \\ y_j \in \{0,1\}}}{\text{Minimize}} \quad \sum_{i\in I} \sum_{j\in J} s_{ij} (\alpha \ln s_{ij} + c_{ij}) \tag{1}$$

$$\text{subject to} \quad \sum_{j\in J} s_{ij} = P_i, \; i\in I \tag{2}$$

$$s_{ij} \leq P_i \, y_j, \; i\in I, \; j\in J \tag{3}$$

$$\sum_{j\in J} a_j \, y_j + \sum_{j\in J} b_j \sum_{i\in I} s_{ij} \leq B \tag{4}$$

where

I = set of residence zones for service clients, indexed $i = 1,2,\cdots,|I|$;

J = set of potential facility locations, indexed $j = 1,2,\cdots,|J|$;

s_{ij} = number of service clients in zone $i \varepsilon I$ that utilize facility at location $j \varepsilon J$;

y_j = $\begin{cases} 1 \text{ if a facility is established at location } j \varepsilon J \\ 0 \text{ if a facility is not established at location } j \varepsilon J; \end{cases}$

$P_i(>0)$ = number of service clients in zone $i \varepsilon I$;

$c_{ij}(\geq 0)$ = cost per client in zone $i \varepsilon I$ for service at facility $j \varepsilon J$;

$a_j(\geq 0)$ = fixed charge for opening a facility at $j \varepsilon J$;

$b_j(\geq 0)$ = variable capacity cost per unit of service at facility $j \varepsilon J$;

$B(\geq 0)$ = total budget for facilities;

$\alpha(>0)$ = inverse spatial discount (or distance decay) rate.

The objective function (1) is the negative of "consumers' surplus" benefits to clients and, as we shall see, implies client travel behavior as in a "gravity" model. Constraints (2) ensure that all clients are served, constraints (3) prevent service at a closed facility, and constraint (4) enforces a budget limit on facilities expenditures.

A more convenient formulation for solution is provided if we employ a Lagrangian relaxation of the budget constraint (4). One can also show, as in Erlenkotter (1980), that such a modified formulation can provide a solution that is preferable to the clients even if they must pay for a possible budget deficit through direct use fees. Employing the Lagrange multiplier $\lambda \geq 0$

for (4) and substituting $x_{ij} = s_{ij}/P_i$, we have

$$\underset{\substack{x_{ij} \\ y_j \in \{0,1\}}}{\text{Minimize}} \left\{ \sum_{i\in I} \sum_{j\in J} P_i \, x_{ij}(\alpha \ln x_{ij} + c_{ij} + \lambda b_j) + \sum_{j\in J} \lambda a_j \, y_j \right\}$$

$$+ \; \alpha \sum_{i\in I} P_i \ln P_i - \lambda B \tag{5}$$

subject to

$$\sum_{j\in J} x_{ij} = 1 \; , \qquad i\in I \tag{6}$$

$$x_{ij} \leq y_j \; , \qquad i\in I \; , \qquad j\in J. \tag{7}$$

The formulation (5) - (7) is now expressed in terms of the fraction of clients at i served at j, x_{ij}. If we add the generally redundant constraints $x_{ij} \geq 0$ and examine the limiting case for $\alpha = 0$ (corresponding to an infinite spatial discount rate), we obtain the classical UFLP of Efroymson and Ray (1966) with total "shipping" costs from i to j of $P_i(c_{ij} + \lambda b_j)$ and facility fixed charges at j of λa_j.

3. CONDITIONAL OPTIMIZATION OF FLOW VARIABLES

The success of solution approaches for the UFLP suggests that we attempt to solve (5) - (7) by relaxing the integrality constraints on the variables y_j. For that problem, such relaxed solutions often are naturally integer as demonstrated in Erlenkotter (1978). Even when non-integer solutions result, bounds on the objective value provided by the relaxed solution are useful in determining a solution by means of a branch-and-bound procedure.

