INVESTIGATION OF THE MORPHOLOGICAL SPACE OF SYSTEMS VARIANTS

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PREFACE

One of the main and difficult problems of systems analysis is that of generating the system's variants. The method of morphological analysis is widely used for the solution of this problem. This paper is concerned with the investigation of the problem of morphological space metrization. The paper includes questions on the theoretical justification of a metric in a morphological space, and contains analytical relations for the calculation of the number of points in neighborhoods of the morphological space. Questions on the determination of medians and antimedians are also discussed here. The above-mentioned results can be useful for decision-makers and systems analysts who apply the method of morphological analysis in practice. The author considers metricized morphological space as also of interest to mathematicians.

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CONTENTS

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Vladimir Iakimets

1. INTRODUCTION

An important and difficult problem of applied systems analysis is that of generating variants of the system under consideration (or variants of solutions of the problem under study). The analysis of the problem of generating variants allows one to observe the following contradiction: on the one hand, it is desirable to study as carefully and completely as possible all opportunities for the construction of system variants. On the other hand, the number of variants to be generated has to be admissable, i.e. should not exceed the abilities of the researcher. Besides, the variants generated should be preferable to others. Therefore we require methods of generating the required number of preferable (in some previously defined sense) variants of the system under study. The same problem arises, for example, in connection with the generation of technological variants within the framework of the Task 2 model being developed by the Food and Agriculture Program (FAP) of IIASA. Task 2 is concerned with the analysis of interactions between technological changes in the food production chain, the environment and resource use in the long run. The main aims and approaches to be taken in the study of these problems are outlined in a draft paper by J. Hirs (1980). The purpose of the Task 2 model is to analyze the limits and consequences of food production over a longer time period. A recursive linear programming model will be employed in the modeling approach for the cast studies. The linear programming activities in the LP matrix include various technologies for the production of different agricultural commodities. At least two paths exist for the generation of these activities (D. Reneau et al., 1981).

The production of each commodity can be considered as a series of steps, each step being a separate technological process. Each step has its own set of activities with the required variable inputs. However, this method will require a large number of transfer rows in order to trace all the available methods of production. This will lead to an increase in computation time and expense.

Another method is one where the production of each commodity is combined beforehand and each combination entered as a separate activity. In this case the number of transfer rows will be reduced significantly and the cost of solving the LP matrix will likewise be reduced.

The morphological analysis method can be used for implementation of the second method. However, the application of the traditional method of morphological analysis for the generation of variants results in a larger number of variants since the number of agricultural commodity technologies is large enough and there are constraints on the LP matrix dimension. In this case it is necessary to generate only the required number of preferable morphological variants. One way to achieve this is to use the notion of a distance between morphological variants.

This paper considers a theoretical investigation of the method of morphological analysis, as well as the development of the problem of metrization of the space of the morphological variants of the system under study.

The idea of introducing the distance function between morphological variants of a system stems from R. Ayres (1969). Ayres suggests a number of terms such as "morphological space", "morphological distance", "morphological neighborhood", "surface of a morphological neighborhood", and "area of the surface" and gives their informal definitions. He builds a model for the analysis of the potential possibilities for developing the technological area under consideration by using these terms. This model allows one to estimate the probability of a breakthrough in the technological area depending on the extent of the area's exploration.

Unfortunately, Ayres does not give a rigorous justification of his idea of the morphological space metrization. In my opinion, a theoretical justification should be made because this idea is fruitful for decision making in the technological area under study. The investigation of metric properties of a morphological space of the system's variants is especially useful when the system under consideration has many functions, and each of these functions can be implemented in many ways.

There are six groups of questions connected with the investigation of the metric morphological space:

1. Terminological aspects;
2. Problems of formalization;
3. Theoretical justification of an approach;
4. Quantative analysis of neighborhoods of the morphological space;
5. Algorithmic aspects;
6. Problems of interpretation and utilization of theoretical results.

In this paper, the author touches upon all these questions, but most attention is given to the problems mentioned in points 2, 3 and 4. The reader interested in acquainting himself with more detailed information on the other problems, can find corresponding results in the thesis (Iakimets, 1980).

This paper is divided into eight sections. In the introduction, the description of the problem under study and the contents of all the sections of the paper are given. The method of morphological analysis and Ayres' approach are described briefly in Section 2. Section 3 is devoted to the theoretical justification of a metric in a morphological space. It includes axiomatics of the metric and proof of the existence and uniqueness of the metric. Section 4 contains formal definitions of the equidistant set and the spherical neighborhood of any points of the morphological space and the description of some of its properties. In Section 5 the formulas for the calculation of the number of points of the equidistant sets and the spherical neighborhoods of any radius are deduced. Section 6 contains the formula for the calculation of the number of points of the intersection of spherical neighborhoods. In Section 7 the terms median and antimedian, and the algorithm "Median" are discussed. In the last section, conclusions are drawn up.

2. Description of the Method of Morphological Analysis

The method of morphological analysis was first suggested by a Swiss astronomer Fritz Zwicky. This method aims to identify, classify and count all the possible variants of a specific problem solution. Zwicky developed a number of specific morphological methods such as "Negation and Construction", "Morphological Box", and "Total Field Coverage", which are described in his book (Zwicky, 1969).

