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A METHOD OF NONDIFFERENTIABLE OPTIMIZATION APPLIED TO THE PROBLEM OF FINNISH FORESTRY AND FOREST INDUSTRIAL SECTOR DEVELOPMENT

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ABSTRACT

Here we consider one special problem of linkage which was used as a framework for the analysis of the steady-state of a large dynamical model describing the processes of growing and using wood on a particular example of Finland.

A short description of the dynamical model is given together with the static linear linkage problem which represents the stationary state of the general model. The technique of nondifferentiable optimization was applied for solving this problem. A new method of subgradient type is discussed, results of computation are given which show its good convergence characteristics.

A Method of Nondifferentiable Optimization Applied to the Problem of Finnish Forestry and Forest Industrial Sector Development

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1. Introduction

There are many models, describing economic and social activities, which consist of several submodels. Examples are industrial or agriculture production models, resources allocation and supply models, manpower and educational planning models, etc. Variables in such models can be divided in two parts: internal variables of subsystems and external ones which link different subsystems in an integral system. The solution of such a problem as a single large-scale model might be difficult or practically impossible because of many reasons. Among these reasons are:

-practical impossibility of putting a large-scale problem on a small computer at hand;

-distributed character of data collected in different places;

-institutional constraints.

So it might be useful to find a way of solving such problems preserving individuality of submodels and leaving them relatively independant.

The whole scope of questions arising in this respect is referred to as a linkage problem.

The general approach to decomposition and linkage problems was studied in(Dantzig64a, Ditrix79a, Ermoliev81a, Ermoliev80a, Nurminski79a).

The objective of this paper is to discuss one special linear linkage problem and the possibilities for applying some methods of nondifferentiable optimization to its solution. One new method is discussed and the computational results are given. This method can be applied to both linear and nonlinear linkage problems, it is also possible to use it in a stochastic case. Our objective is to show how to do it in the case of a special problem of growing and using wood in Finland as a particular example.

2. Case Study.

This chapter describes the linear programming model for studying development of forestry and forest based industries. The data on Finnish forest sector was used for actual numerical calculation.

The detailed account of this model is given in(Kallio80a). , and here we give only basic characteristics of this problem.

The model consists of two subsystems, the forestry and industrial ones, which are linked to each other through the wood supply from the first to the second. The forestry submodel describes planting and harvesting activities, and the volume of various tree species at different ages. The production process is described by a small Leontiev model with substitution. Various production activities are considered; such as the pulp and paper industry, the panel industry, the saw mill industry, and also further processing of primary products Production is restricted through supply of wood and demand for wood products, as well as through labor availability, financial resources and production capacities. The general model is formulated within the framework of the dynamic linear programming approach. Its terminal conditions are determined through an optimal solution of a stationary problem. In this paper one special method of nondifferentiable optimization is discussed for solving this stationary problem.

Here following (6) we give the brief description of the dynamic linear programming model for forest sector.

2.1. The Forestry Subsystem.

Let w(t) be a vector determining the number of trees of various types in different age groups: we denote by $w_{sa}(t)$ the number of trees of species s (s=1,2,...,l) in in age group a (a=1,2,...N) at the beginning of time period t (t=0,1,...,T). Let α_a^s show the ratio of trees of species s and in age group a that will proceed to the age group a+1. We denote by $u^+(t)$ and $u^-(t)$ the vectors of planting and harvesting activities at time period t. The state equation describing the development of the forest is the following

$$w(t+1) = \alpha w(t) + \nu u^+(t) - \omega u^-(t) \tag{1}$$

where matrices ν and ω are so that $\nu u^+(t)$ and $-\omega u^-(t)$ are the incremental change in numbers of trees resulting from planting and harvesting activities, respectively.

Let G_{ad}^s be the area of land type d required by one tree of species s and age group a. We have the land availability restriction

$$Gw(t) \le H(t) \tag{2}$$

where matrix G=(G_{ad}^s) and H(t) is the vector of total amount of different types of land available at time period t.

For harvesting and planting activities we need special resources such as machinery and labor. Let $R_{gn}^+(t)$ and $R_{gh}^-(t)$ be the usage of resource g at the unit level of planting activity n and harvesting activity h, respectively. We have the resource availability constraint as follows:

$$R^+(t)u^+(t) + R^-(t)u^-(t) \le R(t) \tag{3}$$

where matrices $R^+(t)=\{ R_{gn}^+(t) \}$ and $r^-(t)=\{ R_{gh}^-(t)\}$ and $R(t)=\{ R_g(t) \}$ is vector of available resources during period t.

Let x(t) be the vector of requirements for different timber assortments in industry, and matrix S(t) transforms quantities of harvested trees into the volume of different timber assortments. Then the requirements for wood supply to industries can can be written as follows:

$$S(t)u^-(t) = x(t) \tag{4}$$

The objective function is the discounted sum of net income in forestry as follows:

$$IC_F = \sum_{t=0}^{T-1} \beta(t)[J^-(t)u^-(t) - J^+(t)u^+(t)] \tag{5}$$

Here $J^-(t)$ is a price of the wood less transportation and harvesting costs at unit level, $J^+(t)$ describes planting costs at unit level and $\beta(t)$ is a discounting factor.

In summary, the forest model may be stated as follows. Given state equation (1), an initial state $w(0) = w^0$ and a terminal state $w(T) = w^*$ (about the terminal state see below), find such nonnegative controls $\{u^-(t)\}$ and $\{u^+(t)\}$ (t=0,1,...,T-1), which yield nonnegative state vectors w(t), satisfy constraints (2)-(4) and maximize the objective function (5). If we consider the vector x(t) of wood supply as exogenous variable we obtain an independent forestry model, but we shall link it below to an industrial submodel.

2.2. The Industrial Subsystem.

Let y(t) be the vector of production activities (such, for example, as production of sawn wood, panel, pulp, paper, etc.) for period t (t=0,1,... ,T-1). For each product j there may exist several production activities i. Let U be the matrix of wood usage per unit of production activity so that during period t, industry processes the amount of wood Uy(t). Matrix U has one row corresponding to each timber assortment.

We denote by r(t) the vector of wood raw material inventories in the beginning of period t. We have the folloming state equation for it:

$$r(t+1) = r(t) + x(t) - Uy(t) + z^{+}(t) - z^{-}(t) \quad (6)$$

where $z^{+}(t)$ is the vector of import, $z^{-}(t)$ is the vector of export outside the forest sector. For wood import and export we have upper limits, so that

$$z^{+}(t) \le Z(t), z^{-}(t) \le Z^{-}(t) \quad (7)$$

The production process may be described by a simple input-output model with substitution. Let A(t) be an input-output matrix which has one row for each product j and one column for each production activity i. Let m(t) and e(t) be the vectors of import and export .