Here, however, it is obvious that such a relaxed solution will never be integral because all x_{ij} will be positive and no x_{ij} will equal one.

Therefore __all__ y_j will be fractional since each will equal the largest corresponding x_{ij}. Solution of (5) - (7) via such a relaxation would not seem to offer much promise.

However, we can simplify (5) - (7) by optimizing the flow variables x_{ij} conditionally on the site opening variables y_j. Making the substitutions

$$d_{ij} = e^{-(c_{ij} + \lambda b_j)/\alpha} \leq 1$$

and

$$f_j = (\lambda/\alpha)a_j$$

yields the equivalent problem

$$\underset{\substack{x_{ij} \\ y_j \in \{0,1\}}}{\text{Minimize}} \sum_{i\epsilon I} \sum_{j\epsilon J} P_i x_{ij} \ln (x_{ij}/d_{ij}) + \sum_{j\epsilon J} f_j y_j \tag{8}$$

subject to

$$\sum_{j\epsilon J} x_{ij} = 1 \quad , \qquad i\epsilon I \tag{9}$$

$$x_{ij} \leq y_j \quad , \qquad i\epsilon I \quad , \qquad j\epsilon J. \tag{10}$$

Conditional optimization with respect to x_{ij}, assuming some nonzero y_j, yields two cases:

$$y_j = 0 \Rightarrow x_{ij} = 0 \quad , \quad i\epsilon I$$

$$y_j = 1 \Rightarrow x_{ij} = d_{ij}\, e^{(v_i/P_i)-1} \quad , \quad i\epsilon I$$

where the v_i are Lagrange multipliers for constraints (9). Combining the two cases gives

$$x_{ij} = y_j\, d_{ij}\, e^{(v_i/P_i)-1} \quad . \tag{11}$$

Substituting (11) into (9) yields

$$e^{(v_i/P_i)-1} = \frac{1}{\sum_{j\varepsilon J} y_j\, d_{ij}}$$

and hence in (11)

$$x_{ij} = \frac{y_j\, d_{ij}}{\sum_{j\varepsilon J} y_j\, d_{ij}} \leq 1 \tag{12}$$

which satisfies naturally the constraints (10). The form (12) is a "gravity" expression for the flow variables x_{ij} since the coefficients d_{ij} are spatially discounted distance measures. Substitution from (12) into the objective function (8) yields

$$\underset{y_j\,\varepsilon\,\{0,1\}}{\text{Minimize}} \sum_{i\varepsilon I} \sum_{j\varepsilon J} P_i \frac{d_{ij} y_j \ln\, y_j}{\sum_{j\varepsilon J} y_j\, d_{ij}} - \sum_{i\varepsilon I} P_i \ln(\sum_{j\varepsilon J} y_j d_{ij}) \sum_{j\varepsilon J} x_{ij} + \sum_{j\varepsilon J} f_j y_j \tag{13}$$

which, since $y_j \ln y_j = 0$ for $y_j = 0$ or 1, becomes

$$\underset{y_j\,\varepsilon\,\{0,1\}}{\text{Minimize}} \sum_{j\varepsilon J} f_j y_j - \sum_{i\varepsilon I} P_i \ln(\sum_{j\varepsilon J} y_j d_{ij}) \, . \tag{14}$$

The rather succinct form of (14), then, would seem to be more promising for solution.

4. A BRANCH-AND-BOUND SOLUTION PROCEDURE

The simple objective form (14) has several useful characteristics. First, it is easily shown that this function is convex in the variables y_j. Solution of the continuous relaxation of (14) with $0 \leq y_j \leq 1$ therefore may be carried out by standard convex optimization techniques, and this solution will provide a lower bound for the optimal objective value. Second, the negative of this function is submodular as defined by Nemhauser, Wolsey, and Fisher (1978). The submodularity property can be exploited in branch-and-bound search algorithms, and this approach has been explored in Leonardi