The method of a Morphological Box seems to be most adequate in connection with the problem of generating the system's variants. This method is widely used in different fields of science and technology: biology, geography, sociology, economics, technological forecasting, etc.

This method consists of the following stages (Ayres, 1969; Zwicky, 1969; and Jantsch, 1967):

1. Statement of the problem under study.
2. Identification and listing of all characteristic parameters of this problem, or the listing of all specific functions of a system under consideration.
3. Identification and listing of all the possible realizations of each of these parameters (or all the variants of each of the specific system's functions). As a result, we have the morphological box where $N = \prod_{\ell=1}^{L} k_\ell$ variants are contained. Here:

N is the number of the theoretically possible morphological variants of a system;

L is the number of the specific functions of a system;

k_ℓ is the number of all variants of the realization of l - th system's specific function, $\ell \in \overline{1,L}$.

4. Determination of the functional utility of morphological variants.
5. Selection and implementation of the most preferable morphological variants.

The structure of a morphological box containing all the possible variants of some system under study is shown in Figure 1. One of the N theoretically possible morphological variants of a system includes those variants of the realization of the single function which are marked by circles. Later, variants of the realization of the single function will be called "elements".

There are many questions connected with the implementation of each of this method's stages. For example:

-- what ways should be used to state our problem (or to describe a system under study) and by whom can the precision of our statement be evaluated, and by what way?

-- what methods can we use for the evaluation of the completeness of lists of the system's functions and elements?

-- what methods should be used for the selection of a preferable system's variants, taking into consideration the complexity of a system under study and the number of possible variants?

System's Functions	All Possible Realizations of Each of the System's Functions						The Number of Elements
Λ_1	λ_{11}	λ_{12}	λ_{13}	λ_{14}	...	λ_{1k_1}	k_1
Λ_2	λ_{21}	λ_{22}	λ_{23}	...	λ_{2k_2}		k_2
...	...	.	...	...			...
Λ_ℓ	$\lambda_{\ell 1}$	$\lambda_{\ell 2}$	$\lambda_{\ell 3}$	...	$\lambda_{\ell k_\ell}$		k_ℓ
...	...	.	...	...			...
Λ_L	λ_{Li}	λ_{L2}	λ_{L3}	...	λ_{Lk_L}		k_L

Fig. 1 Structure of a Morphological Box

One of the main difficulties of applying the morphological analysis method is connected with the fact that the number of systems variants is usually great. For example, if the morphological box has the following parameters: $L = 5$; $\forall \ell \in \overline{1,5} : k_\ell = 10$, then we have $N = 10^5$ theoretically possible systems variants. It is difficult for a researcher to deal with all of them. The problem of localization of the area of analyzable morphological variants therefore arises.

Ayres' concept of the metrization of the morphological space can be used for the solution of this problem.

In his book (Ayres, 1969) he introduces the term morphological distance between any two morphological systems variants in the morphological space. He describes the model for the analysis of the opportunities for technological progress in the technological area under consideration. This model can be briefly characterized in the following way.

The set of all the system's variants is divided into two groups: the known variants and the unknown ones. The known variants are ones which have already been realized. The unknown variants are those which consist of new combinations of systems elements. Ayres considers research and development devoted to the detailed investigation of the known variants with the objective of improving upon their performance characteristics. If we vary the elements of known variants one at a time, keeping the others constant, then subsequent exploration (determination and analysis of new favorable variants) is achieved.

In order to estimate the probability of a breakthrough in a technological area under study, Ayres introduces the following new terms:

1. The MORPHOLOGICAL SPACE of a broad area of technology consists of a set of discrete points or "coordinates", each corresponding to a particular combination of variables or parameters and each representable by a set of indices $\left\{P_k^j\right\}$ *. The space has as many dimensions as variables.
2. The MORPHOLOGICAL DISTANCE between two points in the space is the number of parameters wherein the two configurations (variants of a system - V.I.) differ from one another.
3. A MORPHOLOGICAL NEIGHBORHOOD is a subset of points, each of which is morphologically close to the other.
4. The SURFACE of a morphological neighborhood is the set of all configurations differing in at most a single parameter from the points in the neighborhood. The AREA of the surface is the number of such points. A WEIGHTED AREA can also be defined by summing up the the numbers of points differing by one, two, three, etc. parameters, multiplied by appropriately decreasing coefficients, α_1, α_2, α_3, etc.
5. Each time a new configuration becomes realizable in actuality, as a result of exploratory research and development, a TECHNOLOGICAL BREAKTHROUGH may be said to have been achieved.

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* P_k^j is the j-th element of the k-th system's function.

According to Ayres "the probability of a breakthrough in a technological area per unit time, is a decreasing function of its morphological distance from existing art, other things being equal." In his book Ayres gives the illustrative example of the investigations of 3-dimensional morphological space which consists of the 27 variants.

However, in the cited book and subsequent publications, Ayres doesn't give the theoretical justification of this metric and also he doesn't give the formal relations for the calculation of the number of points in any neighborhood of the morphological space.

The requirements of the theoretical justification of a metric and the requirement of relations for the calculation of the "volume" of neighborhoods arises in connection with a solution to our problem of generating the required number of preferable systems variants. These results are especially necessary if a researcher has limitations on investments and time for the development of preferable systems variants and if he is interested in the implementation of some defined strategy of set variants investigation. For example, he can be interested in the analysis of all the modifications of any variant, or in the search for all the possible intermediate variants with respect to some marked points of the morphological space, or in the determination of all radically new variants.