If the inventory level is constant we have:

$$A(t)y(t) + m(t) - e(t) = 0 \quad (8)$$

For import and export we have:

$$e(t) \le E(t) \quad (9)$$

$$m(t) \le M(t) \quad (10)$$

Production activities are also restricted through labor and mill capacities. Let us denote L(t) the vector of different types of labor available for the forest industries, $\rho(t)$ denote a coefficient matrix so that $\rho(t)y(t)$ is the vector of labor demand given production activity levels $y(t)$. We have

$$\rho(t)y(t) \le L(t) \quad (11)$$

Let q(t) be the vector of the amount of different types of mill capacity at the beginning of period t. If Q(t) is a coefficient matrix than Q(t)y(t) is the demand for these types of capacity. Thus we have the production capacity restriction:

$$Q(t)y(t) \leq q(t) \tag{12}$$

The state equation for the development of the capacity is as follows:

$$q(t+1) = (I-\delta)q(t)+v(t) \tag{13}$$

where δ is a deprecation matrix and v(t) is a vector of investments (in physical units).

For financial calculations we define a vector $\bar{q}(t)$ of fixed assets which corresponds to the vector $q(t)$ given in physical units. Let $\bar{\delta}(t)$ be such matrix that $(I-\bar{\delta}(t))\bar{q}(t)$ is the vector of fixed assets left at the end of period t when we have no investments. Let K(t) be a matrix of increase in fixed assets per (physical) unit of an investment activity, and let v(t) be the vector of investments (in physical units). Then we have the following state equation:

$$\bar{q}(t+1) = (I-\bar{\delta}(t))\bar{q}(t)+K(t)v(t) \tag{14}$$

The state equation for external financing (long-term debt) is as follows:

$$l(t+1) = l(t)+l^{+}(t)-l^{-}(t) \tag{15}$$

where l(t) is the vector of the balance of external financing at the beginning of the t-th period $l^{+}(t)$ and $l^{-}(t)$ are the drawings of debt and the repayments made during period t.

We have one more restriction:

$$[I-\mu(t)]l(t) \leq \mu(t)b(t) \tag{16}$$

Let $p^{+}(t)$ and $p^{-}(t)$ be vectors of profits and losses for the financial units, let P(t) be a matrix of prices for products, C(t) be matrix of direct unit production costs. Then the vector of revenue from sales e(t) outside the forest industry, is given by P(t)e(t), and the vector of direct production costs is given by

C(t)y(t).

The profit is given as follows:

$$p^{+}(t)-p^{-}(t) = P(t)e(t)-C(t)y(t)-F(t)q(t)-\bar{\delta}(t)\bar{q}(t)-\varepsilon(t)l(t)-D(t) \qquad (17)$$

where D(t) is the vector of cash expenditure, $\varepsilon(t)$ is the matrix of interest rates, vector F(t)q(t) yields the fixed costs of period t.

For b(t) we have the following state equation:

$$b(t+1) = b(t)+[I-\tau(t)]p^{+}(t)-p^{-}(t)+B(t) \qquad (18)$$

The state equation for cash is:

$$c(t+1)=c(t)+[I-\tau(t)]p^{+}(t)-p^{-}(t)+\bar{\delta}(t)\bar{q}(t)+l^{+}(t)-l^{-}(t)-K(t)v(t)+B(t) \qquad (19)$$

In this model we have the initial state given as

$$r(0)=r^{*}, q(0)=q^{*}, \bar{q}(0)=\bar{q}^{*},$$

$$l(0)=l^{*}, c(0)=c^{*}, b(0)=b^{*} \qquad (20)$$

and a terminal state restricted as follows:

$$r(T) \geq r^{*}, q(T) \geq q^{*}, \bar{q}(T) \geq \bar{q}^{*},$$

$$l(T) \leq l^{*}, c(T) \geq c^{*} \qquad (21)$$

The objective function may be chosen as follows:

$$IC_I = \sum_{t=0}^{T-1} \beta(t)[(I-\tau(t))p^{+}(t)-p^{-}(t)] \qquad (22)$$

So the problem is to find nonnegative control vectors $x(t), z^{+}(t), z^{-}(t), m(t), e(t), v(t), l^{+}(t), l^{-}(t), p^{+}(t)$ and $p^{-}(t)$ and nonnegative state vectors $r(t), q(t), \bar{q}(t), l(t), c(t)$ and $b(t)$,for all t, which satisfy constraints and state equations (6)-(19), the terminal requirements (21), and maximize the linear functional (22).

For both of the models above the wood supply x(t) from forestry to industrial submodel is considered as exogenous. For the integrated model we consider x(t) as an endogenous vector of linking variables. The objective function may be written as IC_F+IC_I.

3. Problem Formulation.

Due to the long transient time of forest system planning, the horizon in this model is of 50 to 80 years, and one period has an interval of five years. To the industrial subsystem such a horizon is too long and it is too short for the forestry subsystem. That's why it is desirable to analyze a stationary regime for the forests, i.e. we set w(t+1)=w(t)=w; $u^+(t)=u^+$ and $u^-(t)=u^-$, for all t. The state equation in this case is the following:

$$w = \alpha w + \nu u^+ - \omega u^- \tag{1a}$$

With constraints (2)-(4) we have the static linear programming problem for the forestry subsystem. There are also some simplifications. Equation (1a) presumes that there are separate planning/harvesting activities in every age and species category. It is difficult to imagine, however, that harvesting, for instance, follows this routine. So two generalized harvesting activities were introduced--thinning and final harvesting, which harvest some fixed proportions of trees in age and species categories. We can find an optimal stationary state w* of the forest and corresponding harvesting and planting activities. The solution of a dynamic linear programming problem with terminal constraints

$$w(T) = w^*$$

yields the optimal transition to this stationary state.

Similarly we can determine the terminal state as a stationary solution in the industrial subproblem. Considering the integrated model and corresponding dynamic linear programming problem it is desirable to know its stationary solution. Our objective now is to find this stationary solution.

The correspondent static linear programming problem has the following dual block-angular form:

$$(c_F, z_F) + (c_I, z_I) = \min \tag{23}$$

$$A_F z_F + B_F x \leq d_F \tag{24}$$

$$A_I z_I + B_I x \leq d_I \tag{25}$$

$$z_F \geq 0,\ z_I \geq 0,\ x \geq 0 \tag{26}$$

where blocks F and I represent sets of equations in the forestry submodel and in the industry submodels. The variables Z_f and Z_I are internal variables of the Forestry and Industrial models respectively. Variable x is a linking variable (wood supply from forestry to industry) which links those submodels.

The matrix of the forest part is given in fig. 1 and the matrix of the industry part is given in fig. 2.

4. Relation to Nondifferentiable Optimization

To solve the problem described above one can use the well-known finite methods of decomposition or iterative methods cof nondifferentiable optimization.