(1981). However, such an approach employs only function value calculations, and Nemhauser and Wolsey (1978) have shown that such a "black box" algorithm has significant theoretical limits on its efficiency, at least in worst-case situations. Here, therefore, we shall develop an approach that uses the continuous relaxation of (14):

$$\underset{0 \le y_j \le 1}{\text{Minimize}} \sum_{j\varepsilon J} f_j y_j - \sum_{i\varepsilon I} P_i \ln \left(\sum_{j\varepsilon J} y_j d_{ij} \right) . \qquad (15)$$

The problem (15) is a simple convex programming problem constrained only by bounds on variables, and many solution approaches could be applied. We believe that the method of Frank and Wolfe (1956), also known as the "method of convex combinations" (Wagner, 1975), has several advantages, at least for initial solution steps:

(a) The Frank-Wolfe method always moves in the direction of extreme points, which are integer solutions here. These are precisely the solutions that interest us, and each extreme point encountered provides an upper bound to the optimal solution value.

(b) Direction-finding problems for this approach can be solved by inspection, since they involve only the simple bounds on variables and a linearization of the objective function at the current feasible point.

(c) The Frank-Wolfe method has a rapid initial rate of convergence (Wolfe, 1970).

(d) Lower bounds on the optimal relaxed objective value are easily calculated from any trial feasible point.

The hope is that, if the relaxed solution is naturally integer, the Frank-Wolfe method will move quickly to that point and terminate. If this solution is not integer, we then must face the major drawback of the Frank-Wolfe method: its extremely poor asymptotic convergence behavior (Wolfe, 1970). The method can be modified to improve its asymptotic performance (Wolfe, 1970; Holloway, 1974), but instead we have chosen to switch over to a variant of the cyclic coordinate descent method (Luenberger, 1973) to complete calculation of the optimal relaxed solution.

The linearization of the objective function (15) at a current solution y^0 is obtained from the gradient components

$$z_j(y^0) = f_j - \sum_{i \varepsilon I} \frac{P_i \, d_{ij}}{\sum_{k \varepsilon J} y_k^0 \, d_{ik}} . \tag{16}$$

The linearized problem for the Frank-Wolfe approach is then

$$\underset{0 \leq y_j \leq 1}{\text{Minimize}} \quad \sum_{j \varepsilon J} y_j \, z_j(y^0). \tag{17}$$

The solution to (17) is given by

$$y_j^* = \begin{cases} 1 & \text{if } z_j(y^0) < 0 \\ 0 & \text{otherwise.} \end{cases} \tag{18}$$

A new solution point is then provided by $y^0 + t^* (y^* - y^0)$, where t^* is the solution to the step-finding problem

$$\underset{0 \leq t \leq 1}{\text{Minimize}} \left\{ \sum_{j \varepsilon J} f_j \, [ty_j^* + (1-t) \, y_j^0] - \sum_{i \varepsilon I} P_i \ln \sum_{j \varepsilon J} d_{ij} \, [ty_j^* + (1-t) \, y_j^0] \right\} . \tag{19}$$

The step size is easily optimized in (19) via Newton's method.

A convenient initial starting point is derived by assuming that all y_j^0 are equal; setting $\sum_{j\epsilon J} z_j(y^0) = 0$ we then obtain from (16)

$$y_j^0 = \text{minimum} \left\{ 1, \frac{\sum_{i\epsilon I} P_i}{\sum_{j\epsilon J} f_j} \right\}.$$

If the relaxed continuous solution is non-integer, we proceed via a branch-and-bound approach using the solutions to (15) as lower bounds. This procedure is standard and rather elementary, and we provide just a brief sketch of its details.

(a) As described by Geoffrion (1967), an elementary backtracking scheme with last-in, first-out processing of nodes is used to minimize computer storage requirements and simplify updating of solutions.