The subsequent sections of this paper will be devoted to the investigation of the problem of metrization of a morphological space.

3. Metric in a Space of Morphological Systems Variants

3.1 Morphological Space

Let there be given:

-- a finite set $E = \{e\}$; elements of this set we shall call elements of a system;

-- partitioning $\sigma: E \rightarrow \overline{1,L}$ of the set E on morphological classes $\sigma^{-1}(\ell)$, $\ell \in \overline{1,L}$, $\sigma^{-1}(\ell) \cap \sigma^{-1}(\ell^1) = \emptyset$, if $\ell \neq \ell^1$;

-- morphological variant of a system $\bar{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_L)$.

Definition 1.

Morphological space Λ is called a subset of the set 2^E such that $\forall \bar{\lambda} \in \Lambda, \forall \ell \in \overline{1,L}$, $\bar{\lambda} \cap \sigma^{-1}(\ell)$ is a one-element set.

It follows from definition 1 that the morphological variant $\bar{\lambda} = (\lambda_1, \ldots, \lambda_\ell, \ldots, \lambda_L)$ of a system is the set of the representatives of classes $\sigma^{-1}(\ell)$, $\ell \in \overline{1,L}$.

If each of the classes $\sigma^{-1}(\ell)$ is a linearly ordered set and all classes are also linearly ordered, then the morphological space can be represented in the form of L - dimensional integral lattice. Really, in this case $\forall \ell \in \overline{1,L}$: $\sigma^{-1}(\ell) \simeq \overline{1, |\sigma^{-1}(\ell)|} \overline{\overline{\Delta}} \Lambda_\ell$

and

$$\Lambda \simeq \prod_{\ell=1}^{L} \Lambda_\ell \qquad (1)$$

It should be noted that the identification Λ with the Cartesian product is not correct in a general case, because the ordering of classes and ordering of elements in each of the classes are by convention. However, we shall use the representation of morphological space in form (1), later, taking into account this convention and that the enumeration of coordinates and elements of sets Λ_ℓ provides the equivalent representation of a morphological space.

3.2 Axiomatics

Let $\bar{x} = (x_1, x_2, \ldots, x_L)$, $\bar{x} \in \Lambda$ be a morphological variant and $\rho(\bar{x}, \bar{y})$ be a distance between variants $\bar{x}$ and $\bar{y}$. Function $\rho: \Lambda^2 \to [0, \infty)$ must possess the properties which are customary for a distance function (Korn and Korn, 1968; Kolmogorov and Fomin, 1957) and, in addition, those properties which characterize the distance between morphological variants.

1 Let us consider the axiomatics of a metric in a morphological space. The first three axioms are the known metric axioms

<u>Axiom 1</u> (non-negativity)

$\rho(\bar{x}, \bar{y}) \geq 0.$

$\rho(\bar{x}, \bar{y}) = 0$ if and only if $\bar{x} = \bar{y}$

<u>Axiom 2</u> (symmetry)

$\rho(\bar{x}, \bar{y}) = \rho(\bar{y}, \bar{x}).$

<u>Axiom 3</u> (triangle inequality)

$\rho(\bar{x}, \bar{z}) + \rho(\bar{z}, \bar{y}) \geq \rho(\bar{x}, \bar{y}).$

Let us introduce the additional axioms. First of all it is necessary to define the notion "between".

<u>Definition 2</u>

Ternary relation $R \subset \Lambda^3$ is called the betweenness relation if and only if

$$\forall \bar{x}, \bar{y}, \bar{z} \in \Lambda = \prod_{\ell=1}^{L} \Lambda_\ell : \bar{z}\, R(\bar{x}, \bar{y}) \Leftrightarrow \forall \ell \in \overline{1,L}: (z_\ell = x_\ell) \vee (z_\ell = y_\ell).$$

By definition 2, the morphological variant $\bar{z}$ is considered lying between variants $\bar{x}$ and $\bar{y}$ if it is made up only of elements from variants $\bar{x}$ and $\bar{y}$.

Axiom 4 ("between variant")

$$\rho\ (\bar{x}, \bar{z}) + \rho(\bar{z}, \bar{y}) = \rho(\bar{x}, \bar{y}) \Leftrightarrow \bar{z}\ R(\bar{x}, \bar{y}) \quad .$$

Axiom 5 (segment)

$$\forall\ \bar{x},\bar{y} \in \Lambda \ \doteq\ \rho(\bar{x}, \bar{y}) = \rho(\bar{x}', \bar{y}') \qquad (2)$$

Here $\bar{x}'$ and $\bar{y}'$ denote projections of vectors $\bar{x}$ and $\bar{y}$ respectively in a subspace spanned on the basis vectors with numbers of unequal coordinates of vectors $\bar{x}$ and $\bar{y}$. This axiom means that the distance between $\bar{x}$ and $\bar{y}$ depends only on the existence of not identical elements in these variants. It should be stressed that essentially (2) is the requirement of the concordance of metrics in subsets

$$\prod_{\ell \in L \subset \overline{1,L}} \Lambda_\ell \ .$$

Axiom 6 (gauge axiom)

$(L = 1) \Rightarrow \forall\, x,\ y \in \Lambda : \rho(x, y) = 1.$

Assertion 1

$$\forall\ L,\ \min_{\{\bar{x},\ \bar{y} \in \Lambda\ :\ \bar{x} \neq \bar{y}\}} \rho(\bar{x}, \bar{y}) = 1 \qquad (3)$$

It should be noted that the relation (3) is impossible to use as a gauge axiom because axiom 6 is not deduced from relation (3). The proof of this assertion and some other assertions are not given in this paper (please see Iakimets, 1980).