The number of vertices of the feasible polyhedral set for such problem is, generally speaking, combinatorially large enough, and finite-step methods, based on moving from one vertex to another yield very small steps at each iteration. Empirical evidence shows that the convergence of these methods is slow. Moreover, the finite methods often possess numerical instability, when the number of steps is large and errors are accumulated.

The nondifferentiable approach gives a possibility to develop iterative decomposition schemes. They are easy to implement, and robust, with respect to computational errors. That is why we chose a nondifferentiable approach for solving the problem

The initial problem can be written in such a way

$$\min \{ f_F(x) + f_I(x) \} \tag{27}$$

where

$$f_F(x) = \min(c_F, z_F) \tag{28}$$
$$A_F z_F + B_F x \leq d_F$$
$$z_F \geq 0, x \geq 0$$

$$f_I(x) = min(c_I, z_I) \tag{29}$$
$$A_I z_I + B_I x \leq d_I$$
$$z_I \geq 0, x \geq 0$$

To use the iterative decomposition scheme of the subgradient type to (28) and (29) we must know the feasible set X, such that $f_{F,I}(x)$ are finite. This set has implicit representation and it is difficult to take it into account directly.

We can avoid this difficulty using extra variables y_F and y_I in the objective function:

$$\bar{f}_F(x) = \min\{ (c_F, z_F) + (M_F, y_F) \} \tag{F}$$
$$A_F z_F + B_F x - y_F \leq d_F$$
$$z_F \geq 0, y_F \geq 0$$

$$\bar{f}_I(x) = \min\{ (c_I, z_I) + (M_I, y_I) \} \tag{I}$$
$$A_I z_I + B_I x - y_I \leq d_I$$
$$z_I \geq 0, y_I \geq 0$$

Now any x is feasible in subproblems (F) and (I), so we get rid of the feasibility problem. When x is feasible then (F) is equal to (28) and (I) is equal to (29). M_F and M_I are penalty vectors for violation of constraints. If their components are big enough the value of extra variables is zero at the optimal point and problem (27) has the same solution as the problem $\min(\overline{f}_F(x) + \overline{f}_I(x))$.

Let us denote

$$f(x) = \overline{f}_F(x) + \overline{f}_I(x) \tag{30}$$

$$g(x) = B_F^T u_F(x) + B_I^T u_I(x) \tag{31}$$

Here $u_F(x)$ is the optimal value of dual variables in F-subproblem, $u_I(x)$ the optimal value of dual variables in subproblem I. According to the standard LP theory, g(x) is a subgradient of function f(x).

The problem now can be reformulated as

$$\min f(x) \tag{32}$$

$$x \varepsilon E^2 \tag{33}$$

Once optimal x is found it is easy to solve the initial problem by solving the independent problems (28), (29).

Generally speaking, f(x) is a nondifferentiable piece-wise linear convex function. To find the subgradient g(x) we must know only the optimal value of dual variables in subproblems F and I. This can be done by solving the following dual problems:

$$\begin{aligned} &\max(u_F, B_F x - d_F) \\ &\quad c_F + u_F A_F \geq 0 \\ &\quad M_F - u_F \geq 0 \\ &\quad u_F \geq 0 \end{aligned} \tag{FD}$$

$$\max(u_I, B_I x - d_I) \qquad \text{(ID)}$$
$$c_I + u_I A_I \geq 0$$
$$M_I - u_I \geq 0$$
$$u_I \geq 0$$

In problems (FD) and (ID) only the objective function depends on x, so only the objective function changes with the number of iterations. Therefore the previous solutions $u_F(x^k), u_I(x^k)$ can be used as basic solutions for calculating the solution $u_F(x^{k+1}), u_I(x^{k+1})$ in the next iteration $k + 1$. For this reason it is possible to calculate $u_F(x^{k+1}), u_I(x^{k+1})$ very quickly.

5. One Method of Nondifferentiable Optimization

For minimization of the function f(x) we can use a method of the subgradient type. The simplest one is the following(Ermoliev76a, 78a, 79a).

$$x^{k+1} = x^k - \rho_k g(x^k)$$

where $g(x^k)$ is a subgradient of the function f(x) in the point x^k, $\rho_k \to +0, \sum_{k=0}^{\infty} \rho_k = \infty$. However, the convergence of this method is not very fast especially for ill-behaving functions. The convergence of this method is based on the decrese of the distance from the approximate solution x^k to the minimum point x^* when $k \to \infty$ and when the vector $g(x^k)$ is nearly orthogonal to the vector $x^k - x^*$ then this decrease is small and the convergence is slow. So to minimize function $f(x)$ (as well as any convex function) we can use another subgradient method which finds better descent directions. Its main idea is to use the information about some previous descent directions for obtaining the new one in the case when corresponding points lie not far from one another.

The procedure is the following:

$$x^{k+1} = x^k - \rho_k \frac{p^k}{\|p^k\|} \qquad (34)$$

$$p^k = \sum_{\tau=k_s}^{k} g(x^\tau) \qquad (35)$$

$\|\cdots\|$ is euclidean norm; ρ_k is step-size multiplier.

Points x^{k_s} are the special points in which one of two conditions must be fulfilled: either

$$\|x^{k_s+1} - x^{k_s}\| > \varepsilon_s \qquad (36)$$

or

$$k_{s+1} = k_s + L \qquad (37)$$

There are two versions of this method. The first version is described by the following theorem, which can easily be proved:

Theorem 1. If in method (34)-(37) for the step-size multiplier we have $\sum_{k=0}^{\infty} \rho_k = \infty, \rho_k \to +0$, then any accumulation point of the sequence $\{x^k\}$ belongs to the set $X^* = x^* : f(x^*) \min f(x)$.

There is also another way of choosing the step-size multiplier:

$$\rho_k = \gamma_k \frac{f(x^k) - c}{\|g(x^k)\|} \qquad (38)$$

where $c \geq \min f(x)$.

Theorem 2. Let us suppose that

$$0 < \gamma_* \leq \gamma_k \leq \gamma^* < 2, \varepsilon_s \to 0, s \to \infty$$

Then either such k^* exists that $x^* \varepsilon M(c)$ or any accumulation point belongs to set $M(c) = x \varepsilon X : f(x) \leq c$.

The proof of theorems 1,2 is based on the technique proposed in(79a).

This method is easy in computation and at the same time the results of the tests showed it's good convergence characteristics. The results of the computation are good for ravined-type functions which are difficult to deal with by straightforward subgradient methods.

6. Computer Implementation.

The program for solving the problem (32) - (33) was written and run with F77 compiler under the UNIX operating system on the Institute VAX-11/780. The executable file compiled from Fortran source has a name PROG and so it may be called by the following sentence

PROG1=FOR.DATA2=IND.DATA3=F14=F2

Files FOR.DATA and IND.DATA contain the initial matrices of forestry and industrial subproblems written in a compact form.