(b) For branching, we use a second-order Taylor series approximation of the objective function to estimate the degradation of the objective value from altering a fractional variable y_j^0. Given the current objective value $z(y^0)$, first derivative $z_j(y^0) = 0$ at a continuous optimum, and second derivative $z_{jj}(y^0)$, we have

$$z(y) - z(y^0) = (1/2)\ z_{jj}(y^0)(y_j - y_j^0)^2. \tag{20}$$

The strategy is to select the variable and integer value providing the maximum estimated degradation according to (20), and to branch initially in the direction opposite this value.

(c) Optimization of the partially relaxed problem at each node is carried out via the Frank-Wolfe method until the objective value improvement is less than 0.1%, and is completed by cyclic coordinate descent.

(d) Fathoming of nodes is either by bounding or by integer feasibility.

In some preliminary tests, an alternative branching strategy was tried in (b) of branching on the variable and value from (20) providing the minimum degradation. This strategy is intended to seek a good solution as quickly as possible, while the one in (b) attempts to identify branches that may be bounded off as soon as possible. The alternative strategy gave very poor results, taking more than ten times the total computational effort of that in (b) in some instances. This behavior is consistent with the findings for integer linear programming problems reported in Geoffrion and Marsten (1972).

5. COMPUTATIONAL RESULTS

The algorithm for the spatial interaction facility location model has been developed into a FORTRAN IV computer code named INTLOC. Preliminary computational testing was carried out on the PDP 11/70 computer at IIASA. This testing was conducted in an extremely congested interactive environment, and computational times were too unreliable to report here. However, the results obtained are sufficient to provide a tentative evaluation of the approach.

The first problem investigated is the 8 × 8 shopping center location problem developed by Coelho and Wilson (1976) for the city of Leeds, England. Coelho and Wilson ignored fixed charges in the cost structure for shopping centers but included sizes of existing centers. Here we try different levels of fixed charges but assume no existing centers. Results from INTLOC for this problem are given in Table 1. At the lowest value for the fixed charge,

Table 1. Computational results for Coelho-Wilson 8 × 8 location problem

Fixed Charge	Optimal number of facilities	Initial fractional facilities	Optimal objective value	Relaxed objective value	Tree search: Maximum depth	Tree search: Total nodes	Total iterations: Frank-Wolfe	Total iterations: Cyclic descent
50	8	0	12106.2	12106.2	0	1	1	0
100	5	0	12448.8	12448.8	0	1	2	0
150	4	0	12682.2	12682.2	0	1	2	0
200	4	1	12882.2	12881.1	1	3	6	0
250	4	1	13082.2	13063.6	1	3	6	2
300	3	2	13244.5	13224.7	2	5	10	2
350	3	2	13394.5	13363.5	2	5	10	3
400	2	2	13508.8	13486.6	2	5	9	3
450	2	1	13608.8	13599.3	1	3	5	0
500	2	1	13708.8	13706.3	1	3	5	0
550	2	1	13808.8	13808.8	1	3	5	0
600	2	0	13908.8	13908.8	0	1	2	0

centers are opened at all eight sites; at the higher values, just two sites are utilized.

This problem was solved quite easily in all cases, probably because never more than two facility opening variables were fractional in the initial continuous solution. In several cases the initial continuous solution was integer and optimal, indicated in Table 1 by a single node and maximum tree depth of zero in the tree search. As a consequence of the small number of fractional variables, virtually all the optimization of the relaxed problem was accomplished with Frank-Wolfe iterations, and the cyclic coordinate descent phase was used in just four cases.

The second problem is a 23 × 23 high school location problem for Turin, Italy taken from Leonardi (1981). Results for this problem are given in Table 2. For most cases of this problem, the number of fractional variables in the initial continuous solution was quite large, and the solution process was lengthy. The most difficult case was for a fixed charge of 3000, where the maximum depth of the search tree was just one less than the maximum possible depth of 23. The number of Frank-Wolfe iterations tends to be about three times the total number of nodes, and the number of cyclic descent iterations is as high as three times the number of Frank-Wolfe iterations for these problems.

