DYNAMICS OF MULTIREGIONAL POPULATION SYSTEMS: A MATHEMATICAL ANALYSIS OF THE GROWTH PATH

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FOREWORD

Interest in human settlement systems and policies has been a central part of urban-related work at IIASA since its inception. From 1975 through 1978 this interest was manifested in the work of the *Migration and Settlement Task*, which was formally concluded in November 1978. Since then, attention has turned to dissemination of the Task's results and to the conclusion of its comparative study, which is carrying out a comparative quantitative assessment of recent migration patterns and spatial population dynamics in all of IIASA's 17 NMO countries.

This paper sets out the mathematics of multiregional stable growth theory. It presents an analytical solution that describes a multiregional population's growth path in terms of eigenvalues and eigenvectors.

Reports summarizing previous work on migration and settlement at IIASA are listed at the back of this paper.

Andrei Rogers
Chairman
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ABSTRACT

The multiregional population projection models can be rewritten in terms of eigenvalues and eigenvectors and an analytical solution can be obtained using coefficients that are determined by two different methods. The growth path can then be decomposed showing that it may be divided into five stages. These procedures are discussed in this paper and are illustrated with data for three regions in Belgium: Brussels, Flanders, and Wallonia.

CONTENTS

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INTRODUCTION

The process of multiregional demographic change may be represented as a matrix multiplication, or equivalently, as a system of simultaneous first-order linear difference equations (Rogers 1968, 1975). The advantages of this model are not only that it is compact, but also that it enables the separation of the fundamental components of population changes from the population to which these changes apply, thus allowing for a clearer view of the intrinsic characteristics of a particular growth structure.

The purpose of this paper is to investigate the growth path of a multiregional population. The growth model projects the population into the future, taking into account all interdependencies between the regions. But some of the complexities of multiregional population growth are hidden in the model and can only be revealed by looking at the growth path from a different perspective. As an ordinary light ray may be decomposed into the individual colors by using a prism, the demographic growth path may be decomposed into individual independent subtrajectories by using some mathematical manipulation. The observed growth path is then simply a sum of these individual trajectories.

The decomposition involves the rewriting of the conventional model of multiregional population change in terms of the eigenvectors and eigenvalues of the growth matrix. It implies a change of the coordinate system in which the population distribution vector is expressed. The result is a set of independent equations that replace the simultaneous equation system.

The discussion begins with the conventional growth models rewritten in terms of eigenvectors and eigenvalues; it yields the analytical solution to the growth model. The coefficients of this analytical solution are then determined with the aid of the z-transform and the left-eigenvector methods. These mathematical concepts and techniques are applied to decompose the growth paths of a population disaggregated by region and of a population disaggregated by age and region. The procedure is illustrated with data from three regions in Belgium: Brussels, Flanders, and Wallonia.

1. ANALYTICAL SOLUTION OF THE DEMOGRAPHIC GROWTH PATH

Multiregional demographic change may be represented by the following matrix model (Rogers 1968, 1975):

$$\{k(t+1)\} = \underset{\sim}{G}\{k(t)\} \tag{1}$$

where

$\{k(t)\}$ is an n-dimensional vector denoting the population distribution by region (and age) at time t

$\underset{\sim}{G}$ is the growth matrix

Since the growth matrix is constant, the system described by (1) is said to be time-invariant. The general solution, which expresses the state vector $\{k(t)\}$ at time t in terms of the initial condition is

$$\{k(t)\} = \underset{\sim}{G}^t \{k_o\} = \underset{\sim}{\phi}(t,0) \{k_o\} \qquad (2)$$

where

$\{k_o\}$ is the population distribution in the year

$\underset{\sim}{\phi}(t,0)$ is the state-transition matrix*

The purpose of this section is to characterize the solution (2). This can be done by decomposing (2) in n independent equations or by describing $\underset{\sim}{G}^t$ in terms of some fundamental and demographically meaningful parameters. To do this, we rewrite (2) in terms of the eigenvectors and eigenvalues of $\underset{\sim}{G}$; in other words, we derive a different type of solution to (1).

To obtain an analytical solution to (1), we first assume a solution vector and then derive the conditions that must be satisfied for the solution vector to solve the system. This is the usual practice in differential and difference calculus (see e.g. McFarlane, 1970).

Assume that (1) has the following solution:

$$\{k(t)\} = \lambda^t\{\xi\} \qquad (3)$$

where λ and $\{\xi\}$ are independent of time. Introducing (3) into (1) gives

$$\{k(t + 1)\} = \underset{\sim}{G}[\lambda^t\{\xi\}]$$

Also, (3) gives:

$$\{k(t + 1)\} = \lambda[\lambda^t\{\xi\}]$$

* In the early literature, it was referred to as the matricant (Gantmacher, 1959).

For (3) to solve (1), we must have

$$\underset{\sim}{G}\{\xi\} = \lambda\{\xi\}$$

or (4)

$$[\underset{\sim}{G} - \lambda\underset{\sim}{I}]\{\xi\} = \{0\}$$

Equation (4) is the characteristic equation. It has a nonzero solution vector $\{\xi\}$, if the determinant $|\underset{\sim}{G} - \lambda\underset{\sim}{I}| = 0$. This holds if λ is an eigenvector of $\underset{\sim}{G}$. Hence, the solution of (1) takes the form of (3) if and only if λ is an eigenvalue of $\underset{\sim}{G}$ and $\{\xi\}$ is the associated right eigenvector. The scalar proportionality factor λ denotes that a solution to (1) exists if $\{k(t + 1)\}$ and $\{k(t)\}$ have the same direction in the state space but only differ in magnitude.

Note that there are as many solutions as there are different values of λ for which the determinant $|\underset{\sim}{G} - \lambda\underset{\sim}{I}|$ is zero (and hence $\{\xi\}$ is not zero). Denote the various values of λ by the subscript i. With each value λ_i, there is associated a vector $\{\xi_i\}$. The matrix $\underset{\sim}{G}$ has now the important property that if all the eigenvalues λ_i are distinct, the eigenvectors $\{\xi_i\}$ are linearly independent. They describe therefore the solution (vector) space of dimension n. In other words, the eigenvector set $[\{\xi_1\},\{\xi_2\},\{\xi_3\}\cdots\cdots\{\xi_n\}]$ may be taken as the basis of a new coordinate system. Hence, we call the set of vectors $\{\xi_i\}$ the basis or basic solutions. In our numerical illustration, the observed population vector $\{k_0\}$ has three elements, each of which may be thought of as referring to a dimension. The observed population vector denotes, therefore, a point in the three-dimensional space, spanned by the basic vectors.

Any solution to (1) can be expressed in terms of the basis or coordinate system.* For instance, the state vector $\{k(t)\}$, i.e., the population distribution at time t, may be expressed as a linear combination of the eigenvector set of $\underset{\sim}{G}$ as

$$\{k(t)\} = \sum_{i=1}^{n} \bar{c}_i(t)\{\xi_i\} \tag{5}$$

The coefficients $\bar{c}_i(t)$ are functions of time and have to be determined. They consist of two components. One is a time-independent parameter c_i, the other is function of time λ_i^t. The coefficients of the linear transformation also have particular demographic interpretations. Before determining these coefficients in Section 2, we define a particular matrix to be used later.

Define the n x n matrix $\underset{\sim}{\Xi}$ such that $\{\xi_i\}$ is the i-th column:

$$\underset{\sim}{\Xi} = [\{\xi_1\}\{\xi_2\}\{\xi_3\}\cdots\cdots\{\xi_n\}]$$

or

$$\underset{\sim}{\Xi} = \begin{bmatrix} \xi_{11} & \xi_{12} & \xi_{13} & \cdots\cdots & \xi_{1n} \\ \xi_{21} & \xi_{22} & \xi_{23} & \cdots\cdots & \xi_{2n} \\ \xi_{31} & \xi_{32} & \xi_{33} & \cdots\cdots & \xi_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \xi_{n1} & \xi_{n2} & \xi_{n3} & \cdots\cdots & \xi_{nn} \end{bmatrix} \tag{6}$$

*The property that any solution vector may be expressed as a linear combination of a set of n linearly independent solution vectors is known as the Principle of Superposition. (McFarlane, 1970, p. 396). If $\{\xi_1\}$ and $\{\xi_2\}$ are independent solutions, then $\bar{c}_1\{\xi_1\} + \bar{c}_2\{\xi_2\}$ is also a solution.

The matrix $\underset{\sim}{\Xi}$ is called the fundamental matrix. It has the basic solution vectors as its columns. Since these vectors are linearly independent, the fundamental matrix is nonsingular. In this particular case where the columns of $\underset{\sim}{\Xi}$ are eigenvectors, the fundamental matrix is a modal matrix. The modal matrix will be used in the next section to describe the solution to (1) as n independent equations.

Combining (4) and (6) gives the expression

$$\underset{\sim}{G}\,\underset{\sim}{\Xi} = \underset{\sim}{\Xi}\,\underset{\sim}{\Lambda} \tag{6'}$$

where $\underset{\sim}{\Lambda}$ is the diagonal matrix of the eigenvalues of $\underset{\sim}{G}$; also known as the spectral matrix, since it contains the spectrum of $\underset{\sim}{G}$. These eigenvalues or roots of the characteristics equation are distinct in demographic applications. By (6'), we have that $\underset{\sim}{G} = \underset{\sim}{\Xi}\,\underset{\sim}{\Lambda}\,\underset{\sim}{\Xi}^{-1}$ and

$$\underset{\sim}{G}^{t} = \underset{\sim}{\Xi}\,\underset{\sim}{\Lambda}^{t}\,\underset{\sim}{\Xi}^{-1} \tag{6''}$$

This is a first expression of $\underset{\sim}{G}$ in terms of its eigenvalues and eigenvectors*. Other expressions will be derived in the next section.

2. DETERMINATION OF THE COEFFICIENTS OF THE ANALYTICAL SOLUTION

It has been shown that each population distribution at time t may be expressed as a linear combination of the right eigenvectors of $\underset{\sim}{G}$:

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* A similar expression may be derived using the left eigenvectors $\{v_j\}'$. If the eigenvectors are normalized such that the product $\{v_j\}'\{\xi_i\} = 1$ if $i = j$ and zero if $i \neq j$, then the modal matrix obtained by grouping the left eigenvectors is simply $\underset{\sim}{\Xi}^{-1}$. In other words, the rows of $\underset{\sim}{\Xi}^{-1}$ are normalized left eigenvectors.

$$\{k(t)\} = \sum_{i=1}^{n} c_i \lambda_i^t \{\xi_i\} \tag{5'}$$

where λ_i is the i-th eigenvalue of $\underset{\sim}{G}$. The problem is to determine the coefficients of c_i.

An equivalent problem to determining the coefficients of (5') is to derive expressions for $\underset{\sim}{G}^t$ in terms of the eigenvectors. Both problems will be dealt with in this section.

The first approach to determining the coefficients of the analytical solution uses the z-transform. The second approach introduces the left eigenvectors.

a. Z-Transform

The determination of the coefficients of the linear combination (5') using the z-transform in population analysis is due to Liaw (1975).

The z-transform of (1) is

$$z\{K(z)\} - z\{k(0)\} = \underset{\sim}{G}\{K(z)\} \tag{7}$$

where $\{K(z)\}$ is an n x 1 vector representing the z-transform of $\{k(t)\}$. Solving for $\{K(z)\}$ gives;

$$\{K(z)\} = [z\underset{\sim}{I} - \underset{\sim}{G}]^{-1} z\{k(0)\} \tag{8}$$

The expression $[z\underset{\sim}{I} - \underset{\sim}{G}]^{-1} z$ is the z-transform of the state transition matrix $\underset{\sim}{\phi}(t) = \underset{\sim}{G}^t$. It may also be written as

$$[z\underset{\sim}{I} - \underset{\sim}{G}]^{-1} z = \frac{\mathrm{adj}[z\underset{\sim}{I} - \underset{\sim}{G}]}{|z\underset{\sim}{I} - \underset{\sim}{G}|} z \tag{9}$$

where $|\cdot|$ denotes the determinant and $\mathrm{adj}[\cdot]$ denotes the adjoint matrix. Note that $|z\underset{\sim}{I} - \underset{\sim}{G}| = 0$ if z takes the value λ_i, i.e., an eigenvalue of $\underset{\sim}{G}$. If the eigenvalues of $\underset{\sim}{G}$ are distinct, by partial fraction expansion we have:

$$\frac{\operatorname{adj}[z\underset{\sim}{I} - \underset{\sim}{G}]}{|z\underset{\sim}{I} - \underset{\sim}{G}|} = \frac{\underset{\sim}{A}_1}{z-\lambda_1} + \frac{\underset{\sim}{A}_2}{z-\lambda_2} + \frac{\underset{\sim}{A}_3}{z-\lambda_3} + \cdots \frac{\underset{\sim}{A}_n}{z-\lambda_n}$$

where

$$\underset{\sim}{A}_i = (z - \lambda_i) \left. \frac{\operatorname{adj}[z\underset{\sim}{I} - \underset{\sim}{G}]}{|z\underset{\sim}{I} - \underset{\sim}{G}|} \right|_{z = \lambda_i} \quad (10)$$

for i = 1, 2, ... n, and

$$|z\underset{\sim}{I} - \underset{\sim}{G}| = \prod_i (z - \lambda_i) \quad (11)$$

Hence,

$$\frac{(z - \lambda_i)}{|z\underset{\sim}{I} - \underset{\sim}{G}|} = \prod_{j \neq i} (z - \lambda_j)$$

Taking the inverse z-transform, (9) becomes

$$\{k(t)\} = \sum_{i=1}^{n} \underset{\sim}{A}_i \lambda_i^t \{k(0)\} \quad (12)$$

which is the spectral form of the solution of (1).

From (12) we derive an expression for $\underset{\sim}{G}^t$:

$$\underset{\sim}{G}^t = \sum_{i=1}^{n} \underset{\sim}{A}_i \lambda_i^t \quad (13)$$

b. <u>Introducing the left eigenvector</u>

Another way to arrive at expressions for c_i is to premultiply (5') by the <u>left</u> eigenvector $\{v_j\}'$ (see, for instance, McFarlane, 1970, p. 402). In demography this procedure has been used by Keyfitz (1968, pp. 55-62). Although appealing because of its demographic interpretation, this procedure is not always applicable. It requires the eigenvalues to be distinct and nonzero. Premultiplying (5') by $\{v_j\}'$ gives:

$$\{v_j\}'\{k(t)\} = \sum_{i=1}^{n} c_i \lambda_i^t \{v_j\}'\{\xi_i\} \tag{14}$$

and for the base period

$$\{v_j\}'\{k_o\} = \sum_{i=1}^{n} c_i \{v_j\}'\{\xi_i\} \tag{15}$$

Since $\{v_j\}$ and $\{\xi_i\}$ are orthogonal if $j \neq i$, their inner product is zero. Hence, $\{v_j\}'\{\xi_i\} = 0$, for $j \neq i$.

Therefore, for j = i = 1, (15) reduces to

$$\{v_1\}'\{k_o\} = c_1\{v_1\}'\{\xi_1\}$$

and

$$c_1 = \frac{\{v_1\}'\{k_o\}}{\{v_1\}'\{\xi_1\}}$$

In general, we may write

$$c_i = \frac{1}{d_i}\{v_i\}'\{k_o\} \tag{16}$$

where $d_i = \{v_i\}'\{\xi_i\}$ is a normalizing factor. If the left and right eigenvectors are normalized, their inner product is unity, i.e., $\{v_i\}'\{\xi_i\} = 1$, and $c_i = \{v_i\}'\{k_o\}$.