3.3 Existence and Uniqueness of the Metric.

Theorem 1

Axioms 1 - 6 define identically metric ρ in the morphological space at any L, $L \geq 1$ and

$$\rho(\bar{x}, \bar{y}) = \sum_{\ell=1}^{L} (1-\delta_{x_\ell, y_\ell}) , \qquad (4)$$

$$\delta_{x_\ell, y_\ell} = \begin{cases} 1, & \text{if } x_\ell = y_\ell , \\ 0, & \text{otherwise.} \end{cases}$$

Proof

Let us prove this assertion by induction on L. Λ is represented as $\Lambda = \prod_{\ell=1}^{L} \Lambda_\ell$. If L = 1, then by axiom 1 $\rho(x_i, x_i) = 0$

for pairs (x_i, x_i). By virtue of definition 2 there is no intermediate variant for pairs (x_i, x_j), $i \neq j$. Then by axiom 1 $\rho(x_i, x_j) = 1$.

Let us assume that the distance is identically defined for all morphological spaces of dimension L, L << T. Let $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_t\}$ be the subset of numbers of noncoincident coordinates of vectors $\bar{x}$ and $\bar{y}$, $\lambda_\tau < \lambda_{\tau+1}$, $\tau \in \overline{1, t-1}$.

If $t < T$, then by virtue of axiom 5 $\rho(\bar{x}, \bar{y}) = \rho(x_{\lambda_t}, \ldots, x_{\lambda_t}, y_{\lambda_1}, \ldots, y_{\lambda_t})$ and therefore by assumption of induction it is an identically defined value. If all coordinates of vectors are different: $x_i \neq y_i$, $i \in \overline{1, T}$, then we consider the following sequences of vectors:

$$x^i = (x_i, x_{i+1}, \ldots, x_T) = (x_i, x^{i+1}) \quad ,$$

$$y^i = (y_i, y^{i+1}) \quad ,$$

$$z^i = (x_i, y^{i+1}) \quad .$$

Vector z^i is situated between x^i and y^i by Definition 2 and, hence, by axiom 4

$$\rho_i = \rho\ (x^i, y^i) = \rho\ (x^i, z^i) + \rho\ (z^i, y^i) \quad .$$

By assumption of induction all ρ_i, $i \in \overline{2, T}$ are defined identically.

It follows from axioms 5 and 6 that

$$\rho_i - \rho_{i+1} = 1 \quad . \tag{5}$$

Since $\rho_T = \rho\ (x^T, y^T) = \rho\ (x_T , y_T) = 1$ and $\rho\ (\bar{x}, \bar{y}) = \rho_1$,

then summing (5) over i from 1 to T - 1, we define identically $\rho_i = T$.

Axiom 1 is fulfilled because of $\sum_{\ell=1}^{L} \delta_{x_\ell, y_\ell} \leq L$, and $\delta_{x_\ell, y_\ell} = 1$.

Axiom 2 follows from $\delta_{x_\ell, y_\ell} = \delta_{y_\ell, x_\ell}$.

It is easily seen that

$$1 + \delta_{x_\ell, z_\ell} \geq \delta_{x_\ell, y_\ell} + \delta_{y_\ell, z_\ell} \quad . \tag{6}$$

Summing (6) over $\ell \in \overline{1,L}$, we obtain

$$L - \sum_{\ell=1}^{L} \delta_{x_\ell, y_\ell} + L - \sum_{\ell=1}^{L} \delta_{y_\ell, z_\ell} \geq L - \sum_{\ell=1}^{L} \delta_{x_\ell, z_\ell} \quad ,$$

which corresponds to

$$\rho\ (\bar{x},\bar{y}) + \rho\ (\bar{y},\bar{z}) \geq \rho(\bar{x},\bar{z}) \quad . \qquad (7)$$

Taking into account (7) and also that (6) is turned into equality only at $x_\ell = y_\ell$ or $y_\ell = z_\ell$, we convince ourselves in the fulfillment of axiom 4.

Let $T = |\ \{\ell \in \overline{1,L} : x_\ell = y_\ell\}\ |$. It follows from (4) that

$$\rho\ (\bar{x},\bar{y}) = L - \Big(\sum_{\ell \in T} \delta_{x_\ell, y_\ell} + \sum_{\ell \notin T} \delta_{x_\ell, y_\ell} \Big) = L - T \quad .$$

And if $\bar{x}'$ and $\bar{y}'$ are the same as in axiom 5, then

$p\ (\bar{x}', \bar{y}') = L - T$. Therefore axiom 5 is fulfilled. Axiom 6 is fulfilled also because if there is not an intermediate variant between $\bar{x}$ and $\bar{y}$, then they are distinguished by not more than one element. Hence $\sum_{\ell=1}^{L} \delta_{x_\ell, y_\ell} = L - 1$ and then $\rho\ (\bar{x},\bar{y}) = 1$.