In the file F2 we obtain the values of x(1), x(2) and f(x) on each iteration.

In the file F1 some more information about the computational process is written.Here we have the iteration number, the subgradient value in each point x^k ,the value of the descent direction p^k, the step-size multiplier value and the value of the objective function of each subproblem. If each of the two linear programming subproblems is solved normally, than we have KOUT=2, otherwise these subproblems have no optimal solutions (something is wrong with the initial data). In F1 we also have the information about $\|p^k\|$ and if it is less than 10^{-12} the program inform you about restart. If the subgradient norm is less than 10^{-10} program terminates and informs about the value of this norm. All the information given in F1 we also can see on the screen of the terminal attached to the program as standard output.

In the main program of PROG the descent direction p^k is calculated and

In the main program of PROG the descent direction p^k is calculated and step (11) is realised. In this program step-size muultiplier is chosen according to the theorem 1. The part of the main program provides an interface between a user and a computer. User must specify:

1)the dimension NX of the linking vector X (for this problem NX=2);

2)the initial values of X(1) and X(2);

3)the number of iterations NITER;

4)the initial value RO of the step-size multiplier (on the step k the value of ρ_k is RO/K; in the subroutine PP the value of RO can be changed in order to obtain better convergence);

5)the minimum value of RO, ROMIN ; the maximum value of RO, ROMAX.

6)the value of the penalty coefficient CM (in this realisation all the components of vectors M_F and M_I are equal CM. The convergence of the process depends on this variable. We usually had it from(10^{-5} to 10^{-6});

7) interactive or automatic regulation of RO .

This dialogue is basically self-explanatory and is not particularly bound by formats.

The example of this dialogue:

```
the question        the answer

 NX     =           2
 XO     =           0.0
 XO     =           10.0
 NITER  =            100
 RO     =           1.0
 CM     =            1.0e5
 ROMIN  =             0.2
 ROMAX  =              20.0
 CHANGE RO ?
 ANSWER: YES=1 OR NO=0   0
```

Besides the main program there are also several subroutines. The basic ones are LLP, COPY, YTAB, FUN, GRAD and PP.

Subroutine LLP realises symplex method procedure (it was written by N.Orchard-Hays when he was with IIASA).

Subroutine COPY transforms the data in files FOR.DATA and IND.DATA from compact form into the full symplex tables.

Subroutine YTAB reads the data and forms the symplex tables for subproblems (F) and (I) which includes extra variables y_F, y_I.

Subroutine FUN obtains the value of f(x) in the point x.

Subroutine GRAD defines the subgradient vector GR of the function f(x).

Subroutine PP calculates the vector x^{k+1} given x^k and changes the value of a step-size RO, if necessary. Its input parameters are K, NX, X, P, NN, EN, ROMIN and ROMAX, X1 is an output variable and RO is input-output variable. This subroutine makes not more than NN steps from the point X in the direction P with the step-size multiplier ROK=RO/K. It makes such a step if the function value in the next point is less than in the previous one. If, on the contrary, the function increases than the process terminates and X1 is calculated as the arithmetic mean of two last points. If the number of successive steps is more than NK1 the value of RO is doubled (RO=2*RO). If this number is zero, i.e., the function did not decrease in this direction, than RO=RO/2. When the value of RO becomes less than ROMIN we have RO=ROMIN in the case of automatic regulation (EN=0) or the user must set a new value of RO (EN=1). So is the situation when the value of RO becomes more than ROMAX.

The main process terminates when either the subgradient norm is less than 10^{-10} or when the number of iterations is equal to NITER.

Results of the computation are given in figures 2 to 8 (here we have values of $|f(x)|$ instead of $f(x)$). The initial point was taken $x(1) = 0, x(2) = 0$, and the total number of iterations was taken equal to 80. We can see that on the first 50 - 55 iterations the function decreased quickly and values of x(1), x(2) also changed quickly. Then the process began to oxcillate around the optimal point. Such behavior is typical of the gradient typical of the gradient type methods. The minimum value of the function, (-140471.2), was achieved at the point $x(1) = 20.92$, $x(2) = 167.39$. These results correspond well with results obtained with the help of another approach ($f = -140480$, $x(1) = 20.92, x(2) = 167.43$) .

7. Conclusions

--Nondifferentiable optimization presents an adequate theoretical framework for linkage optimization problems.

--Computational experience with iterative procedures based on nondifferentiable optimization shows that they represent a robust, reliable means of solving linkage problems. They are especially valuable for getting first estimates of the solution, with possible application of other methods to the final solution. Iterative schemes of nondifferentiable optimization are particularly well suited as an algorithmic base of iterative linkage systems. The provide many ways of controlling a solution process which are well suited to the practical user.

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	W1	W2	W3	W4	W5	W6	W7	W8	W9	W10	W11	W12	W13	W14	W15	W16	W17	W18	W19	W20	W21	y	H1	H14	H15	H16	H17	H18	H19	H20	H21	S1	S2	cost / RHS
																																-29	-30	
B				.01	.02	.03	.05	.07	.09	.12	.15	.19	.23	.27	.32	.36	.41	.46	.51	.56	.60													≤ 1800
PL																						-1	1.3	5	5	5	5	5	5	5	5			≤ 0
W1	1																					-1												≤ 0
W2	-.7	1																																= 0
W3		-.95	1																															= 0
W4			-.95	1																			.05											= 0
W5				-1	1																		.05											= 0
W6					-1	1																	.20											= 0
W7						-1	1																.20											= 0
W8							-1	1															.10											= 0
W9								-1	1														.05											= 0
W10									-1	1													.05											= 0
W11										-1	1												.10											= 0
W12											-1	1											.10											= 0
W13												-1	1										.20											= 0
W14													-1	1									.25	1										= 0
W15														-1	1										1									= 0
W16															-1	1										1								= 0
W17																-1	1										1							= 0
W18																	-1	1										1						= 0
W19																		-1	1										1					= 0
W20																			-1	1										1				= 0
W21																				-.99	.05										1			= 0
S1																							.02	.10	.14	.19	.23	.26	.30	.38	.45	-1		≥ 0
S2																							.13	.17	.17	.18	.19	.20	.20	.18	.15		-1	≥ 0

Figure 1. Subproblem "Forestry".