Substituting (16) into (5') yields

$$\{k(t)\} = \sum_{i=1}^{n} \frac{1}{d_i} \{v_i\}'\{k_o\}\lambda_i^t\{\xi_i\} \tag{17}$$

and

$$\{k(t)\} = \sum_{i=1}^{n} \frac{\lambda_i^t}{d_i} \{\xi_i\}\{v_i\}'\{k_o\} \tag{18}$$

Hence,

$$\underset{\sim}{G}^t = \sum_{i=1}^{n} \frac{\lambda_i^t}{d_i} \{\xi_i\}\{v_i\}' \tag{19}$$

The coefficients of (18) depend on two basic components: the initial population distribution $\{k_o\}$ and the left eigenvectors $\{v_i\}$. Note that eigenvectors and eigenvalues are independent of the initial population distribution and only depend on the elements of the growth matrix, $\underset{\sim}{G}$. The expression (18) contains considerable potential for demographic interpretations. For instance, it can be shown that the left eigenvector $\{v_1\}$ associated with the dominant eigenvalue denotes the regional distribution of the reproductive potential of the population. Hence, the product $\{v_1\}'\{k_o\}$ is the total reproductive value of the initial population [for a further discussion of the reproductive value, see Willekens (1977) and Rogers and Willekens (1978)].

A comparison of (18) with (12) shows that

$$\underset{\sim}{A}_i = \frac{1}{d_i} \{\xi_i\}\{v_i\}' = \frac{1}{d_i} \underset{\sim}{Z}_i \tag{20}$$

The matrix $\underset{\sim}{Z}_i = \{\xi_i\}\{v_i\}'$ is the <u>constituent matrix</u> or <u>spectral component</u>. (Lancaster, 1969, p. 63). It has the same features as $\underset{\sim}{A}_i$, to which it is proportional.

We have now three expressions for $\underset{\sim}{G}^t$ which are equivalent:

$$\underset{\sim}{G}^t = \underset{\sim}{\Xi}\,\underset{\sim}{\Lambda}^t\,\underset{\sim}{\Xi}^{-1} = \sum_{i=1}^{n} \underset{\sim}{A}_i\,\lambda_i^t = \sum_{i=1}^{n} \frac{\lambda_i^t}{d_i}\,\underset{\sim}{Z}_i$$

The constituent matrix may be expressed in terms of different matrix expressions:

(i) $\underset{\sim}{Z}_i = d_i \underset{\sim}{A}_i$

(ii) (Morgan, 1966),

$$\underset{\sim}{Z}_i = [\mathrm{tr}\ \underset{\sim}{R}(z)]^{-1}\,\underset{\sim}{R}(z)\Big|_{z = \lambda_i}$$

where tr denotes the trace of a matrix* and $\underset{\sim}{R}(z) = \mathrm{adj}\ (z\underset{\sim}{I} - \underset{\sim}{G})$

(iii) (Lancaster, 1969, p. 174),

$$\underset{\sim}{Z}_i = \prod_{j \neq i}^{n} [\lambda_j \underset{\sim}{I} - \underset{\sim}{G}] \Big/ \prod_{j \neq i} (\lambda_i - \lambda_j)$$

If $\underset{\sim}{G}$ is simple, then (Lancaster, 1969, p. 175),

$$\underset{\sim}{Z}_i = \frac{C(\lambda_i)}{\psi^{(1)}(\lambda_i)}$$

where $C(\lambda_i)$ is the reduced adjoint of $\underset{\sim}{G}$ and $\psi(\cdot)$ its minimal polynomial.

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*Note that the inner product $\{v_i\}'\{\xi_i\}$ is equal to the trace of the constituent matrix. It is equal to unity if the eigenvectors are normalized.

The constituent matrix has the following properties:

(i) The nonzero rows are left eigenvectors of $\underset{\sim}{G}$; the nonzero columns are right eigenvectors of $\underset{\sim}{G}$. Postmultiplying $[\underset{\sim}{G} - \lambda_i \underset{\sim}{I}]\{\xi_i\}$ with $\{v_i\}'$ gives

$$[\underset{\sim}{G} - \lambda_i \underset{\sim}{I}]\{\xi_i\}\{v_i\}' = \{\xi_i\}\{v_i\}'$$

Premultiplying $\{v_i\}'[\underset{\sim}{G} - \lambda_i \underset{\sim}{I}]$ with $\{\xi_i\}$ gives

$$\{\xi_i\}\{v_i\}'[\underset{\sim}{G} - \lambda_i \underset{\sim}{I}] = \{\xi_i\}\{v_i\}'$$

Therefore,

$$[\underset{\sim}{G} - \lambda_i \underset{\sim}{I}]\underset{\sim}{Z}_i = \underset{\sim}{Z}_i[\underset{\sim}{G} - \lambda_i \underset{\sim}{I}]$$

(ii) The rank is one. This is due to the fact that all eigenvectors associated with a given eigenvalue are linearly dependent. Hence, the columns of $\underset{\sim}{Z}_i$ are linearly dependent.

(iii) The constituent matrix is idempotent, i.e., $\underset{\sim}{Z}_i^2 = \underset{\sim}{Z}_i$ (i = 1,2...n). This implies (Lancaster, 1969, pp. 82-83):

- the eigenvalues are all equal to one or zero (if eigenvectors are initially normalized). If the eigenvectors are not normalized, the nonzero eigenvalue is equal to $d_i = tr[\{\xi_i\}\{v_i\}']$.
- $\underset{\sim}{Z}_i$ is simple, i.e., it is similar to a diagonal matrix of its eigenvalues.

(iv) The sum of the constituent matrices is the identity matrix

$$\sum_i \underset{\sim}{Z}_i = \underset{\sim}{I}$$

This can be seen by partitioning $\underset{\sim}{\Xi}\ \underset{\sim}{V}$ (where $\underset{\sim}{V} = \underset{\sim}{\Xi}^{-1}$, the model matrix of left eigenvectors) into vectors and by multiplying the vectors as if they were scalar elements (see also Keyfitz, 1968, p. 62).

3. GROWTH TRAJECTORY OF POPULATION DISAGGREGATED BY REGION: NUMERICAL ILLUSTRATION

Consider the components-of-change model for the three-region system Brussels, Flanders, and Wallonia (Willekens, 1979):

$$\{k(t + 1)\} = \begin{bmatrix} 0.969497 & 0.002615 & 0.004221 \\ 0.017749 & 1.000175 & 0.002383 \\ 0.012907 & 0.001435 & 0.993583 \end{bmatrix} \{k(t)\} \tag{21}$$

The growth matrix describes the pattern of change during one year (projection interval), hence, t + 1 = 1971. The initial population distribution (in 1970) is:

$$\{k_o\} = \begin{bmatrix} 1\ 079\ 520 \\ 5\ 386\ 158 \\ 3\ 155\ 988 \end{bmatrix} = 9621666 \begin{bmatrix} 0.112197 \\ 0.559795 \\ 0.328008 \end{bmatrix} \tag{21'}$$

We derive the analytical solution to this equation system using the z-transform and the left eigenvector.

a. <u>Analytical solution using z-transform</u>

To find the analytical solution of (21) in the form of equation (12), we must first compute the constituent matrices $\underset{\sim}{A}_i$. Recall that

$$\underset{\sim}{A}_i = (z - \lambda_i) \left. \frac{\text{adj}[z\underset{\sim}{I} - \underset{\sim}{G}]}{|z\underset{\sim}{I} - \underset{\sim}{G}|} \right|_{z = \lambda_i} , \qquad \text{for } i = 1,2,\ldots n$$

The determinant $|z\underset{\sim}{I} - \underset{\sim}{G}|$ is equal to

$$|z\underset{\sim}{I} - \underset{\sim}{G}| = \begin{vmatrix} z-0.9695 & -0.0026 & -0.0042 \\ -0.0177 & z-1.0002 & -0.0024 \\ -0.0129 & -0.0014 & z-0.9936 \end{vmatrix}$$

$$= (z-1.00301)(z-0.99393)(z-0.96632)$$

The eigenvalues of $\underset{\sim}{G}$ are solutions to the equation $|z\underset{\sim}{I} - \underset{\sim}{G}| = 0$, hence

$$\lambda_1 = 1.00301$$

$$\lambda_2 = 0.99393$$

$$\lambda_3 = 0.96632$$

All eigenvalues are real. Note that the sum of the eigenvalues is equal to the trace of $\underset{\sim}{G}$ (sum of diagonal elements).

The adjoint matrix adj$[z\underset{\sim}{I} - \underset{\sim}{G}]$ is equal to the transpose of the cofactor matrix cof$[z\underset{\sim}{I} - \underset{\sim}{G}]$, which is derived by replacing each element h_{ij} of the matrix $\underset{\sim}{H} = [z\underset{\sim}{I} - \underset{\sim}{G}]$ by its cofactor H^c_{ij} (Rogers, 1971, p. 82).

The coefficient matrices are equal to

$$A_i = \frac{1}{t_i} \text{adj}[z\underset{\sim}{I} - \underset{\sim}{G}] \Big|_{z = \lambda_i} \tag{22}$$

where $t_i = \prod_{j \neq i} (z - \lambda_j) \Big|_{z = \lambda_i}$ or $t_i = \text{tr}[\text{adj}(z\underset{\sim}{I} - \underset{\sim}{G})] \Big|_{z = \lambda_i}$

In the numerical illustration, the values of t_i are:

$$t_1 = (1.00301-0.99393)(1.00301-0.96632)$$

$$= 0.000333$$

$$t_2 = (0.99393-1.00301)(0.99393-0.96632)$$

$$= -0.000251$$

$$t_3 = (0.96632-1.00301)(0.96632-0.99393)$$

$$= 0.001013$$

The adjoint matrices $\text{adj}(\lambda_i \underset{\sim}{I} - \underset{\sim}{G})$ are computed using the improved Leverrier algorithm (Faddeev and Faddeeva, 1963, pp. 260-265). The algorithm which yields simultaneously the coefficients of the characteristic polynomial and the adjoint matrices, is described in the Appendix 1 (see also Willekens, 1975).

The adjoint matrices are equal to

$$\text{adj}(\lambda_1 \underset{\sim}{I} - \underset{\sim}{G}) = \begin{bmatrix} 0.000023 & 0.000031 & 0.000018 \\ 0.000198 & 0.000262 & 0.000155 \\ 0.000062 & 0.000082 & 0.000049 \end{bmatrix}$$

$$\text{adj}(\lambda_2 \underset{\sim}{I} - \underset{\sim}{G}) = \begin{bmatrix} -0.000006 & 0.000007 & -0.000020 \\ 0.000037 & -0.000046 & 0.000133 \\ -0.000055 & 0.000069 & -0.000199 \end{bmatrix}$$

$$\text{adj}(\lambda_3 \underset{\sim}{I} - \underset{\sim}{G}) = \begin{bmatrix} 0.000920 & -0.000065 & -0.000137 \\ -0.000453 & 0.000032 & 0.000067 \\ -0.000411 & 0.000029 & 0.000061 \end{bmatrix}$$

Note that the values of t_i are equal to the traces (sum of diagonal elements) of the adjoint matrices.

Substituting (22) into (12) yields the analytical solution to (21):

$$\{k(t)\} = (1.00301)^t \begin{bmatrix} 0.07014 & 0.09218 & 0.05463 \\ 0.59456 & 0.78507 & 0.46460 \\ 0.18628 & 0.24567 & 0.14599 \end{bmatrix}$$

$$+ (0.99393)^t \begin{bmatrix} 0.02235 & -0.02778 & 0.08029 \\ -0.14724 & 0.18307 & -0.53108 \\ 0.21993 & -0.27448 & 0.79362 \end{bmatrix}$$

$$+ (0.96632)^t \begin{bmatrix} 0.90783 & -0.06440 & -0.13492 \\ -0.44731 & 0.03183 & 0.06648 \\ -0.40621 & 0.02882 & 0.06043 \end{bmatrix} \begin{bmatrix} 1\ 079\ 520 \\ 5\ 386\ 158 \\ 3\ 155\ 988 \end{bmatrix}$$

Note that $\sum_{i=1}^{n} \underset{\sim}{A}_i \lambda_i^t$ is equal to $\underset{\sim}{G}^t$.

b. Analytical solution using left eigenvector

The left and right eigenvectors of $\underset{\sim}{G}$, associated with the different eigenvalues, are given in Table 1. The eigenvectors are normalized such that their inner product equals unity. (Hence the modal matrix of left eigenvectors is the inverse of the modal matrix of right eigenvectors.)

Table 1. Eigenvalues and Eigenvectors of the Multiregional Population Growth Matrix, $\underset{\sim}{G}$.

	Eigenvalues					
	$\lambda_1 = 1.00301$		$\lambda_2 = 0.99393$		$\lambda_3 = 0.96632$	
Region	Eigenvectors					
	Left	Right	Left	Right	Left	Right
Brussels	0.85068	0.08212	0.38950	0.05715	1.76154	0.51539
Flanders	1.12250	0.69897	-0.48599	-0.37792	-0.12497	-0.25398
Wallonia	0.66486	0.21891	1.40562	0.56493	-0.26180	-0.23063

The coefficients c_i of the analytical solution (17) are:

$$c_1 = \{v_1\}'\{k_o\} = 918,326 + 6,045,962 + 2,098,290$$

$$= 9.062,579$$

$$c_2 = \{v_2\}'\{k_o\} = 420,473 - 2,617,619 + 4,436,120$$

$$= 2,238,974$$

$$c_3 = \{v_3\}'\{k_o\} = 1,901,618 - 673,108 - 826,238$$

$$= 402,272$$

The demographic growth model $\{k(t)\} = \underset{\sim}{G}^t\{k_o\}$ may be replaced by the analytical expression (5'), which for the numerical illustration becomes:

$$\{k(t)\} = 9,062,579 \times (1.00301)^t \times \begin{bmatrix} 0.08212 \\ 0.69897 \\ 0.21891 \end{bmatrix}$$

$$+ 2,238,974 \times (0.99393)^t \times \begin{bmatrix} 0.05715 \\ -0.37792 \\ 0.56493 \end{bmatrix} \qquad (23)$$

$$+ \quad 402,272 \times (0.96632)^t \times \begin{bmatrix} 0.51539 \\ -0.25398 \\ -0.23063 \end{bmatrix}$$

Equation (23) may be written as follows:

$$\{k(t)\} = (1.00301)^t \begin{bmatrix} 744,219 \\ 6,334,471 \\ 1,983,889 \end{bmatrix} + (0.99393)^t \begin{bmatrix} 127,957 \\ -846,153 \\ 1,264,864 \end{bmatrix} + (0.96632)^t \begin{bmatrix} 207,327 \\ -102,169 \\ -92,776 \end{bmatrix} \qquad (23')$$

The above expression decomposes the multiregional population projection into a set of three univariate equations. The growth of Brussels is described by the single equation

$$k_1(t) = 744,219 \times (1.00301)^t + 127,957\ (0.99393)^t + 207,327 \times (0.96632)^t$$

For t = 0 (1970), the formula yields:

$$k_1(0) = 744,219 + 127,957 + 207,327 = 1,079,503$$

which compares with the observed number of 1,079,520.

For t = 1 (1971), the population of Brussels is equal to

$$k_1(1) = 746,459 + 127,180 + 200,344 = 1,073,984$$

which is comparable with the 1,073,998 obtained by multiplying the population vector of the base year with the demographic growth matrix (Willekens, 1979). Deviation is due to rounding errors introduced predominantly in the computation of the eigenvalues and eigenvectors.

Note that as t becomes large, the contribution of the second and third term of the right-hand side to the population vector $\{k(t)\}$ diminishes, since the associated eigenvalues are less than unity. The third term will become zero after 400 steps (years) and the second term after 1500 steps. Values of the three terms for different values of t are given in Appendix 2. The observation that gradually higher terms disappear leads to the stable population concept and will be discussed in the next section.

Once the effect of the last two components disappears, the growth process will completely be described by the first term $c_1 \lambda_1^t\{\xi_1\}$ only. At this stage, the population is said to have reached stability. The first term contains information on the most important features of the stable or steady-state population. Stable-population analysis for the three-region system will be carried out in Section 5. Here it suffices to state that λ_1 denotes the stable growth ratio and is easily converted into an annual growth rate $r = \frac{1}{h} \ln \lambda_1$, where h is the projection interval; in this case $h = 1$. The vector $\{\xi_1\}$ denotes each region's share of the national population.