4. Investigation of Neighborhoods of Morphological Space

The metric in a morphological space allows one to give the precise and identical meaning of notions connected with the similarity and difference of variants by using the quantitative evaluation of distinction from one variant to another. We can estimate the number of variants differing from some marked variant by the fixed number of elements, determine all of the intermediate variants with respect to the set of marked variants, analyze the intersection of variant groups and so on. It allows us to solve the problems of directed generating of morphological variants taking into account the preference structure of a researcher and the limitations in investment and time for the development of a system under study. As a result, a researcher can plan the rational and justified study of new variants.

The problems of evaluation of "volume" of the neighborhoods to be studied are important. It is necessary to be able to calculate the numbers of points of neighborhoods of the systems prototypes determined by the researcher.

Let us introduce the necessary notations.

Let $\Lambda = \prod_{\ell=1}^{L} \Lambda_\ell$ and r and ρ are integral non-negative numbers.

<u>Definition 3</u>

The equidistant set $U_{\rho,L}(\bar{x})$ is called the subset of points $\bar{y}, \bar{y} \in \Lambda$ situated on the distance ρ from the point $\bar{x}, \bar{x} \in \Lambda$:

$$U_{\rho,L}(\bar{x}) = \{\bar{y} : \rho(\bar{x},\bar{y}) = \rho\} \tag{8}$$

<u>Definition 4</u>

The spherical neighborhood $W_{r,L}(\bar{x})$ of radius r about point $\bar{x}$, $\bar{x} \in \Lambda$ is called the subset of points $\bar{y}$, $\bar{y} \in \Lambda$ situated at a distance of no more than r from point $\bar{x}$, $\bar{x} \in \Lambda$:

$$W_{r,L}(\bar{x}) = \{\bar{y} : \rho(\bar{x},\bar{y}) \leq r\} \tag{9}$$

Using the representation of morphological space Λ as an L-dimensional integral parellel-piped, we shall give examples of these neighborhoods. Examples of the equidistant sets and of the spherical neighborhoods in the case L = 2 are shown in Figures 2 and 3. The equidistant set $U_{1,2}(\bar{x})$ contains all elements marked by points, and $U_{2,2}(\bar{x})$ contains elements marked by crosses (fig. 2), $W_{1,2}(\bar{x})$ contains elements marked by points and $W_{2,2}(\bar{x})$

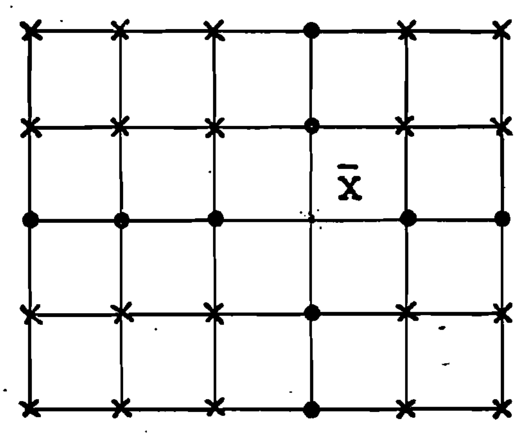


Figure 2.

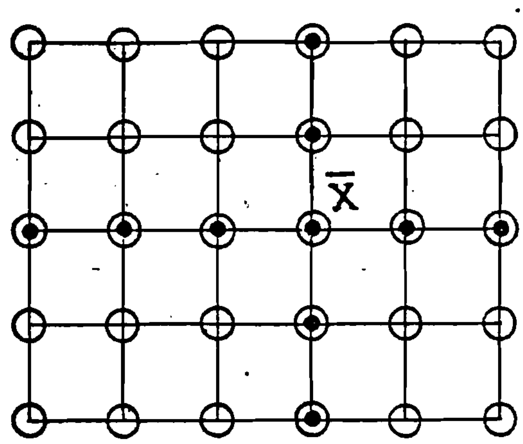


Figure 3.

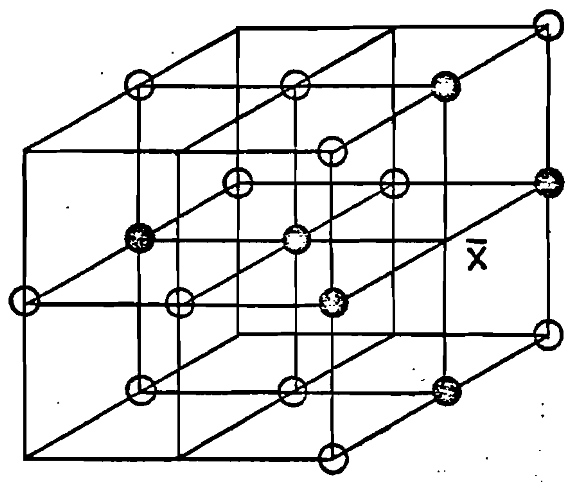


Figure 4.