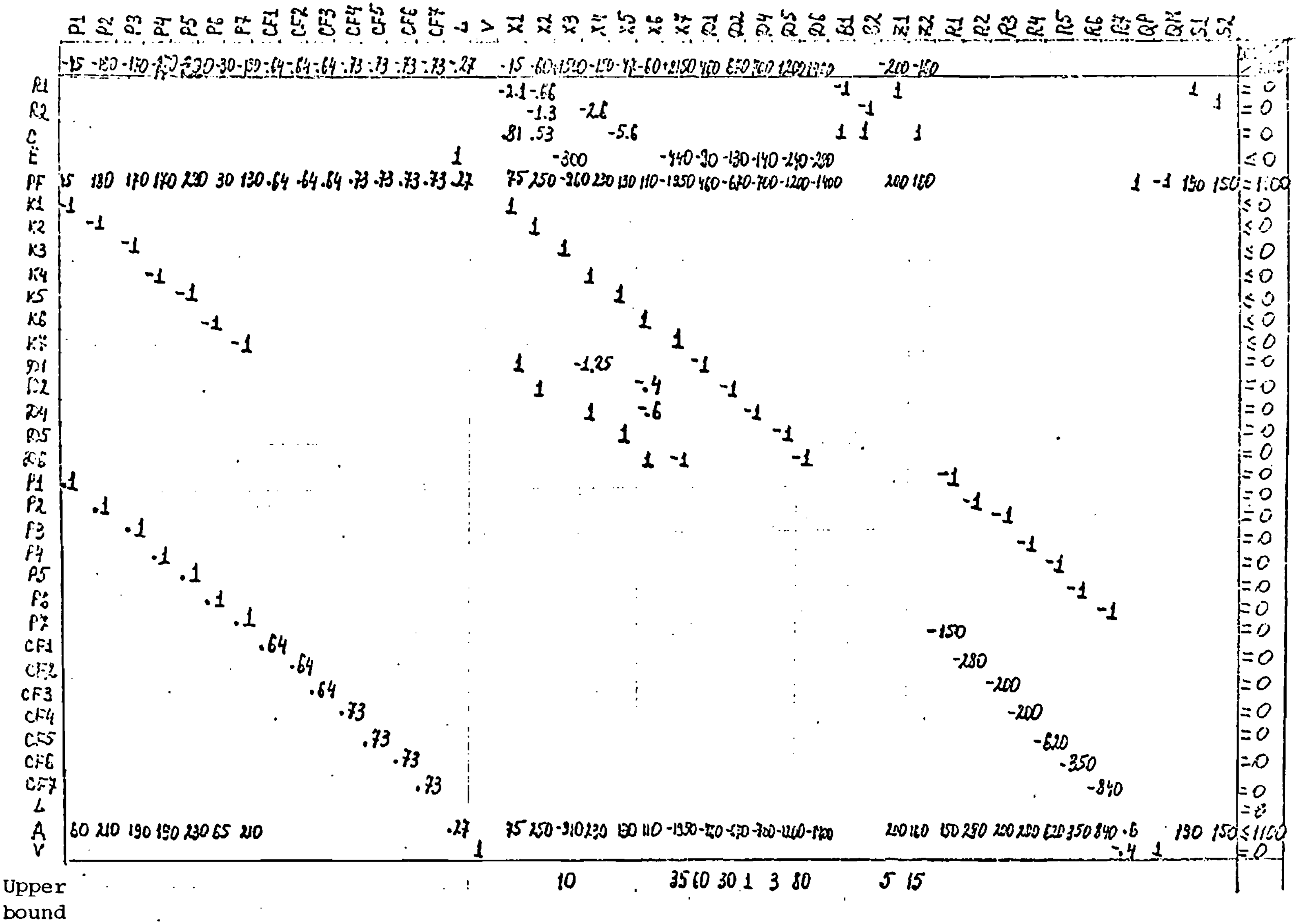

Figure 2. Subproblem "Industry".

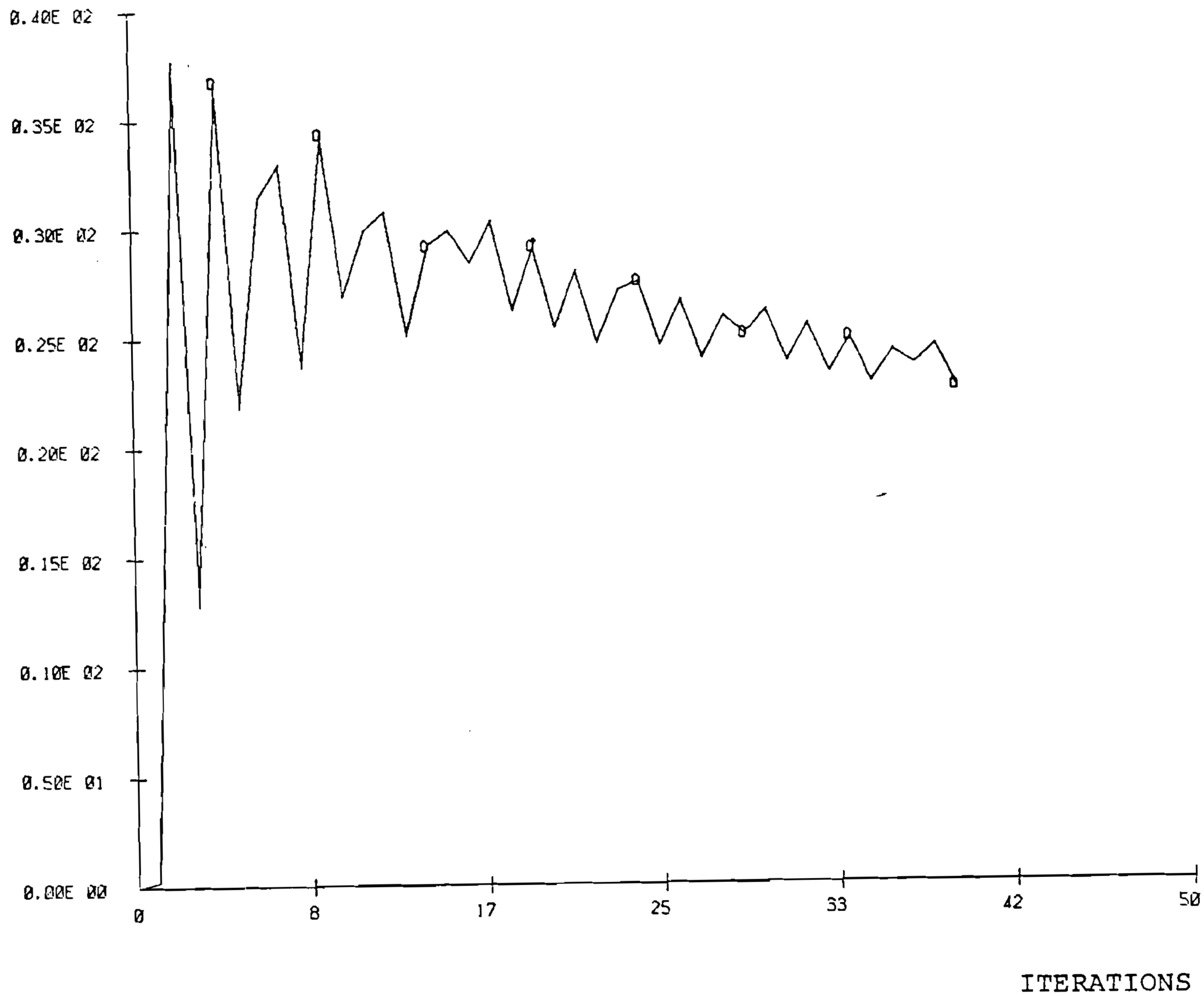

Figure 3. Convergence of the X1 linking variable.