Comparison of the observed (1970) and of the stable regional shares shows that the region of Flanders will be gaining population relative to the other two regions. Such comparisons can be useful in a study of the demographic consequences of migration. More interesting than this comparative static analysis is, however, a dynamic analysis, which focuses on the growth path from the observed to the stable population. For this reason, a short section will now be devoted to the investigation of the way the multiregional population converges towards stability.

4. CONVERGENCE PATH TOWARDS STABILITY

The multiregional demographic growth path was described by equations (5'). For a three-region system, the growth trajectory is represented by the following three independent equations:

$$k_i(t) = c_1 \lambda_1^t\{\xi_{i1}\} + c_2 \lambda_2^t\{\xi_{i2}\} + c_3 \lambda_3^t\{\xi_{i3}\} \qquad (24)$$

$$i = 1, 2, 3$$

For large values of t, the three functions are monotonically increasing and convex. In this section, the shape of the growth path will be investigated for small t as well.

The condition for a monotonic increasing population is that the first derivative of (24) is positive:

$$\frac{dk_i(t)}{dt} = c_1\lambda_1^t\{\xi_{i1}\} \ln \lambda_1 + c_2\lambda_2^t\{\xi_{i2}\} \ln \lambda_2 + c_3\lambda_3^t\{\xi_{i3}\} \ln \lambda_3 > 0$$

If this condition holds for $t = 0$, it will hold for any $t > 0$, and hence, population growth of region i will be monotonic. Not t, but the relative differences between λ_1 and λ_2 and λ_3 determine whether the condition is met. For $t = 0$, the values of $\frac{dk_i(0)}{dt}$ are as follows:

$$\frac{dk_1(0)}{dt} = -7,882 < 0$$

$$\frac{dk_2(0)}{dt} = 27{,}690 < 0$$

$$\frac{dk_3(0)}{dt} = 1{,}440 > 0$$

Therefore, the populations of the last two regions will begin to increase right from the beginning, while the population of the first region (Brussels) will decrease over some period of time and only later will begin to increase. (The increase of the other two regions will resemble the graph in Figure 1a). The period of increase will begin when the effect of the last two members on the right-hand side of (23) is smaller than that of the first one.

The function $k_1(t)$ is decreasing at first, hence one may ask if it is convex or has a different shape. The second derivative can be used to investigate the problem, keeping in mind that when it is positive, the function is convex, and when negative, the function is concave.

$$\frac{d^2k_1(t)}{dt^2} = c_1\lambda_1^t\xi_{11}(\ln\lambda_1)^2 + c_2\lambda_2^t\xi_{21}(\ln\lambda_2)^2 + c_3\lambda_3^t\xi_{31}(\ln\lambda_3)^2$$

For $t = 0$, it was estimated that

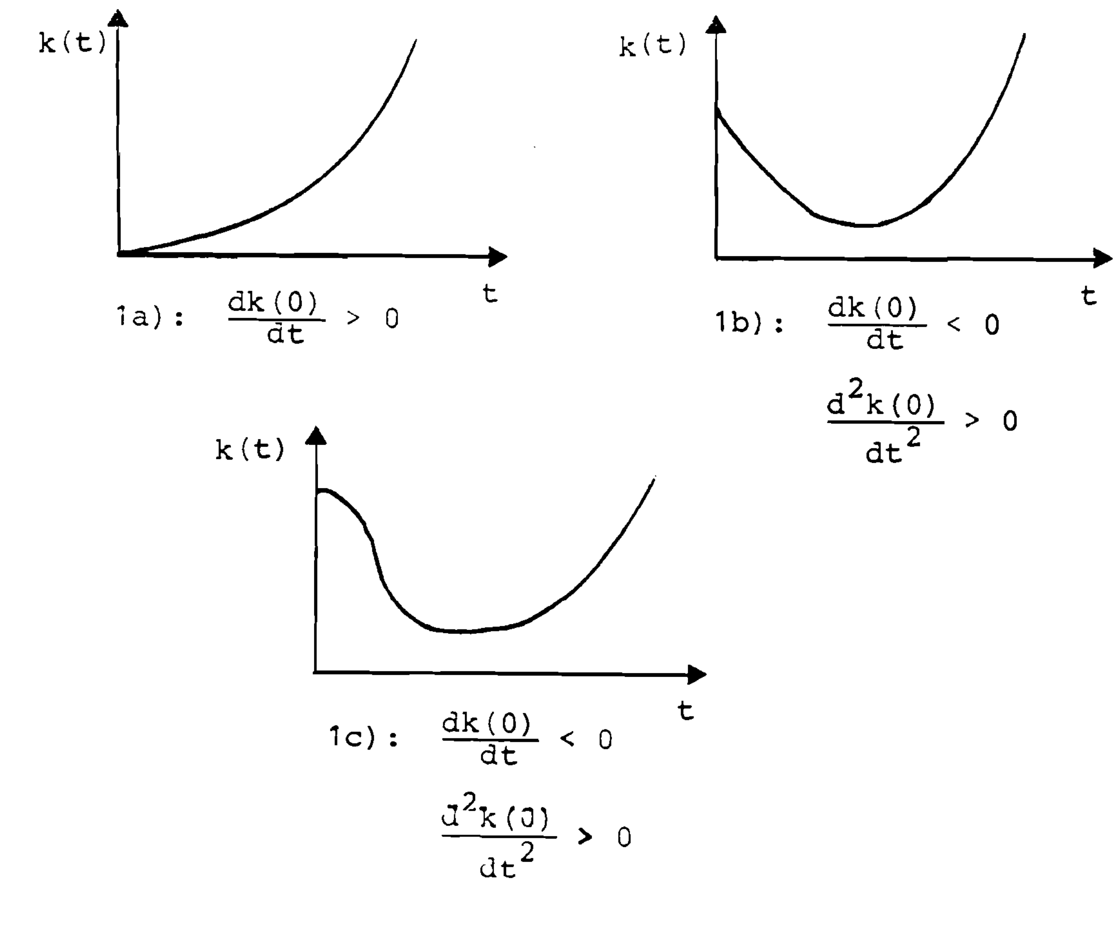
$$\frac{d^2k_1(0)}{dt^2} = 255 > 0$$

hence, $k_1(t)$ is convex at the point $t = 0$. In such a case, the region 1's population growth will have the shape as in Figure 1b. The point of minimum population can be easily found: simply compute this value of t, for which $\frac{dk_1(t)}{dt} = 0$ holds. This t is between 39 and 40 time periods and $k_1(40) = 992252$.

Figure 1c shows a third kind of projection path which is not observed in the case of the Belgium regions. It takes place when $\frac{d^2k(t)}{dt^2} < 0$. The demographic meaning of this case is that the population will decrease slowly at first, more rapidly later, but finally will increase. It is virtually the same process as when $\frac{d^2k(0)}{dt^2} > 0$. Therefore, it is enough usually to know if $\frac{dk(0)}{dt} > 0$, because then the projection path can be identified.

On the basis of this analysis, the projection paths were identified for three regions of Belgium, without carrying out the population projection itself. It was shown that the populations of Flanders and Wallonia will increase from the very beginning, while that of Brussels will decrease from 1,079,520 down to 992,252 during the first 40 years but will continuously increase afterwards. Note that the populations were studied without taking into account the age composition and assuming a closed system (no external migration) and constant demographic parameters.

Figure 1. Three different shapes of regional growth in a three-regional population projection.



5. STABLE POPULATION ANALYSIS

Stable population analysis investigates the long-term impact of current (base year) demographic behavior. The basic question is: what will $\{k(t)\}$ be if t becomes very large? In other words, stable population theory studies the asymptotic behavior of population growth and distribution.

Let $\{^s k(t)\}$ be the stable population at time t, i.e.,

$$\{^s k(t)\} = \lim_{t\to\infty} \{k(t)\} \tag{25}$$

or

$$\{^s k(t)\} = \lim_{t\to\infty} \underset{\sim}{G}^t \{k_o\}$$

or

$$\{^s k(t)\} = \sum_{i=1}^{n} \frac{1}{d_i} \{v_i\}' \{k_o\} \left[\lim_{t\to\infty} \lambda_i^t\right] \{\xi_i\}$$

Because $\{k_o\}$ is fixed, the study of the asymptotic properties of the projection is equivalent to the investigation of $\lim_{t\to\infty} \underset{\sim}{G}^t$.

Therefore, most properties of the stable population depend on the growth matrix, and stable population analysis is largely an analysis of the growth matrix $\underset{\sim}{G}$. The application of fundamental theorems of matrix algebra underlies stable population theory.

In this section we first describe the properties of the growth matrix, then formulate the Perron-Frobenius theorem, which is the main theorem behind stable theory, and finally characterize the stable equivalent population.

a. Properties of the growth matrix $\tilde{G}$

Recall the growth matrix for the three-region system Brussels-Flanders-Wallonia.

$$\tilde{G} = \begin{bmatrix} 0.969497 & 0.002615 & 0.004221 \\ 0.017749 & 1.000175 & 0.002383 \\ 0.012907 & 0.001435 & 0.993583 \end{bmatrix}$$

It is a square matrix of dimension 3 x 3 (or in general, n x n, where n is the number of regions). The growth matrix $\tilde{G}$ and all realistic growth matrices that may be designed have the following properties:

(i) Nonnegative: a matrix $\tilde{G}$ is said to be nonnegative if each of its elements is nonnegative, i.e., $g_{ij} \geq o$ for all i and j.

(ii) Indecomposable or irreducible: a matrix $\tilde{G}$ is irreducible if no permutation matrix $\tilde{P}$ exists such that

$$\tilde{P}' \tilde{G} \tilde{P} = \begin{bmatrix} \tilde{G}_{11} & \tilde{G}_{12} \\ \tilde{0} & \tilde{G}_{22} \end{bmatrix}$$

where $\tilde{G}_{11}$, $\tilde{G}_{22}$ are square matrices of an order less than n (Lancaster, 1969, p. 280).

(iii) Primitive: a square, indecomposable, nonnegative matrix is primitive if there exists a positive integer T such that $\tilde{G}^T > 0$ (Lancaster, 1969, pp. 289-291). Every positive matrix is necessarily primitive. A primitive matrix has a dominant eigenvalue which is unique in absolute value. (The absolute value of the dominant eigenvalue is known as the spectral radius.)

(iv) Distinct eigenvalues λ_i: this is less a property of $\underset{\sim}{G}$ than an assumption in demographic research. Empirically, no cases of multiple roots have turned up (Liaw, 1975, p. 231). This property has, however, important implications. Recall that if the eigenvalues are distinct, the eigenvectors are linearly independent and the modal matrix is nonsingular (i.e., has an inverse). As a consequence, there exists a similarity transformation between $\underset{\sim}{G}$ and a diagonal matrix $\underset{\sim}{\Lambda}$, the diagonal elements of which are the distinct eigenvalues:

$$\underset{\sim}{G} = \underset{\sim}{\Xi}\,\underset{\sim}{\Lambda}\,\underset{\sim}{\Xi}^{-1} \tag{26}$$

where

$$\underset{\sim}{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

is known as the <u>spectral matrix</u>. A matrix, for which a similarity transformation (26) exists, is said to be <u>diagonalizable</u>. Such a matrix is <u>simple</u> (Lancaster, 1969, p. 63). Note that $\underset{\sim}{\Xi}$ converts the population vector $\{k(t)\}$ in terms of the new basis, made up of the right eigenvectors of $\underset{\sim}{G}$. Formally, the vector

$$\{\hat{k}(t)\} = \underset{\sim}{\Xi}^{-1}\{k(t)\} \tag{27}$$

represents the population distribution in terms of the coordinate system made up of the right eigenvectors of $\underset{\sim}{G}$. Similarly,

$$\{\hat{k}(t+1)\} = \underset{\sim}{\Xi}\{k(t+1)\}$$

Therefore,

$$\{\hat{k}(t+1)\} = \underset{\sim}{\Xi}^{-1} \underset{\sim}{G} \underset{\sim}{\Xi}\{\hat{k}(t)\}$$

$$\{\hat{k}(t+1)\} = \underset{\sim}{\Lambda}\{\hat{k}(t)\} \tag{28}$$

Since $\underset{\sim}{\Lambda}$ is diagonal, the set of n simultaneous equations (1) describing the multiregional population growth, is replaced by n independent univariate equations. The value of $\hat{k}_i(t+1)$ only depends on $\hat{k}_i(t)$, i.e., the population of the same region in the previous time period.

In our three-region case, the similarity transformation $\underset{\sim}{G}^t = \underset{\sim}{\Xi}\ \underset{\sim}{\Lambda}\ \underset{\sim}{\Xi}^{-1}$ is:

$$\underset{\sim}{G}^t = \begin{bmatrix} 0.08212 & 0.05715 & 0.51539 \\ 0.69897 & -0.37792 & -0.25398 \\ 0.21891 & 0.56493 & -0.23063 \end{bmatrix} \begin{bmatrix} (1.00301)^t & o & o \\ o & (0.99393)^t & o \\ o & o & (0.96632)^t \end{bmatrix}$$

$$\begin{bmatrix} 0.850678 & 1.122505 & 0.664862 \\ 0.389505 & -0.485986 & 1.405618 \\ 1.761544 & -0.124966 & -0.261801 \end{bmatrix}$$

and the base year population distribution $\{\hat{k}(o)\}$ is $\underset{\sim}{\Xi}^{-1}\{k(o)\}$:

$$\{\hat{k}(o)\} = \begin{bmatrix} 9{,}062{,}611 \\ 2{,}238{,}993 \\ 402{,}297 \end{bmatrix}$$

These properties of the population growth matrices obey a most important theorem of matrix algebra, a theorem which is basic to discrete stable analysis: the Perron-Frobenius theorem.

b. <u>Perron-Frobenius theorem</u>

Let $\underset{\sim}{G}$ be a square, nonnegative, indecomposable, primitive matrix. Then there exists an eigenvalue λ_1 of $\underset{\sim}{G}$ such that

(i) λ_1 is real and positive.

(ii) $\lambda_1 > |\lambda_i|, i = 2 \ldots n$. λ_1 exceeds the absolute value of any other eigenvalue of $\underset{\sim}{G}$. Therefore, λ_1 is generally known as the dominant eigenvalue, or Perron-root, of the matrix.

(iii) λ_1 can be associated with strictly positive left and right eigenvectors.

(iv) λ_1 is a simple root of the characteristic equation, i.e., λ_1 is unique.

There are many proofs of this theorem. The interested reader is referred to Seneta (1973, pp. 2-6) or Gantmacher (1959, vol. 2, pp. 53-62) among others.

The Perron-Frobenius theorem tells us something very important about the asymptotic behavior of the growth process. Recall (5), in which $\bar{c}_i\ (t) = c_i\ \lambda_i^t$:

$$\{k(t)\} = \sum_{i=1}^{n} c_i\ \lambda_i^t\{\xi_i\} \tag{5}$$

Since the eigenvalue λ_1 exceeds all the others, the linear combination in (5) is dominated by the first element if t becomes large. Hence, we may write

$$\{{}^s k(t)\} = \lim_{t\to\infty} \{k(t)\} \doteq c_1\ \lambda_1^t\{\xi_1\} \tag{29}$$

The asymptotic behavior of the population growth process is determined by the dominant eigenvalue λ_1 of the growth matrix $\underset{\sim}{G}$ and by the associated right eigenvector. What this means is that, regardless of the initial population, the ultimate population will grow exponentially and its relative distribution by region will remain constant. The ultimate population is called the stable (or steady state) population. The stable growth rate and the relative stable distribution are independent of the initial population but depend only on the entries of the population growth matrix $\underset{\sim}{G}$, i.e., on the observed rates of fertility, mortality, and migration. This is the ergodic property in demography: the tendency of a population to forget its past (initial distribution).