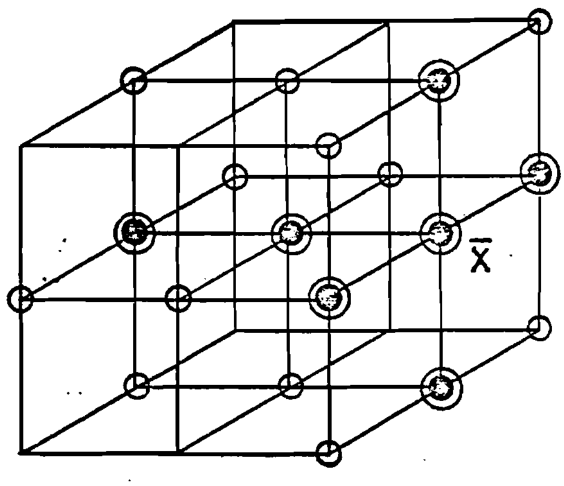


Figure 5.

- elements marked by circles (see Figure 3). The equidistant set and the spherical neighborhood in a case of three-dimensional space (L=3) are given in Figs. 4 & 5. $U_{1,3}(\bar{x})$ contains elements marked by points and $U_{2,3}(\bar{x})$ - elements marked by circles (see Figure 4). Elements of $W_{1,3}(\bar{x})$ are marked by points, and elements of $W_{2,3}(\bar{x})$ - by circles (see Figure 5).

Below some properties of the above mentioned objects of morphological space are given:

1. $\forall\ \bar{x} \in \Lambda : U_{\rho,L}(\bar{x}) \neq \emptyset \iff 0 \leq \rho \leq L$.

2. $\forall\ \bar{x} \in \Lambda : W_{o,L}(\bar{x}) = U_{o,L}(\bar{x}) = \bar{x}$.

3. $\forall\ \bar{x} \in \Lambda : W_{L,L}(\bar{x}) = \Lambda = \{\bar{x}\}$,

However, if $o < r < L$ (remembering that r is an integral number) then the center of the spherical neighborhood is unique.

4. $\forall\ \bar{x} \in \Lambda ,\ W_{r,L}(\bar{x}) = \bigcup_{\rho=o}^{r} U_{\rho,L}(\bar{x})$.

5. The Number of Points of the Equidistant Set and the Spherical Neighborhood of Any Radius

The number of points of the equidistant set and the spherical neighborhoods are important topological characteristics of a morphological space. This section of my paper contains results connected with the analytical and computer calculation of these numbers.

Let us consider the family of sets $\{D_{\rho,L},\ \rho \in \overline{1,L}\}$, in which each of these is a ρ-subset of set $\overline{1,L}$, whose elements are placed in increasing order.

$$D_{\rho,L} = \{\bar{\ell} : 1 \leq \ell_\rho < \ell_{\rho-1} < \ldots < \ell_1 \leq L\} ,$$

where

$$\bar{\ell} = \bar{\ell}\,(\rho) = (\ell_1, \ell_2, \ldots, \ell_j, \ldots, \ell_\rho) \quad .$$

Theorem 2

The number of points $N_{r,L}(\bar{x})$ of the spherical neighborhood of radius r, $1 \leq r \leq L$ about any point $\bar{x} \in \Lambda$ is determined by the relation:

$$N_{r,L} = 1 + \sum_{\rho=1}^{r} \sum_{D_{\rho,L}} \prod_{j=1}^{\rho} (k_{\ell_j} - 1) \quad , \tag{10}$$

where $k_\ell = |\Lambda_\ell|$ and ρ is the same as in (8).

Proof

It can be considered without losing generality that $\bar{x}$ has all coordinates equal to 1, i.e.

$$\bar{x} \equiv \bar{1} = \{1, \underbrace{1, \ldots, 1}_{L \text{ times}}\} \tag{11}$$

Taking into account (11) and the relations between the spherical neighborhood and equidistant sets of points enclosed in it (see properties 2 and 4, Section 4), we obtain the number of points of the spherical neighborhood which is determined by summing up numbers of points of corresponding equidistant sets

$$N_{r,L}\,(\bar{x}) = |\{\bar{y} \in \Lambda : \rho\,(\bar{1},\bar{y}) \leq r\}| = \sum_{\rho=0}^{r} M_{\rho,L}\,(\bar{x}) \tag{12}$$

where $M_{\rho,L}\,(\bar{x}) = |U_{\rho,L}\,(\bar{x})|$ is the number of points of the equidistant set $U_{\rho,L}\,(\bar{x})$.

Let $\bar{x}$, which has only coordinates $\ell_1, \ell_2, \ldots, \ell_\rho$ differing from 1 (i.e. $\bar{x}$ has only ρ noncoincidencing coordinates with $\bar{1}$), belong to the equidistant set $U_{\rho,L}(\bar{x})$. There are only $(k_{\ell_j} - 1)$ ways to choose the value of the coordinate ℓ_j. Hence, when the equidistant set $U_{\rho,L}(\bar{x})$ has only such points that contain coordinates $\ell_1, \ldots, \ell_\rho$ and only those which differ from 1, then the number of points of this equidistant set is equal to $\prod_{j=1}^{\rho} (k_{\ell_j} - 1)$. Summing up all $\bar{\ell} \in D_{\rho,L}$ we obtain:

$$M_{\rho,L}(\bar{x}) = \sum_{\bar{\ell} \in D_{\rho,L}} \prod_{j=1}^{\rho} (k_{\ell_j} - 1) \quad . \qquad (13)$$

Taking into account that the equidistant set $U_{\rho,L}(\bar{x})$ contains only one point, we will obtain (10) after substituting (13) for (12).