The final problem is a 44 × 69 hospital location problem for the London region as described in Mayhew and Taket (1980). Results for this problem are given in Table 3. Even though much larger than the previous problem, solution requirements here were rather modest. The number of fractional variables in the initial continuous solution tends to remain small relative to the total number of potential facilities, and the optimal objective value remains very close to the initial lower bound provided by the initial relaxed objective value. However, the solution effort increases with the value of the fixed

Table 2. Computational results for Leonardi 23 × 23 location problem

Fixed Charge	Optimal number of facilities	Initial fractional facilities	Optimal objective value	Relaxed objective value	Tree search: Maximum depth	Tree search: Total nodes	Total iterations: Frank-Wolfe	Total iterations: Cyclic descent
500	23	0	25899.2	25899.2	0	1	1	0
1000	23	1	37399.2	37257.1	1	3	4	0
1500	22	10	48685.0	47924.9	9	29	76	66
2000	20	14	59000.0	56947.9	18	153	418	512
2500	17	17	68096.8	64389.7	18	881	2483	4118
3000	15	21	76430.0	70583.2	22	5169	13844	29100
3500	11	20	82725.5	75809.6	20	3193	9337	26154
4000	10	20	87921.4	80326.1	19	1963	6097	17561
4500	8	21	92730.7	84300.9	20	1887	6020	17841
5000	8	21	96730.7	87845.0	20	1481	4976	15020

Table 3. Computational results for Mayhew and Taket 44 x 69 location problem.

Fixed Charge	Optimal number of facilities	Initial fractional facilities	Optimal objective value	Relaxed objective value	Tree search		Total iterations	
					Maximum depth	Total nodes	Frank-Wolfe	Cyclic descent
5000	68	1	6363018	6363009	1	3	5	3
10000	62	4	6693082	6692755	4	9	14	8
15000	56	4	6985426	6985036	5	17	31	31
20000	50	8	7247814	7245861	11	45	80	189
25000	44	12	7477704	7474196	19	79	157	294

charge, and problems with higher fixed charges could be more difficult to solve.

6. CONCLUSIONS

We have developed and tested an algorithm for solving a spatial interaction facility location model. Since the spatial flow variables may be optimized analytically, the solution approach considers explicitly only the facility opening variables. The algorithm seems to be quite successful for some problems and rather tedious for others.

Since this class of problems is known to be a difficult one, it is not surprising that some cases are hard to solve. An important characteristic seems to be the number of fractional variables in the initial continuous solution. It is easy to construct examples in which all variables initially will be fractional: e.g., for equal client populations $P_i \equiv P$ for all i and a sufficiently high uniform fixed charge let $d_{jj} = 1$ for all j and $d_{ij} = K < 1$ for all $i \neq j$. Such problems would seem to be inherently difficult for this approach. Even though other nonlinear optimization approaches or branch-and-bound strategies could be tried, it is not clear that they could overcome this difficulty.

Perhaps more puzzling is the relatively greater success of dual-based solution approaches for the uncapacitated facility location problem (Erlenkotter, 1978), which is a limiting case of the problem explored here. Naturally integer solutions to the continuous relaxation of that problem seem to be much more prevalent than for the examples solved here. In the dual-based approaches, dual variables are eliminated by preliminary optimization in contrast to the elimination of primal variables here. But no natural extention of those approaches to the spatial interaction model seems evident.

Given the difficulty of obtaining proven optimal solutions for some instances of this problem, the need for good heuristics is apparent. Nemhauser, Wolsey, and Fisher (1978) have provided some general results for heuristics that could be applied to this problem, and Leonardi (1981) has developed some heuristics and applied them to the Turin high school location problem with very encouraging results. Further exploration of these approaches seems desirable.

Beyond the problem considered here, there is need to extend the approaches to include additional problem aspects such as existing facility capacities and limits on the capacities of new facilities. Some of these extensions have been explored recently by Coelho (1980).

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