The relation between the relative distribution, expressed by $\{\xi_1\}$ and the distribution in absolute terms, $\{{}^s k(t)\}$, is given by (29):

$$\{{}^s k(t)\} \doteq \frac{1}{d_1} \lambda_1^t \{\xi_1\}\{v_1\}'\{k_o\} \tag{30}$$

with

$$\frac{1}{d_1} \lambda_1^t \{\xi_1\}\{v_1\}' = \lim_{t\to\infty} \underset{\sim}{G}^t \tag{31}$$

The value of $\underset{\sim}{G}^t$ for large values of t only depends on the dominant root λ_1 and on the constituent matrix, which is completely determined by the left and right eigenvectors of $\underset{\sim}{G}$ associated with λ_1.

Equation (30) leads directly to a particularly useful concept: the stable equivalent population.

c. The stable equivalent (SE) population $\{{}^s k(o)\}$

The SE population is that population which, if distributed as the stable population and growing at the stable growth ratio λ_1, would lead to the same stable population as the observed population. This can easily be seen by:

$$\{{}^{s}k(t)\} = \lim_{t\to\infty} \underset{\sim}{G}^{t}\{k_o\}$$

$$\{{}^{s}k(t)\} = \lim_{t\to\infty} \lambda_1^t\{{}^{s}k(o)\}$$

where $\{{}^{s}k(o)\}$ is the vector of regional stable equivalent populations. It is equal to $\{{}^{s}k(t)\}/\lambda_1^t$ or, by (31):

$$\{{}^{s}k(o)\} = \frac{1}{d_1} \{\xi_1\}\{v_1\}'\{k_o\} \tag{32}$$

Hence, the matrix

$$\underset{\sim}{T} = \frac{1}{d_1} \{\xi_1\}\{v_1\}' = \frac{1}{d_1} \underset{\sim}{Z}_1 \tag{33}$$

transforms the observed population into the stable equivalent population. If the eigenvectors are normalized ($d_1 = 1$), then the transformation matrix is identical to the constituent matrix. In other words, the stable equivalent population by region may be written as a linear transformation of the observed population by region, the transformation matrix being proportional to the constituent matrix. Therefore, the SE population does not depend <u>directly</u> on the stable growth ratio; however, an indirect relationship exists.

Equation (32) converts the observed population into the stable equivalent population. Another relationship that is of particular interest is between the relative stable distribution $\{\xi_1\}$ and the SE population. Whereas $\{\xi_1\}$ expresses the stable population distribution in <u>relative</u> terms, the SE expresses the stable distribution in <u>absolute</u> terms. The relation between both is given by the proportionality factor (c_1) introduced earlier. Rewriting (32) gives:

$$\{{}^{s}k(o)\} = \frac{1}{d_1} \{v_1\}'\{k_o\}\{\xi_1\}$$

$$= c_1\{\xi_1\}$$

If the eigenvectors are normalized, c_1 is simply $\{v_1\}'\{k_o\}$. If $\{\xi_1\}$ is scaled such that the elements sum up to unity, then c_1 is equal to the total SE population of the multiregional system. The total stable equivalent population is proportional to the total reproductive value $V = \{v_1\}'\{k_o\}$, the proportionality factor being

$$\frac{1}{d_1} = \frac{1}{\{v_1\}'\{\xi_1\}} = \frac{1}{tr[\{\xi_1\}\{v_1\}']}$$

(see also Willekens, 1977, p. 24).

Hence, we have found an interesting demographic interpretation for the proportionality factor d_1: <u>the proportionality factor d_1 is equal to the ratio of the total reproductive value of the system to the total SE population</u>. It only depends on the scaling of the eigenvectors associated with λ_1.

6. GROWTH TRAJECTORY OF POPULATION DISAGGREGATED BY AGE AND REGION

Now the investigations will be repeated for age-disaggregated populations. The same three-region population system will be considered, but the population will now be disaggregated into 5-year age groups. As a consequence, the projection interval will be 5 years ($h = 5$). It was noted earlier that the solution using the left eigenvectors is demographically more meaningful and mathematically easier to follow. Therefore, the disaggregated-by-age case of this section only considers this solution. The matrix $\underset{\sim}{G}$ and the observed population vectors will not be exhibited here, because they are very large.

We shall consider only the age-groups until the end of the reproduction period, i.e., 0 to 50 years--ten age groups. Since there are three regions, $\underset{\sim}{G}$ will be a 30 × 30 matrix. Then $\underset{\sim}{G}$ will have 30 eigenvalues λ_i; and associated with each λ_i, a right and a left eigenvector. The eigenvalues are presented in Table 2, and the first three right eigenvectors in Table 3. The eigenvalues refer to a 5-year period since the projection interval is 5 years. They may be classified into four types: the dominant eigenvalue λ_1; the other real positive eigenvalues (n_1 in number, here $n_1 = 2$); the real negative eigenvalues (n_2 in number, here $n_2 = 3$); and the complex eigenvalues (n_3 in number, here $n_3 = 24$). Since complex eigenvalues are a particular feature of age-disaggregated growth operators; they will receive particular attention in this section.

As in the previous sections, the growth path of the multiregional population system may be expressed in terms of the eigenvalues and eigenvectors of the growth matrix $\underset{\sim}{G}$. Analogously to (5'), the analytical solution of the disaggregate growth path is

$$\{k(t)\} = \sum_{i=1}^{30} c_i \lambda_i^t \{\xi_i\} \tag{34}$$

where $\{\xi_i\}$ is a right eigenvector of $\underset{\sim}{G}$ with 30 elements, 10 for each region.

Given the classification of the eigenvalues, the terms of (34) may be grouped accordingly (Liaw 1980:593). The term associated with the dominant eigenvalue is the <u>dominant component</u>. It determines the system's stable (long-run or steady-state) growth rate and stable age-by-region population distribution. The terms associated with the remaining positive eigenvalues

Table 2. Eigenvalues λ_i, i = 1,...,30, of the multiregional growth matrix (age-disaggregated).

	5-year period		1-year period	
i	Real part[a] (u)	Imaginary[a] part (v)	Real part (x)	Imaginary part (y)
1	1.01158	0.00000	0.00230	0.00000
2	0.79916	0.00000	-0.04484	0.00000
3	0.96325	0.00000	-0.00749	0.00000
4	0.34536	0.74634	-0.03911	0.22748
5	0.34536	-0.74634	-0.03911	-0.22748
6	0.32316	0.70640	-0.05051	0.22835
7	0.32316	-0.70640	-0.05051	-0.22835
8	0.25483	0.58843	-0.08887	0.23242
9	0.25483	-0.58843	-0.08887	-0.23242
10	-0.00883	0.48218	-0.14585	-0.31050
11	-0.00883	-0.48218	-0.14585	0.31050
12	0.00261	0.46130	-0.15474	0.31303
13	0.00261	-0.46130	-0.15474	-0.31303
14	0.00541	0.36713	-0.20039	0.31121
15	0.00541	-0.36713	-0.20039	-0.31121
16	-0.38827	0.39562	-0.11800	-0.15895
17	-0.38827	-0.39562	-0.11800	0.15895
18	-0.37499	0.37075	-0.12799	-0.15594
19	-0.37499	-0.37075	-0.12799	0.15594
20	-0.31168	0.29513	-0.16915	-0.15163
21	-0.31168	-0.29513	-0.16915	0.15163
22	-0.40276	0.09450	-0.17652	-0.04609
23	-0.40276	-0.09450	-0.17652	0.04609
24	-0.39305	0.10387	-0.18001	-0.05167
25	-0.39305	-0.10387	-0.18001	0.05167
26	-0.30057	0.09228	-0.23141	-0.05958
27	-0.30057	-0.09228	-0.23141	0.05958
28	-0.08220	0.00000	-0.49972	0.00000
29	-0.09923	0.00000	-0.46206	0.00000
30	-0.09501	0.00000	-0.47075	0.00000

[a] An explanation of the decomposition of complex numbers into real and imaginary parts is given in Appendix 4.

Table 3. The right eigenvectors of the age-disaggregated growth matrix, corresponding to the three positive eigenvalues.

		Eigenvalues		
		$\lambda_1 = 1.01158$	$\lambda_2 = 0.79916$	$\lambda_3 = 0.96325$
		Right eigenvectors		
Region	Age Group			
Brussels	0-4	0.00837	0.05324	-0.00284
	5-9	0.00805	0.05139	-0.00284
	10-14	0.00790	0.05511	-0.00305
	15-19	0.00781	0.06030	-0.00321
	20-24	0.00813	0.05939	-0.00305
	25-29	0.00866	0.05143	-0.00273
	30-34	0.00881	0.04733	-0.00270
	35-39	0.00874	0.04772	-0.00284
	40-44	0.00857	0.05036	-0.00297
	45-49	0.00833	0.05538	-0.00302
Flanders	0-4	0.06785	-0.02899	0.05302
	5-9	0.06624	-0.02784	0.05174
	10-14	0.06540	-0.03012	0.05210
	15-19	0.06444	-0.03322	0.05251
	20-24	0.06308	-0.03192	0.05080
	25-29	0.06162	-0.02552	0.04774
	30-34	0.06046	-0.02170	0.04612
	35-39	0.05934	-0.02084	0.04568
	40-44	0.05802	-0.02120	0.04564
	45-49	0.05632	-0.02276	0.04565
Wallonia	0-4	0.03210	-0.02229	-0.05191
	5-9	0.03136	-0.02114	-0.05061
	10-14	0.03091	-0.02198	-0.05082
	15-19	0.03047	-0.02331	-0.04112
	20-24	0.02990	-0.02278	-0.04961
	25-29	0.02920	-0.02006	-0.04690
	30-34	0.02851	-0.01835	-0.04534
	35-39	0.02785	-0.01781	-0.04470
	40-44	0.02717	-0.01789	-0.04447
	45-49	0.02640	-0.01865	-0.04429

are denoted by Liaw as spatial components, since they seem to determine the spatial redistribution of the population. The complex and negative eigenvalues are cyclical components, as they determine the transmission of population waves. Each of these categories has a particular contribution to the path of population growth. The study of the contributions is the subject of the remainder of this section.

To study the growth path, we will decompose the right-hand side of (34) into the four types of terms. We also rewrite λ^t as a function of r, the annual growth rate: $\lambda^t = e^{5rt}$, where 5 represents the width of the projection interval. If λ is complex* ($\lambda = u + iv$), then r is complex ($r = x + iy$); we have

$$u + iv = e^{5(x + iy)} \tag{35}$$

The magnitude (modulus) and amplitude (argument) of λ are, respectively (Table 4),

$$|\lambda| = \sqrt{u^2 + v^2} = e^{5x}$$

$$\arg(\lambda) = \text{arctg}\,\frac{v}{u} = 5y \quad **$$

Note that by these equations, x and y may also be expressed in terms of u and v. The values of x and y are given in Table 2. The magnitudes and amplitudes are shown in Table 4.

By the theorem of De Moivre, λ^t for complex λ may be written as follows:

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*Appendix 4 reviews some relevant features of complex numbers.
**Arctg z denotes the angle whose tangent is z.

$$\lambda^t = (v + iv)^t = \sigma^t(\cos t\mu + i \sin t\mu)$$

where $\sigma = |\lambda|$ and $\mu = \arg(\lambda)$.
Equivalently,

$$e^{5rt} = e^{5t(x+iy)} = e^{5xt}(\cos t5y + i \sin t5y) \qquad (36)$$

Table 4. Magnitude and amplitude of eigenvalues of the multiregional growth matrix.[a]

i	Magnitude	Amplitude	
		In radials	In degrees
1	1.012	0.000	0.0
2	0.799	0.000	0.0
3	0.963	0.000	0.0
4	0.822	1.137	65.2
6	0.777	1.142	65.4
8	0.641	1.162	66.6
10	0.482	-1.552	91.1
12	0.461	1.565	89.68
14	0.367	1.556	89.16
16	0.554	-0.795	134.4
18	0.527	-0.780	135.3
20	0.429	-0.758	136.6
22	0.414	-0.230	166.8
24	0.407	-0.258	165.2
26	0.314	-0.298	162.9
28	0.082	0.000	180.0
29	0.099	0.000	180.0
30	0.095	0.000	180.0

[a]Of the complex eigenvalues, only the ones with a positive imaginary part are considered. Extracted from Table 3.

Distinguishing the various elements of (34) associated with different eigenvalue sets, the analytical solution of the population growth path may be written as follows:

$$\{k(t)\} = c_1\lambda_1^t\{\xi_1\} + \sum_{i=1}^{n_1} c_i\lambda_i^t\{\xi_i\} + \sum_{j=1}^{n_2} c_j\lambda_j^t\{\xi_j\}$$

$$+ \sum_{\ell=1}^{n_3} c_\ell \sqrt{u_\ell^2 + v_\ell^2} \left[\cos\left(t \text{ arctg } \frac{v_\ell}{u_\ell}\right)\right. \quad (37)$$

$$\left. + i \sin\left(t \text{ arctg } \frac{v_\ell}{u_\ell}\right)\right]\{\xi_\ell\}$$

or, in terms of r,

$$\{k(t)\} = c_1 \ e^{5r_1t}\{\xi_1\} + \sum_{i=1}^{n_1} c_i \ e^{5r_it}\{\xi_i\}$$

$$+ \sum_{j=1}^{n_2} c_j \ e^{5r_jt}\{\xi_j\} + \sum_{\ell=1}^{n_3}\left[c_\ell \ e^{5x_\ell t} \ \cdot\right. \quad (38)$$

$$\left.(\cos 5y_\ell t + i \sin 5y_\ell t)\{\xi_\ell\}\right]$$

The coefficients c_i are shown in Table 5. The population growth or the pattern of population change with increasing t may be studied using (38). The first component of (38) determines the long-run implications of population growth (stable population characteristics); the second component provides information on how the population is redistributed over space as it converges towards stability; the third and fourth component tells about the fluctuations in the convergence path. The overall population wave is the sum of the individual waves. According to the theory of vibrations, if the individual waves are periodic, the sum of the waves is also periodic, but its length will be much longer. Moreover the sum of periodic waves is itself a composite wave, since each of its periodic movements consists of shorter, aperiodic ones. The waves are damped since all values of x_s are negative; hence, their effects will eventually vanish.