Corollary

The recurrent relation

$$M_{\rho,L} = M_{\rho,L-1} + (k_L - 1)\, M_{\rho-1,\, L-1} \qquad (14)$$

is correct.

Proof

Let us separate in (13) the first multiplier and represent $M_{\rho,L}$ as:

$$M_{\rho,L} = \sum_{\ell_1=\rho}^{L} (k_{\ell_1} - 1) \sum_{\ell_2=\rho-1}^{\ell_1-1} \sum_{\ell_3=\rho-2}^{\ell_2-1} \cdots \sum_{\ell_j=1}^{\ell_{\rho-1}-1} \prod_{j=2}^{\rho} (k_{\ell_j} - 1) \qquad (15)$$

In (15) the expression written under the second sum is $M_{\rho-1,L-1}$ in accordance with (13) and hence:

$$M_{\rho,L} = \sum_{\ell_1=\rho}^{L} (k_{\ell_1} - 1)\, M_{\rho-1,\ell_1-1} \quad . \tag{16}$$

If the last term is removed from the summation in (16), then:

$$M_{\rho,L} = (k_L - 1)\, M_{\rho-1,L-1} + \sum_{\ell_1=\rho}^{L-1} (k_{\ell_1} - 1)\, M_{\rho-1,\ell_1-1} \tag{17}$$

In accordance with (13)

$$\sum_{\ell_1=\rho}^{L-1} (k_{\ell_1} - 1)\, M_{\rho-1,\ell_1-1} = M_{\rho,L-1} \quad .$$

Then finally we obtain $M_{\rho,L} = (k_L-1)\, M_{\rho-1,L-1} + M_{\rho,L-1}$.

Now we introduce into consideration the generating function

$M_L\,(z) = \sum_{\rho=o}^{\infty} M_{\rho,L}\, z^{\rho}$. By multiplying z^{ρ} and summing up both parts of the relation (14) with initial conditions $M_{1,o} = 0$, $M_{o,L} = \delta_{o,L}$ we obtain:

$$M_o\,(z) = 1$$
$$M_1\,(z) = z(k_1-1) + 1$$
$$M_L\,(z) = [z(k_L-1) + 1]\, M_{L-1}\,(z) \quad . \tag{18}$$

Iterating (18) we obtain

$$M_L\,(z) = \prod_{\ell=1}^{L} (z(k_L - 1) + 1) \quad .$$

In particular, if $k_\ell \equiv k$, we have the following generating function:

$$M_L\,(z) = (1 + z(k - 1))^L \tag{19}$$

and correspondingly

$$M_{\rho,L} = \binom{L}{\rho} (k - 1)_\rho \quad . \tag{20}$$

For the spherical neighborhood we obtain correspondingly the following relation:

$$N_{r,L} = \sum_{\rho=0}^{r} \binom{L}{\rho} (k - 1)^\rho \quad . \tag{21}$$

The values of $N_{r,L}$ can be determined by using tables of B-distribution (Sudakov, 1975), if relation (21) is rewritten in the following equivalent form:

$$N_{r,L} = k^L \, I_{(1-\frac{1}{k})} (L - r, r + 1) \tag{22}$$

Using Chernoff's estimate (Erdos and Spencer, 1977) we obtain the following upper estimate for the approximate determination of the number of points of the spherical neighborhood

$$N_{r,L} \leq \exp \left[(L - r) \ln \frac{L}{k(L - r)} + r \ln \frac{L (k - 1)}{k \cdot r} + L \ln k \right] \tag{23}$$

Thus in this section different relations are determined for the calculation of the number of points of the equidistant set and the spherical neighborhood: analytical relations (10) and (13), recurrent relation (14) and relation for upper estimate of this number (23). The relation (14) was used to obtain the computer calculation of these numbers. The corresponding computer program was written. The results of computation of the number of points of equidistant set at different values of ρ, L, k_ℓ are reduced in table (see Appendix 1). The following notations are used there:

-- M (R,L) is the number of points in the equidistant set $U_{R,L}(\bar{x})$.

-- K (L) is the number of variants of the implementation of L-th system's function.

It should be noted that the results obtained can be considered as upper estimates in the more general situation of morphological analysis. An example of one of these situations is one where fulfilled combinations of elements are eliminated at the initial stages of analysis. Such combinations are called incompatible ones. In such a case we have some prohibited points in our morphological space. Thus, the number of points of neighborhoods determined in accordance with the formula described above will be the upper estimate.

Some formal aspects of the representation of the set of prohibited variants by matrixes of binary relations are discussed in the paper (Kats and Iakobson, 1975). Let us explain the main points of this paper using our notations.

Let us assume that the morphological space

$$\Lambda = \prod_{\ell=1}^{L} \Lambda_\ell = \{(\lambda_1, \lambda_2, \ldots, \lambda_L)\}, \lambda_\ell \in \Lambda_\ell = \overline{1, k_\ell}$$

is defined. The indicator of the compatibility of elements λ_i and λ_k is:

$$b(\lambda_i, \lambda_k) = \begin{cases} 1, \text{ if } \lambda_i \text{ and } \lambda_k \text{ are compatible,} \\ 0, \text{ otherwise} \end{cases}$$

is put into accordance with each pair of coordinates i and k of the vector $\bar{\lambda} \in \Lambda$. And

$$N(\bar{\lambda}) = \prod_{i \geq k} b(\lambda_i, \lambda_k) = 1$$

for variant $\bar{\lambda}$, in which all components are compatible in pairs.