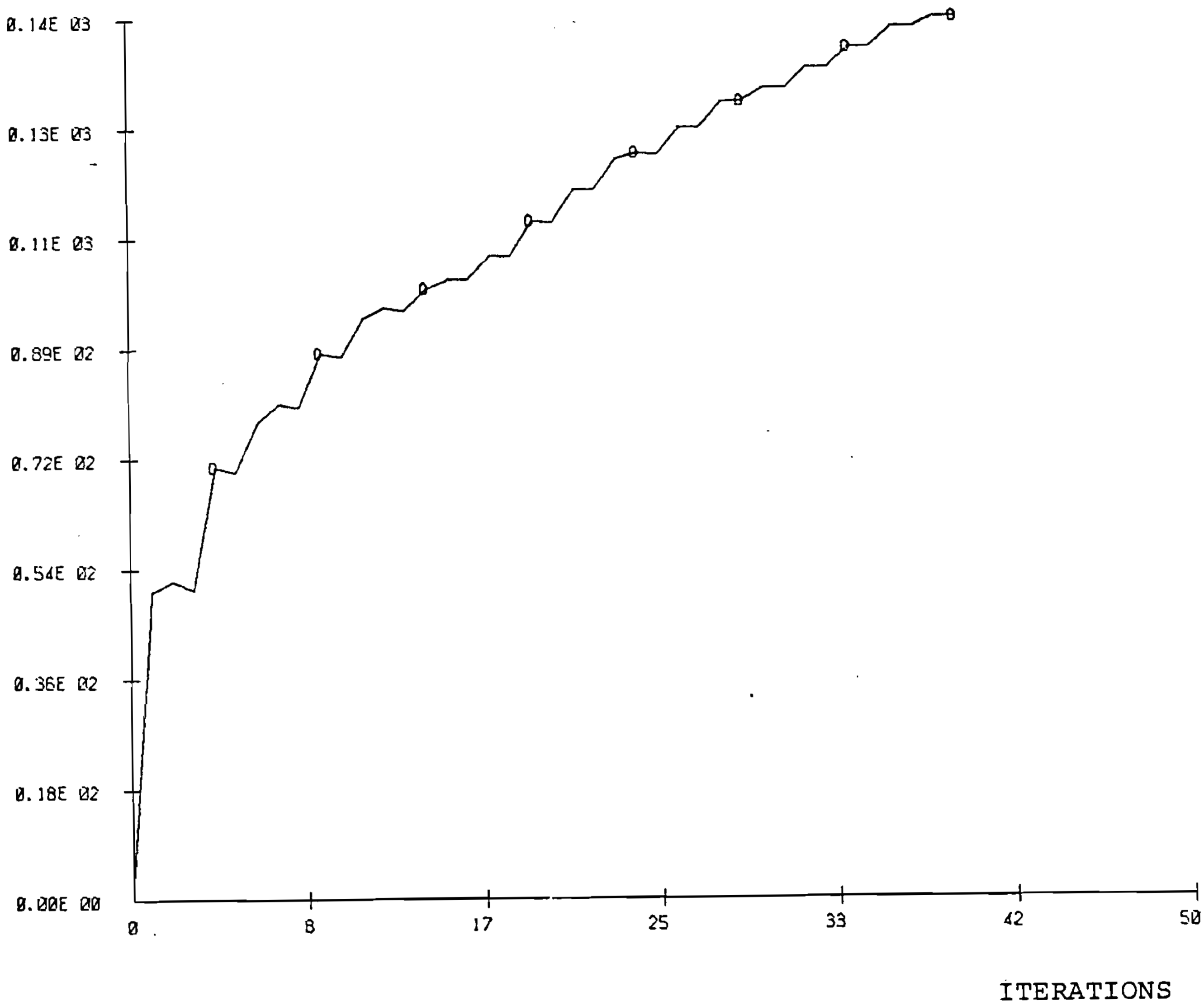

Figure 4. Convergence of the X2 linking variable

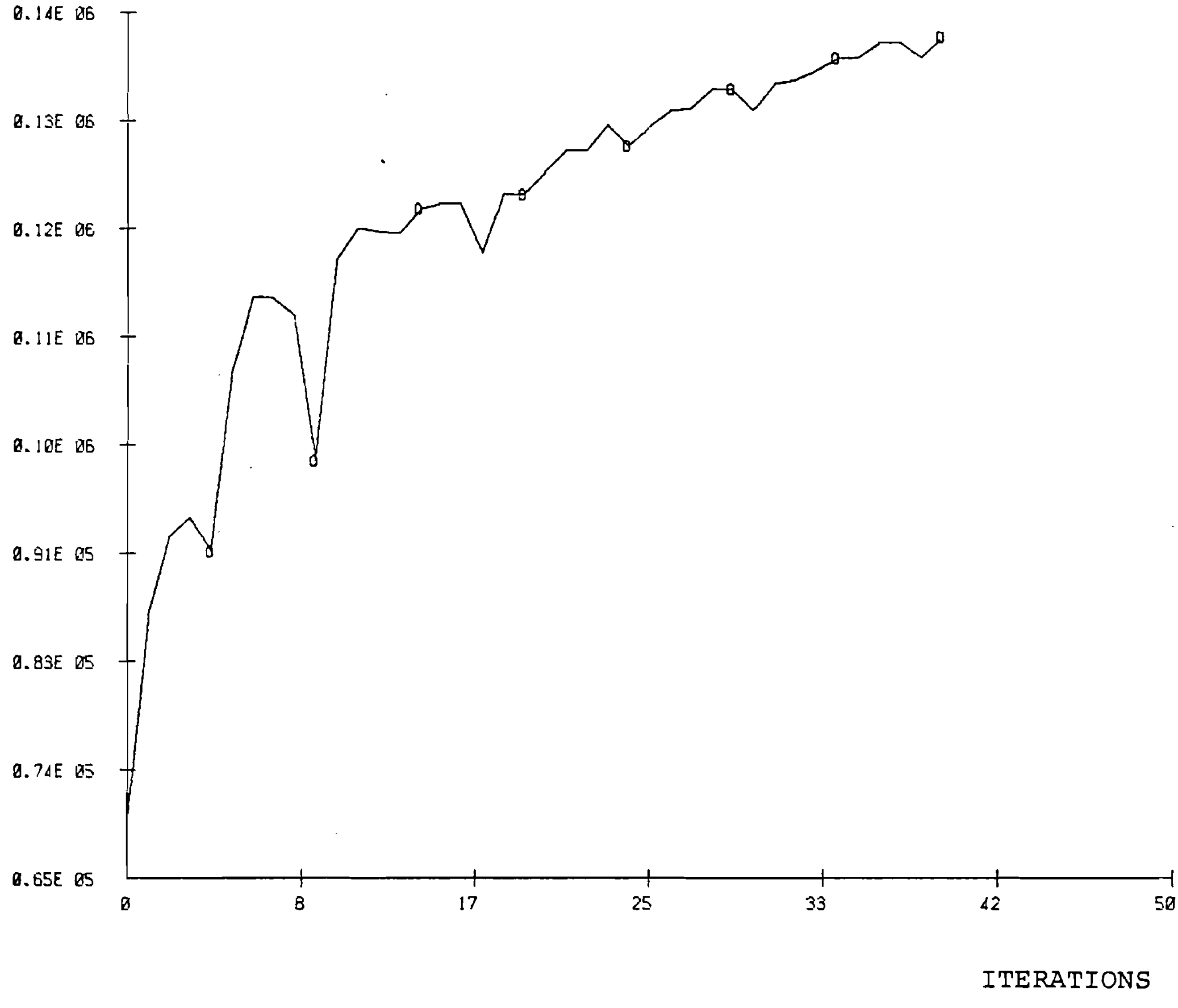