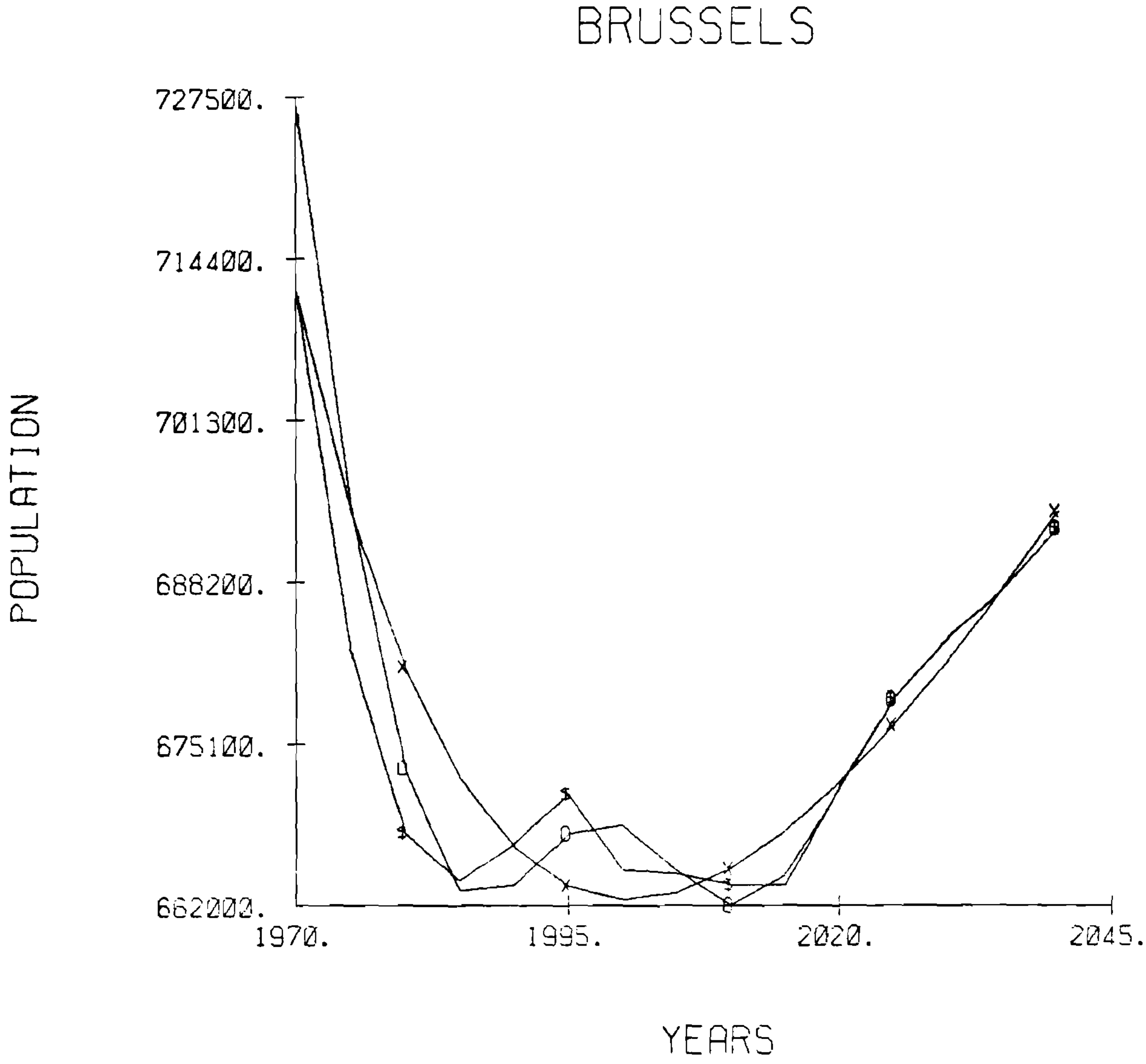
Table 5. Constants of the age-disaggregated linear decomposition.

constants	real part	imaginary part
c 1 =	6957574.	0.
c 2 =	225087.	0.
c 3 =	-420679.	0.
c 4 =	-39956.	-458018.
c 5 =	-39956.	458018.
c 6 =	62037.	-57990.
c 7 =	62037.	57990.
c 8 =	33913.	57719.
c 9 =	33913.	-57719.
c10 =	1388794.	-365713.
c11 =	1388794.	365713.
c12 =	366631.	-247822.
c13 =	366631.	247822.
c14 =	-110008.	135749.
c15 =	-110008.	-135749.
c16 =	1490303.	620626.
c17 =	1490303.	-620626.
c18 =	410895.	12298.
c19 =	410895.	-12298.
c20 =	81030.	239335.
c21 =	81030.	-239335.
c22 =	-1581426.	3599535.
c23 =	-1581426.	-3599535.
c24 =	-1021143.	-409766.
c25 =	-1021143.	409766.
c26 =	-1203104.	758682.
c27 =	-1203104.	-758682.
c28 =	-1470758.	0.
c29 =	-1324177.	0.
c30 =	-2214633.	0.

The projection paths for the three regions of Belgium are shown in Figure 2. For each region, three trajectories are given.

a) the trajectory corresponding to the three positive eigenvalues (—x—x—). The growth path is generated by taking into account only the effect of the dominant and spatial components. The trajectory has no cyclical parts. The term associated with a positive eigenvalue grows or vanishes monotonically.

b) the trajectory corresponding to the first nine eigenvalues: three positive and six complex eigenvalues (—o—o—). From Appendix 3, it can be seen that only the first three pairs of complex conjugate eigenvalues



—⊟—⊟— $\sum_{i=1}^{9} c_i \lambda_i^t \{\xi_i^1\}$

—×—×— $\sum_{i=1}^{3} c_i \lambda_i^t \{\xi_i^1\}$

—$—$— $\sum_{i=1}^{30} c_i \lambda_i^t \{\xi_i^1\}$

Figure 2a. Multiregional Population Projections for Three Regions of Belgium, 1970-2045.

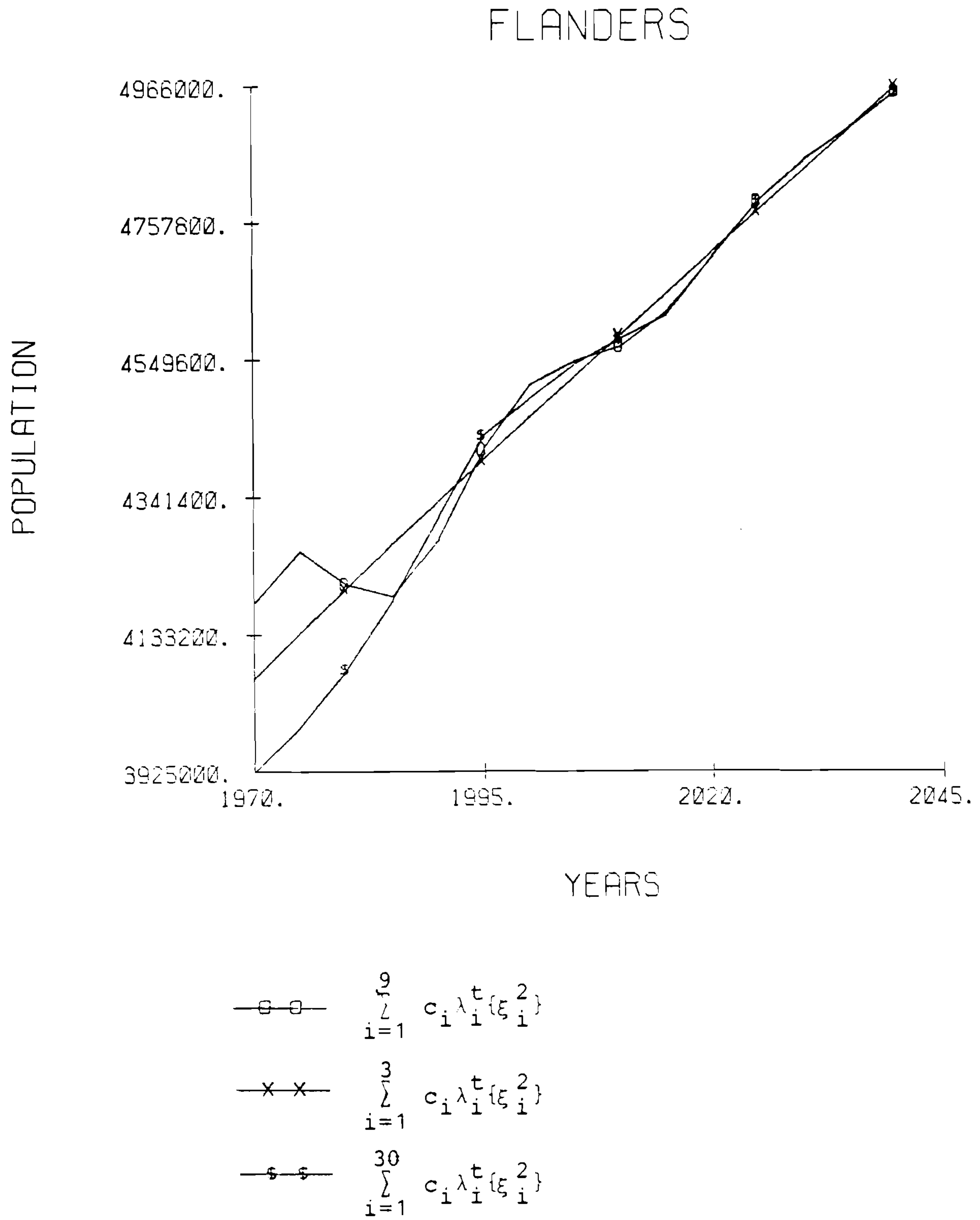


Figure 2b. Multiregional Population Projections for Three Regions of Belgium, 1970-2045.

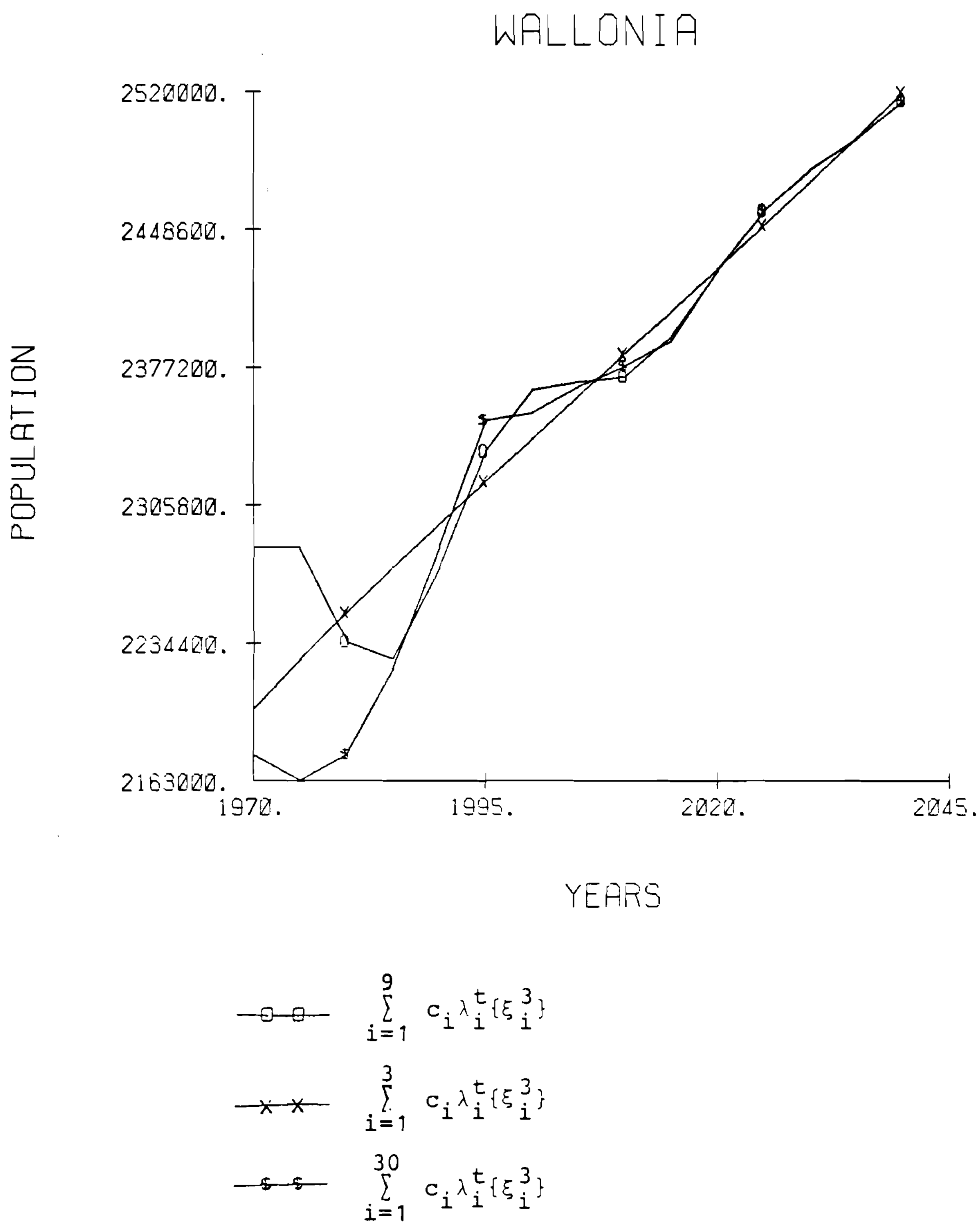


Figure 2c. Multiregional Population Projections for Three Regions of Belgium, 1970-2045.

could be significant for population growth. The others are very small in absolute value.

c) the trajectory corresponding to all the 30 eigenvalues (—\$—\$—). The effect of the terms associated with the small eigenvalues is significant only in the short run. After 25-30 years, the effect will vanish. Hence, one may conclude that the first three pairs of complex eigenvalues adequately determine the population wave. The effect of the small complex eigenvalues is compatible with the effect of the negative eigenvalues. To demonstrate the declining contribution to population change of the small eigenvalues, consider the term associated with λ_{29}, i.e., the 29th member of the right-hand side of (34). The largest element of this term was estimated to be the one corresponding to the tenth age group of the second region, i.e., the twentieth element of $\{\xi_{29}\}$, $\xi_{29}^{20} = 0.74674$. The element is

$$c_{29}\lambda_{29}^{t}\xi_{29}^{20} = -13{,}663{,}787(-0.09923)^{t}\ 0.74674$$

At time t = 0, this gives -10,203,296. This amount is mainly compensated for by other similar elements. After 25 years (t = 5), the amount drops by an absolute value to only 98. It is evident that the element has lost its significance in less than twenty-five years. Since the elements of $\{\xi_{29}\}$ were found to be very close to zero for the other age groups, the future population will be unaffected by the 29-th member of (34).

This example clearly shows that the effect of the element corresponding to the negative eigenvalues disappears in a very short period of time.

We now turn to a more elaborate investigation of the complex eigenvalues. The contribution of complex eigenvalues and their associated terms to the population growth path may be studied by decomposing each eigenvalue into two factors: magnitude and amplitude.* Recall that the magnitude (modulus) is $\sigma = \sqrt{u^2 + v^2} = e^{5x}$ and that the amplitude (argument) is $\mu = \text{arctg} \frac{v}{u} = 5y$. For each eigenvalue, the amplitude (measured in degrees per 5 years) is plotted in Figure 3 against the magnitude. Each complex conjugate pair is represented by the eigenvalue with the positive imaginary part. Particular periods (wavelengths) and half-lives are also shown. Recall that the period is measured by $360^0/\mu$. Hence, an amplitude of 60^0 is associated with a period of 30 years.** Half-lives (or doubling times) measure the time, T, necessary to decrease by half (or double) the population size. The half-life of a particular eigenvalue is given by

$$T = -5 \frac{\ln 2}{\ln \sigma}$$

where σ is the magnitude of the eigenvalue considered. The formula shows that a σ-value of $\frac{1}{2}$ implies a half-life of 5 years.

The eigenvalues are clustered in six groups, each with three members, around a particular amplitude. The first group (or cluster) is located on the vertical axis (amplitude 0^0), the second group at around 60^0, and the last one at 180^0. From the numerical values given in Table 4, the periods of the second set of eigenvalues may be estimated as 27.6, 27.5, and 27.0 years. Analogously, the third group clusters around a period of 20 years: the fourth, 13; the fifth, 11; and the sixth (the negative real eigenvalues), 10 years.

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*The authors acknowledge the recommendation of K.L. Liaw to follow this analytic approach. For an illustration of Liaw's analysis of the Canadian multiregional population system, see Liaw (1978a, 1978b, 1980) and Liaw, Aresta, and George (1979).

**The ratio $360/\mu$ gives the period in unit time intervals of 5 years. To obtain the period in single years, we simply multiply by five.

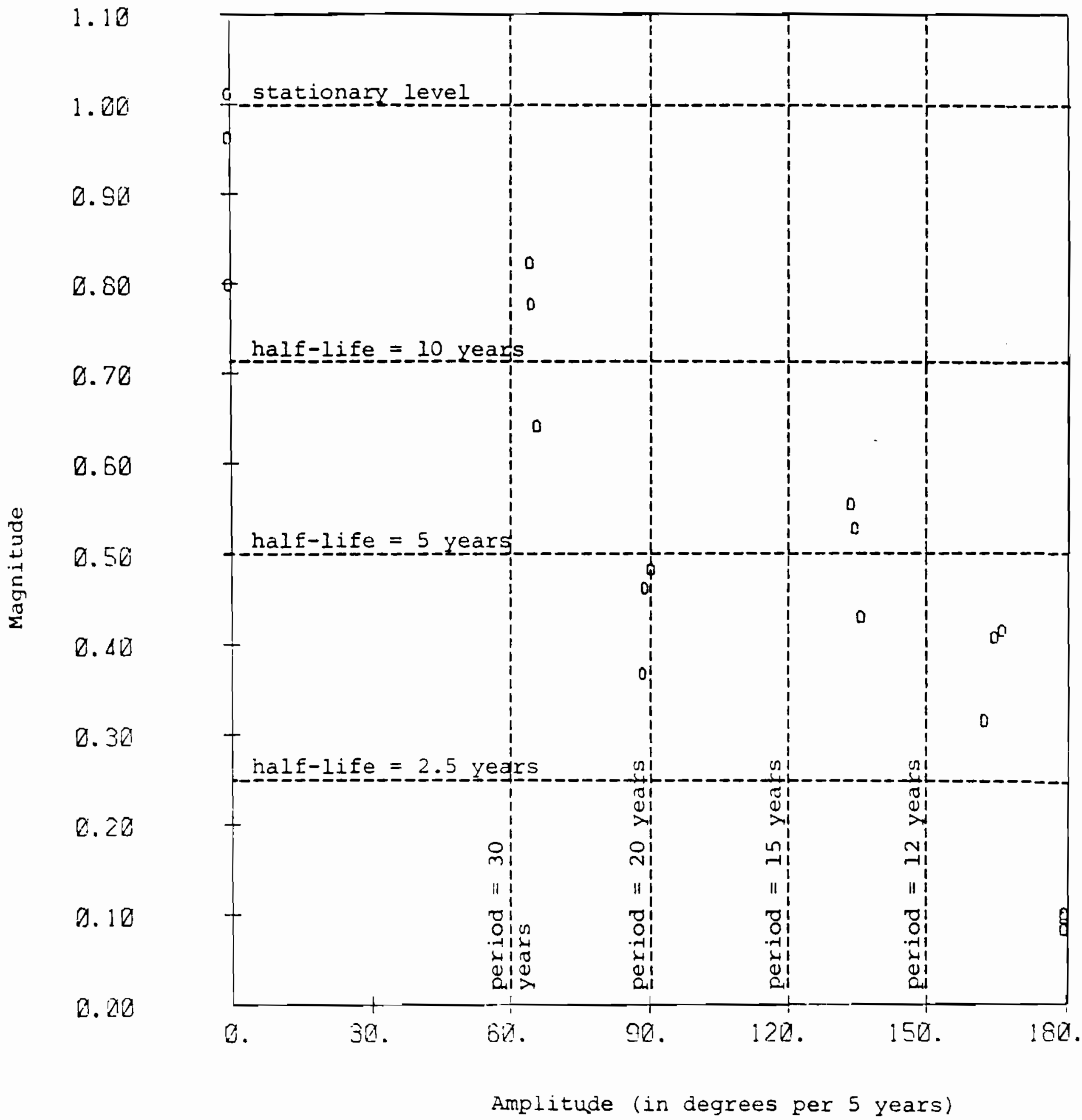

Figure 3. The eigenvalues of $\underset{\sim}{G}$. (Only the eigenvalues with a positive imaginary part are represented.)

The population wave is determined by the joint effect of all complex eigenvalues. As was demonstrated in Figure 3, some eigenvalues contribute more to the wave than others. The contribution of an eigenvalue depends on its magnitude or on its half-life (or doubling time). Only one eigenvalue has a magnitude greater than one and could therefore double the population. The half-life of most eigenvalues is very short. The effect of the eigenvalues in clusters 3 to 6 vanishes in not more than 20-25 years.

The half-lives of the eigenvalues from the second cluster are 17.7, 13.7, and 8.0 years, respectively. Thus the first half-life causes a wave that will dominate over the others by lasting a longer period of time. In order to evaluate how strong it is, the magnitudes of the constants c_i and of the elements of eigenvectors [see equation (34)] must be considered.

The half-lives of the positive eigenvalues (from the first cluster) are not connected with the effect of a wave. The doubling time of the dominant eigenvalue is around 300 years, and the half-lives of the other two are 93 and 15.5 years. Their effect, jointly with the constants and the eigenvectors, is numerically illustrated in Appendix 3.

In Section 4 of this paper, we studied the pattern of change of a population by making use of the first and second derivates of the growth equation. It was shown that the regional shares tend toward stability, and the path to stable growth depends on factors associated with the second and third eigenvalues.

These results will now be used again; for $t > 100$, the first derivative should be positive, hence $\{k(t)\}$ will be an increasing function ($\lambda_1 > 1$). It is known how the regional shares will change in the future, but we would like to know if the age distribution will continue to change, in spite of the fact that the effect of the complex eigenvalues has extinguished.

Consider, for example, the j-th element of the vector $\{k(t)\}$:

$$k_j(t) = c_1\lambda_1^t\xi_1^j + c_2\lambda_2^t\xi_2^j + c_3\lambda_3^t\xi_3^j \quad , \quad j = 1,\ldots,30, \; j \neq 10,20,30$$

If the age composition were already constant, the ratio $k_j(t)/k_{j+1}(t)$, say, must not depend on t, i.e., $k_j(t) = A\,k_{j+1}(t)$ for large values of t. Then,

$$c_1\lambda_1^t(\xi_1^j - A\,\xi_1^{j+1}) + c_2\lambda_2^t(\xi_2^j - A\,\xi_2^{j+1}) + c_3\lambda_3^t(\xi_3^j - A\,\xi_3^{j+1}) = 0$$

which holds if and only if the following three equalities hold:

$$\xi_1^j = A\,\xi_1^{j+1} \; , \quad \xi_2^j = A\,\xi_2^{j+1} \; , \quad \xi_3^j = A\,\xi_3^{j+1}$$

Table 3 shows that these equalities do not hold. For instance, for j = 3, the corresponding elements of the first and the second vector are not proportional. Hence, the above equations do not hold and the age composition is still not constant. It will stabilize only when the second and the third eigenvalues extinguish, i.e., together with the stabilization of the regional shares.

To summarize, the projection process of the multiregional population may be divided into the following stages:

Stage 1. 0-5 years after the initial year. All the eigenvalues are of interest.

Stage 2. Next 20-25 years. The negative eigenvalues are of no interest. Strong waves are to be observed due to all complex eigenvalues.

Stage 3. 25-100 years from the start. The waves that are due to the largest complex eigenvalues gradually damp out.

Stage 4. 100-300, 400 years from the start. The waves have disappeared, but regional shares and age compositions continue to change, approaching their stable values. The change is slow, and is represented as a sum of exponentials. The effect of the positive eigenvalues only counts.

Stage 5. 300-500 years from the start. Stable growth with constant age and regional distributions.

7. CONCLUSION

This paper investigates the growth path of a multiregional population, its constituents, and stages of development. The growth trajectory is a result of various forces. It is made up of a number of relatively independent growth paths, which are not observed in practice. The individual growth paths become visible if one changes the coordinate system in which the population distribution vector is expressed.

Decomposition of the growth path into individual trajectories poses the problem of the relative weight of each trajectory. In this paper, two techniques were considered: the z-transform and the introduction of the left eigenvector. It is the latter procedure that is demographically more attractive since some of the weights obtained have interesting demographic interpretations. Also, this procedure is easier to follow and requires less mathematical techniques.

The decomposition of the growth path into individual trajectories is a useful way to investigate how the multiregional population system converges towards stability. By adopting an analytic procedure, originally proposed by Liaw in his study of a multiregional population system in Canada, we were able to express the complex eigenvalues of the demographic growth matrix in meaningful indicators and to show that only a few of the many complex eigenvalues are responsible for most of the fluctuations or waves in the path towards convergence. As time progresses, the growth path becomes simpler since the effect of many of the eigenvalues vanishes. As a consequence, the path a multiregional population will follow, if projected with constant demographic parameters, may be divided into five stages.

APPENDIX 1: IMPROVED LEVERRIER ALGORITHM

This appendix reviews the improved Leverrier-algorithm to determine simultaneously the coefficients of the characteristic polynomial and the elements of the adjoint matrix. The appendix is adapted from Willekens (1975, Appendix).

Let $\underset{\sim}{R}(\lambda_i)$ be the adjoint matrix of the characteristic matrix $(\underset{\sim}{G} - \lambda_i \underset{\sim}{I})$. The definition of $\underset{\sim}{R}(\lambda_i)$ implies that

$$(\underset{\sim}{G} - \lambda_i \underset{\sim}{I})\ \underset{\sim}{R}(\lambda_i) = |\underset{\sim}{G} - \lambda_i \underset{\sim}{I}|\ \underset{\sim}{I}$$

$$\underset{\sim}{R}(\lambda_i)(\underset{\sim}{G} - \lambda_i \underset{\sim}{I}) = |\underset{\sim}{G} - \lambda_i \underset{\sim}{I}|\ \underset{\sim}{I}$$

Since $|\underset{\sim}{G} - \lambda_i \underset{\sim}{I}| = g(\lambda) = \lambda^n - c_1\lambda^{n-1} - c_2\lambda^{n-2} \ldots - c_n$, we may write

$$(\underset{\sim}{G} - \lambda_i \underset{\sim}{I})\ \underset{\sim}{R}(\lambda_i) = g(\lambda_i)\ \underset{\sim}{I}$$

$$\underset{\sim}{R}(\lambda_i) = (\underset{\sim}{G} - \lambda_i \underset{\sim}{I})^{-1}\ g(\lambda_i) \qquad \text{(A1)}$$

$\underset{\sim}{R}(\lambda_i)$ is a polynomial matrix. It can be represented in the form of a polynomial arranged with respect to the powers of λ_i.

$$\underset{\sim}{R}(\lambda_i) = \underset{\sim}{R}_o\ \lambda_i^{n-1} + \underset{\sim}{R}_1\ \lambda_i^{n-2} + \ldots + \underset{\sim}{R}_{n-1} \qquad \text{(A2)}$$

$$g(\lambda_i) = \lambda_i^n - c_1\ \lambda_i^{n-1} - \ldots\ldots\ldots - c_n \qquad \text{(A3)}$$

Equating the coefficients gives (Gantmacher, 1959, p.85):

$$\underset{\sim}{R}_o = \underset{\sim}{I}$$

$$\underset{\sim}{R}_1 = \underset{\sim}{G} - c_1\ \underset{\sim}{I}$$

$$\underset{\sim}{R}_2 = \underset{\sim}{G}\ \underset{\sim}{R}_1 - c_2\ \underset{\sim}{I} = \underset{\sim}{G}^2 - c_1\ \underset{\sim}{G} - c_2\ \underset{\sim}{I} \qquad \text{(A4)}$$

$$\vdots$$

$$\underset{\sim}{R}_k = \underset{\sim}{G}\ \underset{\sim}{R}_{k-1} - c_k\ \underset{\sim}{I} = \underset{\sim}{G}^k - c_1\ \underset{\sim}{G}^{k-1} - c_2\ \underset{\sim}{G}^{k-2} \ldots - c_k\ \underset{\sim}{I}$$

$$k = 1 \ldots n-1$$

If $\underset{\sim}{G}$ is nonsingular

$$c_n = (-1)^{n-1} \; |\underset{\sim}{G}| \neq 0$$

This leads to an alternative method to compute the inverse of $\underset{\sim}{G}$. Since

$$\underset{\sim}{G} \; \underset{\sim}{R}_{n-1} - c_n \; \underset{\sim}{I} = 0$$

we have $\underset{\sim}{G}^{-1} = \frac{1}{c_n} \underset{\sim}{R}_{n-1}$. (A5)

If λ_i is a characteristic root of $\underset{\sim}{G}$,

$$|\underset{\sim}{G} - \lambda_i \; \underset{\sim}{I}| = 0$$

and (A6)

$$(\underset{\sim}{G} - \lambda_i \; \underset{\sim}{I}) \; \underset{\sim}{R}(\lambda_i) = 0$$

Assume $\underset{\sim}{R}(\lambda_i) \neq 0$ and denote by $\{r\}$ an arbitrary nonzero column of $\underset{\sim}{R}(\lambda_i)$. Then by (A6):

$$(\underset{\sim}{G} - \lambda_i \; \underset{\sim}{I}) \; \{r\} = 0$$

or (A7)

$$\underset{\sim}{G}\{\underset{\sim}{r}\} = \lambda_i \; \{r\}$$

Each nonzero column of $\underset{\sim}{R}(\lambda_i)$ is a characteristic vector corresponding to the characteristic root λ_i.

The set of formulas (A5) to (A7) gives a method to determine $\underset{\sim}{R}(\lambda_i)$, $\underset{\sim}{G}^{-1}$ and the characteristic vector associated with λ_i, if the coefficients of the characteristic polynomial are known. Faddeev proposes a method to determine simultaneously the coefficients of the characteristic polynomial and the adjoint matrix $\underset{\sim}{R}(\lambda_i)$ (improved Leverrier algorithm) (Gantmacher, 1959, pp. 87-89; Faddeev and Faddeeva, 1963, pp. 260-265). Instead of computing $\underset{\sim}{G}$, $\underset{\sim}{G}^2$, $\underset{\sim}{G}^k$ required by the system (A4), a sequence $\underset{\sim}{G}_1$, $\underset{\sim}{G}_2, \ldots, \underset{\sim}{G}_k$ is computed in the following way:

$$
\begin{array}{lll}
\underset{\sim}{G}_1 = \underset{\sim}{G} & c_1 = \mathrm{tr}\, \underset{\sim}{G}_1 & \underset{\sim}{R}_1 = \underset{\sim}{G}_1 - c_1 \underset{\sim}{I} \\
\underset{\sim}{G}_2 = \underset{\sim}{G}\, \underset{\sim}{R}_1 & c_2 = \frac{1}{2}\, \mathrm{tr}\, \underset{\sim}{G}_2 & \underset{\sim}{R}_2 = \underset{\sim}{G}_2 - c_2 \underset{\sim}{I} \\
\vdots & \vdots & \vdots \\
\underset{\sim}{G}_k = \underset{\sim}{G}\, \underset{\sim}{R}_{k-1} & c_k = \frac{1}{k}\, \mathrm{tr}\, \underset{\sim}{G}_k & \underset{\sim}{R}_k = \underset{\sim}{G}_k - c_k \underset{\sim}{I} \\
\vdots & \vdots & \vdots \\
\underset{\sim}{G}_n = \underset{\sim}{G}\, \underset{\sim}{R}_{n-1} & c_n = \frac{1}{n}\, \mathrm{tr}\, \underset{\sim}{G}_n & \underset{\sim}{R}_n = \underset{\sim}{G}_n - c_n \underset{\sim}{I} = 0
\end{array}
\qquad \text{(A8)}
$$

It has been proved that

a) c_i is a coefficient of the characteristic polynomial $g(\lambda_i) = \lambda_i{}^n - c_1 \lambda_i{}^{n-1} - c_2 \lambda_i{}^{n-2} \ldots - c_n$

b) $\underset{\sim}{R}_n$ is a null matrix. This may be used to check the computations.

c) if $\underset{\sim}{G}$ is nonsingular, then

$$\underset{\sim}{G}^{-1} = \frac{1}{c_n} \underset{\sim}{R}_{n-1}$$

If $\underset{\sim}{G}$ is singular, then $(-1)^{n-1}\, \underset{\sim}{R}_{n-1}$ will be the matrix adjoint to $\underset{\sim}{G}$.