Figure 5. Convergence of the value of the objective function.

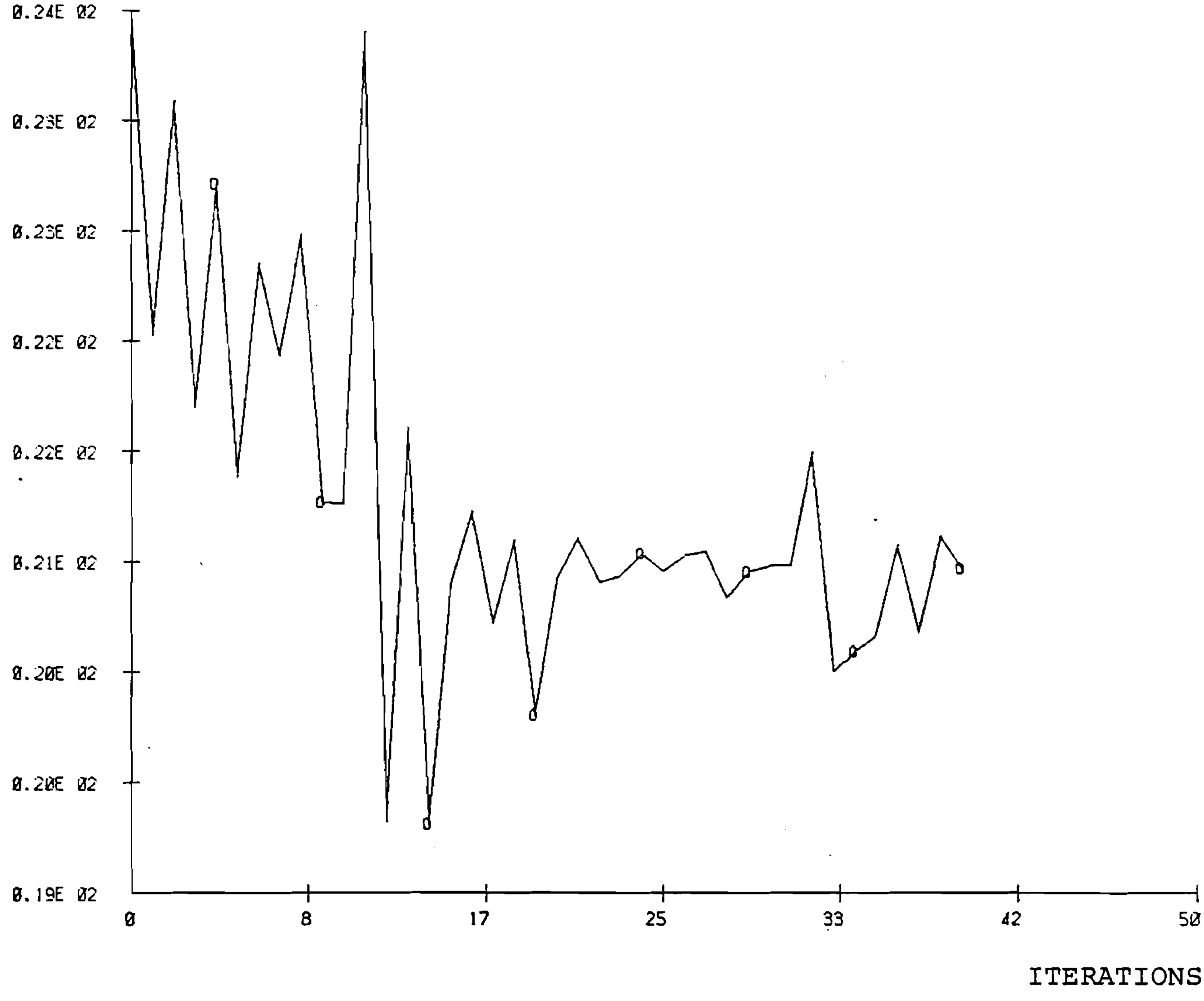

Figure 6. Convergence of the X1 linking variable.