Numerical Illustration

Recall the growth matrix of the multiregional population system consisting of Brussels, Flanders, and Wallonia.

$$\tilde{G} = \begin{bmatrix} 0.969497 & 0.002615 & 0.004221 \\ 0.017749 & 1.000175 & 0.002383 \\ 0.012907 & 0.001435 & 0.993583 \end{bmatrix}$$

Application of the improved Leverrier algorithm yields the following results:

$$\tilde{G}_1 = \begin{bmatrix} 0.969497 & 0.002615 & 0.004221 \\ 0.017749 & 1.000175 & 0.002383 \\ 0.012907 & 0.001435 & 0.993583 \end{bmatrix} \quad c_1 = 2.963255 \quad \tilde{R}_1 = \begin{bmatrix} -1.993758 & 0.002615 & 0.004221 \\ 0.017749 & -1.963080 & 0.002383 \\ 0.012907 & 0.001435 & -1.969672 \end{bmatrix}$$

$$\tilde{G}_2 = \begin{bmatrix} -1.932842 & -0.002592 & -0.004216 \\ -0.017604 & -1.963374 & -0.002235 \\ -0.012884 & -0.001357 & -1.956975 \end{bmatrix} \quad c_2 = -2.926595 \quad \tilde{R}_2 = \begin{bmatrix} 0.993754 & -0.002592 & -0.004216 \\ -0.017604 & 0.963222 & -0.002235 \\ -0.012884 & -0.001357 & 0.969620 \end{bmatrix}$$

$$\tilde{G}_3 = \begin{bmatrix} 0.963341 & 0.000000 & 0.000000 \\ 0.000000 & 0.963341 & 0.000000 \\ 0.000000 & 0.000000 & 0.963341 \end{bmatrix} \quad c_3 = 0.963341 \quad \tilde{R}_3 = \begin{bmatrix} 0.000000 & 0.000000 & 0.000000 \\ 0.000000 & 0.000000 & 0.000000 \\ 0.000000 & 0.000000 & 0.000000 \end{bmatrix}$$

The characteristic equation (A3) is equal to

$$g(\lambda_i) = \lambda_i^3 - 2.963255\lambda^2 + 2.926595\lambda - 0.963341$$

The roots of this equation are the eigenvalues

$$\lambda_1 = 1.00301$$

$$\lambda_2 = 0.99393$$

$$\lambda_3 = 0.96632$$

The adjoint matrices $\underset{\sim}{R}(\lambda_i)$ are, by (A2)

$$\underset{\sim}{R}(\lambda_i) = \lambda_i^3 + \underset{\sim}{R}_1\lambda_i^2 + \underset{\sim}{R}_2$$

$$\underset{\sim}{R}(\lambda_1) = \begin{bmatrix} 0.000023 & 0.000031 & 0.000018 \\ 0.000198 & 0.000262 & 0.000155 \\ 0.000062 & 0.000082 & 0.000049 \end{bmatrix}$$

$$\underset{\sim}{R}(\lambda_2) = \begin{bmatrix} -0.000006 & 0.000007 & -0.000020 \\ 0.000037 & -0.000046 & 0.000133 \\ -0.000055 & 0.000069 & -0.000199 \end{bmatrix}$$

$$\underset{\sim}{R}(\lambda_3) = \begin{bmatrix} 0.000920 & -0.000065 & -0.000137 \\ -0.000453 & 0.000032 & 0.000067 \\ -0.000411 & 0.000029 & 0.000061 \end{bmatrix}$$

APPENDIX 2: THE REGION-DISAGGREGATED POPULATION GROWTH PATH*

init. year 1970	brussels	744219.	+	127957.	+	207327.	=	1079503.	11.22 %
	flanders	6334471.	+	-846153.	+	-102169.	=	5386149.	55.98 %
	wallonia	1983889.	+	1264864.	+	-92776.	=	3155977.	32.80 %
after 1 year	brussels	746459.	+	127181.	+	200344.	=	1073984.	11.13 %
	flanders	6353539.	+	-841017.	+	-98728.	=	5413794.	56.13 %
	wallonia	1989861.	+	1257186.	+	-89651.	=	3157396.	32.74 %
after 2 years	brussels	748706.	+	126409.	+	193597.	=	1068711.	11.05 %
	flanders	6372662.	+	-835912.	+	-95403.	=	5441347.	56.28 %
	wallonia	1995850.	+	1249555.	+	-86632.	=	3158773.	32.67 %
after 3 years	brussels	750960.	+	125641.	+	187076.	=	1063677.	10.97 %
	flanders	6391845.	+	-830838.	+	-92190.	=	5468817.	56.42 %
	wallonia	2001858.	+	1241970.	+	-83714.	=	3160114.	32.60 %
after 4 years	brussels	753220.	+	124879.	+	180776.	=	1058875.	10.90 %
	flanders	6411086.	+	-825795.	+	-89085.	=	5496207.	56.57 %
	wallonia	2007884.	+	1234431.	+	-80895.	=	3161421.	32.54 %
after 5 years	brussels	755488.	+	124121.	+	174687.	=	1054295.	10.82 %
	flanders	6430384.	+	-820782.	+	-86084.	=	5523517.	56.71 %
	wallonia	2013928.	+	1226938.	+	-78170.	=	3162696.	32.47 %
after 6 years	brussels	757762.	+	123367.	+	168804.	=	1049932.	10.75 %
	flanders	6449740.	+	-815800.	+	-83185.	=	5550755.	56.85 %
	wallonia	2019990.	+	1219490.	+	-75537.	=	3163943.	32.40 %
after 7 years	brussels	760043.	+	122618.	+	163118.	=	1045779.	10.68 %
	flanders	6469154.	+	-810848.	+	-80383.	=	5577923.	56.98 %
	wallonia	2026071.	+	1212088.	+	-72993.	=	3165166.	32.33 %
after 8 years	brussels	762330.	+	121874.	+	157624.	=	1041829.	10.62 %
	flanders	6488627.	+	-805926.	+	-77676.	=	5605025.	57.12 %
	wallonia	2032169.	+	1204731.	+	-70535.	=	3166365.	32.27 %
after 9 years	brussels	764625.	+	121134.	+	152316.	=	1038075.	10.55 %
	flanders	6508158.	+	-801034.	+	-75060.	=	5632064.	57.25 %
	wallonia	2038286.	+	1197418.	+	-68159.	=	3167545.	32.20 %
after 10 years	brussels	766927.	+	120399.	+	147186.	=	1034511.	10.49 %
	flanders	6527748.	+	-796172.	+	-72532.	=	5659045.	57.38 %
	wallonia	2044422.	+	1190150.	+	-65864.	=	3168708.	32.13 %
after 15 years	brussels	778539.	+	116789.	+	124014.	=	1019342.	10.21 %
	flanders	6626586.	+	-772299.	+	-61113.	=	5793173.	58.01 %
	wallonia	2075376.	+	1154464.	+	-55495.	=	3174346.	31.79 %
after 20 years	brussels	790327.	+	113287.	+	104490.	=	1008104.	9.97 %
	flanders	6726922.	+	-749143.	+	-51492.	=	5926287.	58.59 %
	wallonia	2106801.	+	1119849.	+	-46758.	=	3179892.	31.44 %
after 25 years	brussels	802293.	+	109890.	+	88040.	=	1000224.	9.76 %
	flanders	6828776.	+	-726681.	+	-43385.	=	6058710.	59.14 %
	wallonia	2138700.	+	1086272.	+	-39397.	=	3185575.	31.10 %
after 40 years	brussels	839291.	+	100299.	+	52662.	=	992252	9.32 %
	flanders	7143687.	+	-663255.	+	-25951.	=	6454481	60.59 %
	wallonia	2237327.	+	991460.	+	-23565.	=	3205222	30.09 %
after 50 years	brussels	864900.	+	94374.	+	37386.	=	996660.	9.11 %
	flanders	7361652.	+	-624077.	+	-18423.	=	6719152.	61.43 %
	wallonia	2305591.	+	932896.	+	-16730.	=	3221758.	29.46 %

*The components on the left-hand side are the terms of the growth equation (23').

after	brussels	1005150.	+	69606.	+	6741.	=	1081497.	8.63	%
100	flanders	8555401.	+	-460286.	+	-3322.	=	8091793.	64.54	%
years	wallonia	2679461.	+	688054.	+	-3017.	=	3364498.	26.83	%
after	brussels	1168143.	+	51337.	+	1216.	=	1220696.	8.45	%
150	flanders	9942725.	+	-339482.	+	-599.	=	9602644.	66.48	%
years	wallonia	3113956.	+	507472.	+	-544.	=	3620884.	25.07	%
after	brussels	1357565.	+	37864.	+	219.	=	1395648.	8.36	%
200	flanders	11555007.	+	-250384.	+	-108.	=	11304515.	67.72	%
years	wallonia	3618906.	+	374284.	+	-98.	=	3993091.	23.92	%
after	brussels	1577705.	+	27926.	+	40.	=	1605671.	8.31	%
250	flanders	13428743.	+	-184670.	+	-19.	=	13244054.	68.51	%
years	wallonia	4205740.	+	276052.	+	-18.	=	4481775.	23.18	%
after	brussels	1833541.	+	20597.	+	7.	=	1854145.	8.27	%
300	flanders	15606312.	+	-136203.	+	-4.	=	15470105.	69.01	%
years	wallonia	4887732.	+	203601.	+	-3.	=	5091330.	22.71	%
after	brussels	2476401.	+	11204.	+	0.	=	2487605.	8.24	%
400	flanders	21078054.	+	-74091.	+	-0.	=	21003964.	69.54	%
years	wallonia	6601423.	+	110754.	+	-0.	=	6712177.	22.22	%
after	brussels	3344651.	+	6095.	+	0.	=	3350745.	8.22	%
500	flanders	28468222.	+	-40303.	+	-0.	=	28427918.	69.75	%
years	wallonia	8915945.	+	60247.	+	-0.	=	8976192.	22.02	%
after	brussels	15031445.	+	290.	+	0.	=	15031735.	8.21	%
1000	flanders	127941176.	+	-1920.	+	-0.	=	127939256.	69.90	%
years	wallonia	40069816.	+	2870.	+	-0.	=	40072684.	21.89	%

APPENDIX 3: THE REGION- AND AGE-DISAGGREGATED POPULATION GROWTH PATH

Key

The quantities in the first three columns are due to each of the positive eigenvalues. The quantity in the fourth column is due to the first complex eigenvalue. The quantity due to its conjugate is not presented. Analogously, the quantities in the fifth and sixth columns are due to the second and third complex eigenvalue. (Their conjugates are also disregarded.)

The first column after the equality sign gives the sum of the quantities due to the first nine eigenvalues (including the conjugates); the second column is associated with the three positive eigenvalues; and the third column with all the eigenvalues.

For each year, the first row gives numbers for the population of Brussels; the second, for Flanders; and the third, for Wallonia.

```
initial  year  =   1970
 580053.+  119667.+   12305.+    9115.+     1286.+    -2777.=   727273.  712025.  712295.
4332969.+  -59448.+ -206553.+   68142.+   -10206.+    -1204.=  4180432. 4066968. 3925425.
2044622.+  -45976.+  201829.+   26807.+    13448.+     1476.=  2283937. 2200475. 2176969.

after    5   years
 586770.+   95633.+   11853.+    8202.+      270.+    -8158.=   694882.  694256.  682949.
4383144.+  -47508.+ -198963.+   61426.+    -2177.+     2386.=  4259945. 4136674. 3992841.
2068299.+  -36742.+  194412.+   24154.+     2564.+     2198.=  2283801. 2225969. 2163247.

after    10   years
 593565.+   76426.+   11417.+    -499.+     -602.+    -3016.=   673174.  681408.  667940.
4433901.+  -37967.+ -191651.+   -3657.+     4752.+     1711.=  4209898. 4204284. 4080401.
2092250.+  -29363.+  187267.+   -1446.+    -6458.+      514.=  2235374. 2250154. 2176705.

after    15   years
 600438.+   61077.+   10997.+   -5892.+     -552.+     1817.=   663261.  672513.  664020.
4485246.+  -30341.+ -184608.+  -44068.+     4385.+     -109.=  4190713. 4270297. 4185785.
2116478.+  -23466.+  180385.+  -17334.+    -5721.+     -642.=  2226003. 2273398. 2221059.

after    20   years
 607391.+   48810.+   10593.+   -3732.+        7.+     2166.=   663677.  666795.  667005.
4537185.+  -24248.+ -177823.+  -27965.+      -34.+     -759.=  4277598. 4335115. 4311879.
2140987.+  -18753.+  173756.+  -10995.+      199.+     -538.=  2273321. 2295990. 2284051.

after    25   years
 614425.+   39007.+   10204.+    1407.+      337.+      357.=   667838.  663636.  671069.
4589726.+  -19378.+ -171288.+   10486.+    -2668.+     -342.=  4414014. 4399060. 4435318.
2165780.+  -14987.+  167371.+    4128.+     3581.+      -10.=  2333562. 2318164. 2349592.

after    30   years
 621540.+   31173.+    9829.+    3496.+      214.+     -709.=   668543.  662542.  664946.
4642875.+  -15486.+ -164993.+   26156.+    -1704.+      138.=  4511575. 4462395. 4490691.
2190859.+  -11977.+  161220.+   10288.+     2194.+      216.=  2365498. 2340102. 2353428.

after    35   years
 628737.+   24912.+    9468.+    1463.+      -65.+     -508.=   664897.  663117.  664600.
4696640.+  -12376.+ -158930.+   10975.+      508.+      211.=  4548722. 4525334. 4543928.
2216230.+   -9571.+  155295.+    4314.+     -743.+      114.=  2369324. 2361953. 2367075.

after    40   years
 636018.+   19909.+    9120.+   -1353.+     -171.+       33.=   662063.  665047.  663721.
4751027.+   -9890.+ -153089.+  -10109.+     1357.+       51.=  4570646. 4588048. 4581940.
2241894.+   -7649.+  149588.+   -3978.+    -1804.+      -31.=  2372208. 2383832. 2377657.

after    45   years
 643383.+   15910.+    8785.+   -1924.+      -71.+      226.=   664538.  668078.  663685.
4806043.+   -7904.+ -147463.+  -14405.+      570.+      -61.=  4622885. 4650676. 4617665.
2267855.+   -6113.+  144090.+   -5665.+     -718.+      -63.=  2392942. 2405832. 2390607.

after    50   years
 650834.+   12715.+    8462.+    -414.+       57.+      102.=   671500.  672010.  671653.
4861697.+   -6316.+ -142044.+   -3113.+     -450.+      -52.=  4706106. 4713337. 4706923.
2294116.+   -4885.+  138795.+   -1223.+      625.+      -19.=  2426792. 2428026. 2427105.

after    55   years
 658370.+   10161.+    8151.+    1016.+       80.+      -41.=   678792.  676682.  678818.
4917996.+   -5048.+ -136824.+    7592.+     -635.+       -1.=  4790034. 4776124. 4790297.
2320682.+   -3904.+  133694.+    2987.+      837.+       16.=  2458152. 2450473. 2458237.

after    60   years
 665994.+    8120.+    7851.+     981.+       17.+      -63.=   683838.  681966.  683789.
4974946.+   -4034.+ -131796.+    7349.+     -139.+       21.=  4853579. 4839117. 4853177.
2347556.+   -3120.+  128781.+    2890.+      164.+       16.=  2479356. 2473217. 2479207.
```

after	65 years							
673706.+	6490.+	7563.+	-9.+	-37.+	-15.=	687636.	687759.	687684.
5032556.+	-3224.+	-126952.+	-58.+	294.+	11.=	4902873.	4902380.	4903226.
2374740.+	-2493.+	124048.+	-24.+	-399.+	2.=	2495454.	2496296.	2495617.
after	70 years							
681508.+	5186.+	7285.+	-670.+	-34.+	18.=	692607.	693979.	692583.
5090834.+	-2576.+	-122287.+	-5010.+	273.+	-3.=	4956492.	4965971.	4956341.
2402240.+	-1993.+	119490.+	-1971.+	-357.+	-6.=	2515071.	2519737.	2514989.
after	75 years							
689400.+	4145.+	7017.+	-457.+	0.+	15.=	699680.	700562.	699678.
5149785.+	-2059.+	-117793.+	-3421.+	-0.+	-6.=	5023078.	5029934.	5023052.
2430058.+	-1592.+	115098.+	-1345.+	10.+	-4.=	2540887.	2543564.	2540882.
after	80 years							
697383.+	3312.+	6759.+	138.+	21.+	0.=	707772.	707455.	707781.
5209420.+	-1645.+	-113464.+	1025.+	-165.+	-2.=	5096026.	5094311.	5096095.
2458198.+	-1273.+	110869.+	404.+	222.+	1.=	2569047.	2567794.	2569076.
after	85 years							
705459.+	2647.+	6511.+	404.+	13.+	-6.=	715439.	714617.	715433.
5269745.+	-1315.+	-109294.+	3022.+	-107.+	1.=	5164970.	5159135.	5164931.
2486664.+	-1017.+	106794.+	1189.+	137.+	2.=	2595096.	2592441.	2595080.
after	90 years							
713628.+	2115.+	6272.+	186.+	-4.+	-3.=	722372.	722015.	722373.
5330769.+	-1051.+	-105277.+	1394.+	31.+	2.=	5227293.	5224440.	5227301.
2515460.+	-813.+	102869.+	548.+	-45.+	1.=	2618523.	2617516.	2618527.
after	95 years							
721892.+	1691.+	6041.+	-145.+	-11.+	1.=	729314.	729623.	729315.
5392499.+	-840.+	-101408.+	-1081.+	84.+	0.=	5288256.	5290250.	5288259.
2544589.+	-649.+	99089.+	-425.+	-112.+	-0.=	2641953.	2643028.	2641954.
after	100 years							
730251.+	1351.+	5819.+	-226.+	-4.+	2.=	736965.	737421.	736964.
5454944.+	-671.+	-97682.+	-1689.+	36.+	-1.=	5353283.	5356591.	5353277.
2574055.+	-519.+	95447.+	-664.+	-45.+	-0.=	2667563.	2668983.	2667561.
after	105 years							
738708.+	1080.+	5605.+	-58.+	4.+	1.=	745285.	745392.	745285.
5518112.+	-536.+	-94092.+	-436.+	-28.+	-0.=	5422555.	5423483.	5422559.
2603862.+	-415.+	91940.+	-171.+	38.+	-0.=	2695122.	2695387.	2695123.
after	110 years							
747262.+	863.+	5399.+	113.+	5.+	-0.=	753758.	753524.	753758.
5582012.+	-429.+	-90634.+	841.+	-40.+	0.=	5492553.	5490949.	5492552.
2634015.+	-331.+	88561.+	331.+	52.+	0.=	2723011.	2722245.	2723011.
after	115 years							
755915.+	690.+	5201.+	117.+	1.+	-0.=	762041.	761806.	762041.
5646651.+	-343.+	-87303.+	876.+	-9.+	0.=	5560740.	5559006.	5560740.
2664517.+	-265.+	85306.+	344.+	10.+	0.=	2750269.	2749559.	2750269.
after	120 years							
764669.+	551.+	5010.+	5.+	-2.+	-0.=	770234.	770229.	770234.
5712040.+	-274.+	-84095.+	36.+	18.+	0.=	5627780.	5627672.	5627781.
2695372.+	-212.+	82171.+	14.+	-25.+	-0.=	2777310.	2777332.	2777310.
after	125 years							
773524.+	440.+	4826.+	-76.+	-2.+	0.=	778634.	778790.	778634.
5778185.+	-219.+	-81004.+	-568.+	17.+	-0.=	5695861.	5696962.	5695861.
2726585.+	-169.+	79152.+	-223.+	-22.+	-0.=	2805076.	2805567.	2805076.
after	130 years							
782481.+	352.+	4648.+	-56.+	-0.+	0.=	787370.	787481.	787370.
5845097.+	-175.+	-78027.+	-416.+	0.+	-0.=	5766061.	5766894.	5766061.
2758159.+	-135.+	76243.+	-164.+	0.+	-0.=	2833940.	2834266.	2833940.

after	135	years						
791542.+	281.+	4477.+	13.+	1.+	-0.=	796329.	796301.	796329.
5912783.+	-140.+	-75160.+	96.+	-10.+	-0.=	5837656.	5837484.	5837656.
2790098.+	-108.+	73441.+	38.+	14.+	0.=	2863534.	2863431.	2863534.
after	140	years						
800708.+	225.+	4313.+	47.+	1.+	-0.=	805340.	805246.	805340.
5981253.+	-112.+	-72398.+	348.+	-7.+	0.=	5909427.	5908744.	5909427.
2822408.+	-86.+	70742.+	137.+	9.+	0.=	2893354.	2893063.	2893354.
after	145	years						
809980.+	180.+	4154.+	23.+	-0.+	-0.=	814361.	814314.	814361.
6050516.+	-89.+	-69737.+	175.+	2.+	0.=	5981045.	5980690.	5981045.
2855091.+	-69.+	68142.+	69.+	-3.+	0.=	2923296.	2923164.	2923296.
after	150	years						
819360.+	144.+	4002.+	-15.+	-1.+	0.=	823473.	823505.	823473.
6120581.+	-71.+	-67174.+	-114.+	5.+	-0.=	6053116.	6053335.	6053116.
2888153.+	-55.+	65638.+	-45.+	-7.+	-0.=	2953631.	2953735.	2953631.
after	155	years						
828848.+	115.+	3855.+	-26.+	-0.+	0.=	832764.	832817.	832764.
6191457.+	-57.+	-64706.+	-198.+	2.+	-0.=	6126305.	6126695.	6126305.
2921598.+	-44.+	63226.+	-78.+	-3.+	-0.=	2984618.	2984779.	2984618.
after	160	years						
838446.+	92.+	3713.+	-8.+	0.+	0.=	842235.	842251.	842235.
6263154.+	-46.+	-62328.+	-59.+	-2.+	-0.=	6200660.	6200781.	6200660.
2955430.+	-35.+	60902.+	-23.+	2.+	-0.=	3016255.	3016297.	3016255.
after	165	years						
848155.+	73.+	3577.+	12.+	0.+	-0.=	851831.	851805.	851831.
6335681.+	-36.+	-60037.+	93.+	-2.+	0.=	6275788.	6275608.	6275788.
2989654.+	-28.+	58664.+	36.+	3.+	0.=	3048369.	3048289.	3048369.
after	170	years						
857977.+	59.+	3445.+	14.+	0.+	-0.=	861509.	861481.	861509.
6409049.+	-29.+	-57831.+	104.+	-1.+	0.=	6351396.	6351189.	6351396.
3024274.+	-22.+	56508.+	41.+	1.+	0.=	3080843.	3080759.	3080843.
after	175	years						
867912.+	47.+	3319.+	1.+	-0.+	0.=	871280.	871278.	871280.
6483266.+	-23.+	-55705.+	9.+	1.+	0.=	6427557.	6427537.	6427557.
3059295.+	-18.+	54431.+	4.+	-2.+	-0.=	3113713.	3113709.	3113713.
after	180	years						
877963.+	37.+	3197.+	-9.+	-0.+	0.=	881179.	881197.	881179.
6558342.+	-19.+	-53658.+	-64.+	1.+	-0.=	6504539.	6504665.	6504539.
3094722.+	-14.+	52431.+	-25.+	-1.+	-0.=	3147085.	3147138.	3147085.
after	185	years						
888130.+	30.+	3079.+	-7.+	-0.+	0.=	891225.	891239.	891225.
6634287.+	-15.+	-51686.+	-50.+	0.+	-0.=	6582485.	6582585.	6582485.
3130558.+	-11.+	50504.+	-20.+	0.+	-0.=	3181012.	3181051.	3181012.
after	190	years						
898414.+	24.+	2966.+	1.+	0.+	-0.=	901406.	901404.	901406.
6711113.+	-12.+	-49787.+	8.+	-1.+	-0.=	6661330.	6661314.	6661330.
3166811.+	-9.+	48648.+	3.+	1.+	0.=	3215458.	3215450.	3215458.
after	195	years						
908818.+	19.+	2857.+	5.+	0.+	-0.=	911705.	911694.	911705.
6788828.+	-9.+	-47957.+	40.+	-0.+	0.=	6740940.	6740861.	6740940.
3203482.+	-7.+	46860.+	16.+	1.+	0.=	3250368.	3250335.	3250368.
after	200	years						
919342.+	15.+	2752.+	3.+	-0.+	-0.=	922115.	922109.	922115.
6867442.+	-8.+	-46195.+	22.+	0.+	0.=	6821284.	6821240.	6821284.
3240579.+	-6.+	45138.+	9.+	-0.+	0.=	3285728.	3285711.	3285728.

APPENDIX 4: REVIEW OF COMPLEX NUMBER THEORY*

Definitions and Operations

A complex number z is a pair of real numbers a and b, written as (a, b), which obeys the following rules:

- equality: $z = z'$ or $(a, b) = (a', b')$ only if $a = a'$ and $b = b'$
- addition: $z + z'$ or $(a, b) + (a', b') = (a + a', b + b')$
- multiplication: $z \cdot z'$ or $(a, b) \cdot (a', b') = (aa' - bb', ab' + a'b)$

Real numbers are a subset of complex numbers. The real number may be written as the complex number (a, 0). Because of this identification of real numbers, we may write

$$
\begin{aligned}
z = (a, b) &= (a, 0) + (0, b) \\
&= (a, 0) + (b, 0) \cdot (0, 1) = a + b \cdot (0, 1)
\end{aligned}
$$

*This review is based on Kuipers and Timman (1969:28-32).

The complex number z is decomposed into a real part a and an imaginary part b $\cdot$ (0, 1); the complex number (0, 1) is always denoted by i.* Hence,

$$z = a + ib \tag{A9}$$

Note that a real number may be represented by a complex number with the imaginary part 0. The complex number $\bar{z} = a - bi$ is the conjugate complex number of z.

The Complex Plane

A complex number may be understood more easily if given a geometric interpretation. A complex number can be associated with a point, P, on a plane (complex plane). The coordinates of P with respect to the orthogonal axes are (a, b). The location of P is fully determined by the knowledge of a and b (Figure A1).

The location of P in the plane may be expressed not only with reference to a cartesian coordinate system, but also in terms of polar coordinates (σ, μ) , where σ is the distance from the origin to P and μ is the angle of z with the horizontal axis. The polar coordinates may easily be derived from the rectangular coordinates a and b. The distance σ is the absolute value, magnitude or modulus of z and is given by the Pythagorian theorem as

$$\sigma = |z| = \sqrt{a^2 + b^2}$$

*i is the imaginary number. It is the number which, multiplied by itself gives -1, i.e., $i^2 = (0, 1) \cdot (0,1) = -1$. Hence, $i = \sqrt{-1}$. This is a highly exceptional number in mathematics, since a square of either a positive or a negative number is positive. Because of this, Italian mathematicians in the early renaissance have called i such things as an absurd, fictitious or imaginary number.

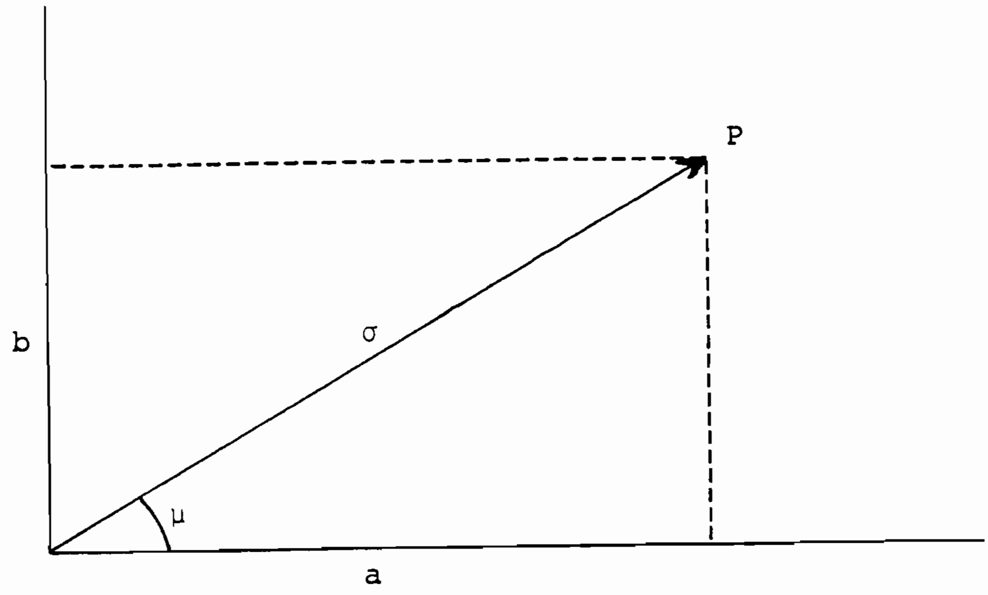

Figure A1. Orthogonal (a, b) and polar (σ, μ) coordinates of a point in a complex plane.

To determine the angle μ, we write

$$z = a + ib = \sigma(\frac{a}{\sigma} + i\frac{b}{\sigma}) \tag{A10}$$

from which follows that

$$(\frac{a}{\sigma})^2 + (\frac{b}{\sigma})^2 = 1$$

and hence,

$$\cos\mu = \frac{a}{\sigma} \quad , \quad \sin\mu = \frac{b}{\sigma} \quad , \quad \mathrm{tg}\,\mu = \frac{b}{a} \tag{A11}$$

The value of μ is, therefore, given by $\mu = \text{arctg}\,\frac{b}{a}$. The angle μ is also called the amplitude or the argument of z and is written as arg(z). It measures the difference between the peak of the oscillation and its average level. If, apart from (A11), μ also satisfies the condition

$$-\pi < \mu \leq \pi$$

then this value of μ is known as the principal value of the argument of z and

$$z = \sigma(\cos\mu + i\sin\mu) \qquad \text{(A12)}$$

The argument is the factor determining the wavelength or period of the time path. The period is the length of a complete oscillation in units of time (years, say) and is calculated as $2\pi/\mu$ or $360^0/\mu$. The reciprocal of the period is the frequency of oscillation ($\mu/2\pi$), i.e., the number of complete oscillations per unit of time.

The complex number z may either be expressed in terms of rectangular coordinates (A9) or in terms of polar coordinates (A12). In demographic applications, the second approach is generally more useful.

Arguments of Products and Quotients

Consider two complex numbers z and z':

$$z = a + ib = \sigma(\cos\mu + i\sin\mu)$$

$$z' = a' + ib' = \sigma'(\cos\mu' + i\sin\mu')$$

Since the product z z' is

$$z\,z' = \sigma\,\sigma'[\cos(\mu + \mu') + i\sin(\mu + \mu')]$$

we may conclude that the argument of the product of two complex numbers is equal to the sum of the arguments of the factors

$$\arg(z\ z') = \arg(z) + \arg(z') \tag{A13}$$

Similarly, the argument of the quotient of two complex numbers is equal to the difference of the arguments of the factors

$$\arg\left(\frac{z}{z'}\right) = \arg(z) - \arg(z') \tag{A14}$$

By (A13), the argument of z^n, with n being a natural number, is

$$\arg(z^n) = n \arg (z) \tag{A15}$$

In demographic analysis, n is generally a time interval. Since the modulus of z^n is simply σ^n, the following expression for z^n may be derived (Theorem of De Moivre):

$$z^n = \sigma^n(\cos n\ \mu + i \sin n\ \mu) \tag{A16}$$

From the theorem of De Moivre follows that

$$z^{-n} = \sigma^{-n}[\cos(-n\ \mu) + i \sin (-n\ \mu)]$$

The theorem may also be applied to derive $\arg(z^n)$ in terms of $\arg(z)$ (in other words to relate $n\mu$ to μ). For $n = 4$, we have

$$(\cos\mu + i\sin\mu)^4 = \cos 4\mu + i\sin 4\mu$$

and therefore

$$\cos^4\mu - 6\cos^2\mu\,\sin^2\mu + \sin^4\mu = \cos 4\mu$$
$$-4\cos^3\mu\,\sin\mu + 4\cos\mu\,\sin^3\mu = \sin 4\mu$$

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