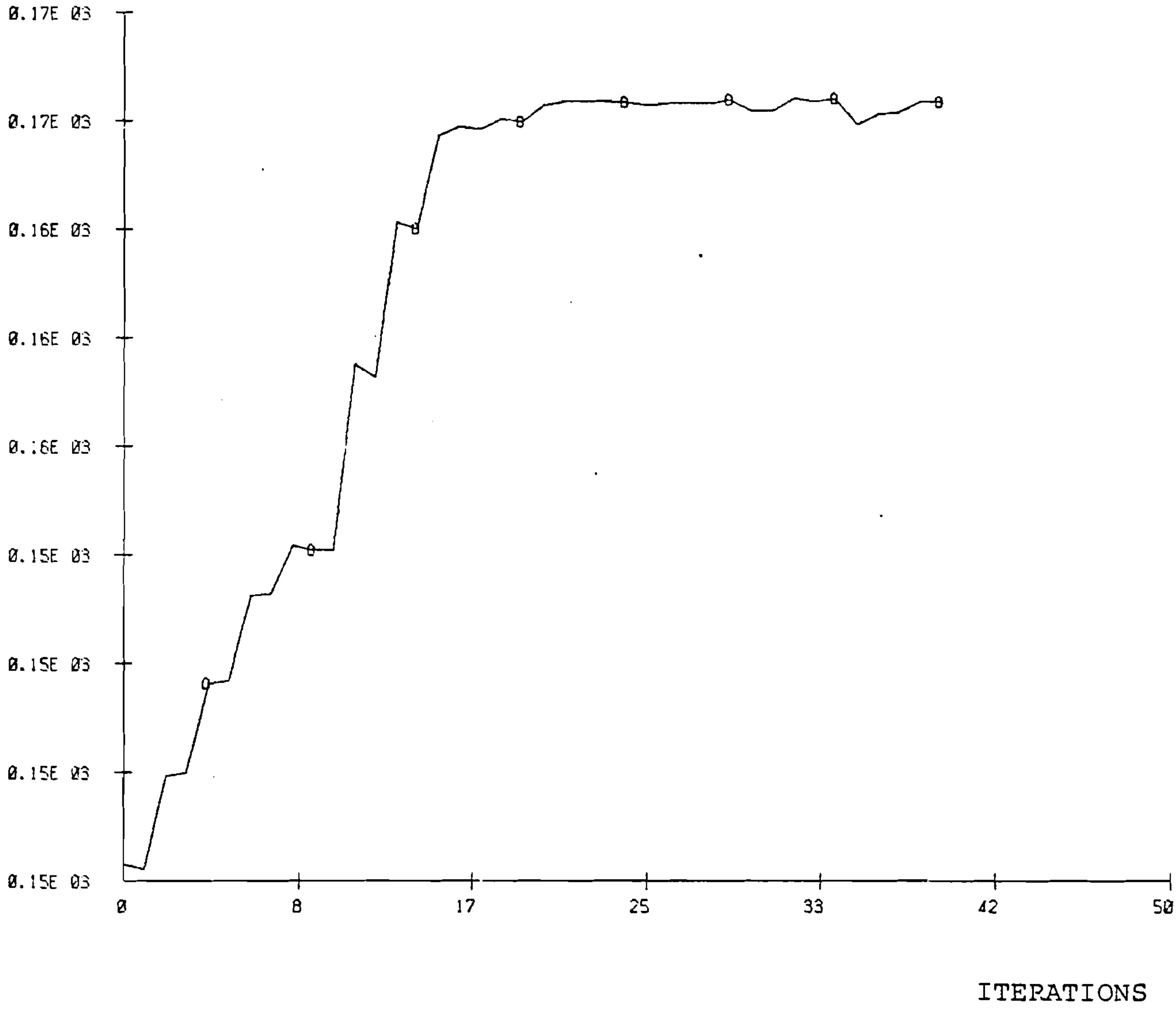

Figure 7. Convergence of the X2 linking variable.

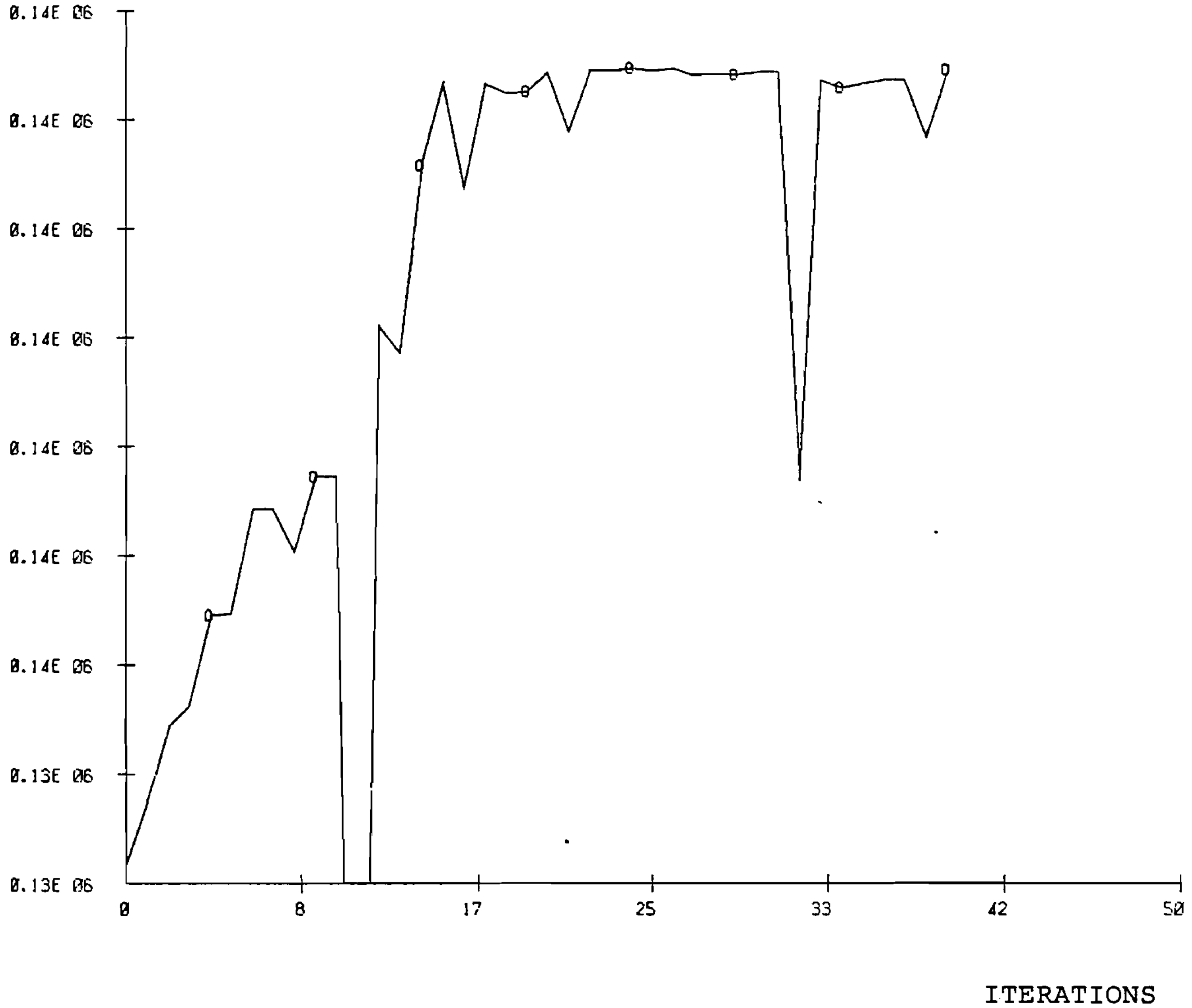

Figure 8. Convergence of the value of the objective function.