Hence, the number N of all permissible variants is determined in accordance with the following relation:

$$N = \sum_{\overline{\lambda} \in \Lambda} N(\overline{\lambda}) \qquad (24)$$

A special case is considered in a cited paper when from all C_L^2 matrixes only L matrixes $B_{i,k} = ((b(\lambda_i, \lambda_k)))$ of dimension $k_i \times k_k$ (namely $B_{L,1}$ and $B_{i,i+1}$, $i \in \overline{1, L-1}$) contain zero elements.

In this case N is equal to the sum of diagonal elements of matrix

$$\prod_i B_{i,i+1} \quad . \qquad (25)$$

It is stressed that the case considered is a specific one of the general situation and that formulas (24) and (25) are the upper estimations.

6. The Intersection of Spherical Neighborhoods.

Usually there are some variants in the morphological space which attract the greater attention of a researcher. Each of these variants posseses advantages and deficiencies. A researcher is interested in the selection of all intermediate variants in order to subject these to more detailed investigation. As usual, he would like to know how many such intermediate variants he would be required to investigate.

The following theorem gives us an analytical relation for the calculation of the number of points belonging to the intersection of two spherical neighborhoods of morphological space. Let s, r_1, r_2 be integral non-negative numbers non-exceeding L.

Theorem 3

The number of points N_s (r_1, r_2) of the intersection of two spherical neighborhoods of radius r_1 and r_2 respectively (whose centers are placed at the distance s) is determined by the relation:

$$N_s(r_1,r_2) = \sum_{\mu=s-r_2+1}^{r_1} \sum_{\nu=s-\mu+1}^{r_2} \sum_{\zeta=0}^{s-\max(\mu,\nu)} \binom{2(s+\zeta)-\mu-\nu}{s+\zeta-\mu} \sum_{D_o^1} \sum_{D_s^2} \prod_{\alpha=1}^{\mu+\nu-2\zeta-s} \prod_{\beta=1}^{\zeta} (k_{j_\beta}-1)(k_{j_\beta}-2) \qquad (26)$$

where

$$D_o^1 = \{j_\alpha, \ \alpha\in\overline{1,\ \mu+\nu-2\zeta-s} : o < j_1 < j_2 < \dots < j_{\mu+\nu-2\zeta-s}\} \ ,$$

$$D_s^2 = \{j_\beta, \ \beta\in\overline{1,\ \zeta} : s < j_1 < j_2 < \dots < j_\zeta\} \ .$$

Corollary

If $\forall \ell\in\overline{1,\ L} : k_\ell \equiv k$, then

$$N_s(r_1,r_2) = \binom{L}{s}^{-1} \sum_{\mu=s-r_2+1}^{r_1} \sum_{\nu=s-\mu+1}^{r_2} \sum_{\zeta=0}^{s-\max(\mu,\nu)} \binom{2(s+\zeta)-\mu-\nu}{s+\zeta-\mu}^2 \cdot$$

$$\cdot \binom{L}{2(s+\zeta)-\mu-\nu,\ L-s-\zeta,\ \zeta} (k-2)^{\mu+\nu-2\zeta-s} (k-1)^\zeta \qquad (27)$$

Proof of Theorem 3 and the Corollary can be found in the thesis (Iakimets, 1980). It should be noted that if $s = o$, $r_1 = r_2$, then (27) is the same as (10).

7. Median and Antimedian

If a researcher marks several variants of a system (more than 2) in a morphological space and he would like to know how many intermediate variants will be in this case, then it is necessary to be able to determine such variants. There are two possibilities:

1. to select all variants which are most similar with respect to the marked variants in a morphological space;
2. to select all variants which are most different from the marked variants.

The implementation of these possibilities is connected with the terms median and antimedian.

<u>Definition 5</u>

The median of a set of marked points Y, $Y \subset \Lambda$ is called any point $\bar{x} \in \Lambda$ where the distance from all points of set Y is minimal on the average.

By Definition 5 the set of all medians M of set Y is:

$$M = \underset{\bar{x} \in \Lambda}{\mathrm{Argmin}} \quad \frac{1}{|Y|} \sum_{\bar{y} \in Y} \rho(\bar{x},\bar{y}) \tag{28}$$

The test of this definition's correctness (i.e. the independence of the set of medians from the selected representation of morphological space in the form $\Lambda = \prod_{\ell=1}^{L} \Lambda_\ell$ does not represent difficulties.

7.1 Algorithm "Median"

Determination of all medians of set Y, $Y \subset \Lambda$ is realized in correspondence with the following algorithm.

1. The enumeration of elements of the set Y:

   $Y = \{\bar{y}_j,\ j \in \overline{1, N}\}$

2. The construction of classes of equivalence with the

relation of the equality R on sets of l-th coordinates of vectors $\overline{Y}_j$:

$$Y_\ell = \{y_{j,\ell},\ j \in \overline{1, N}\} : \{Y_{\ell/R},\ \ell \in \overline{1, L}\} .$$

3. The determination of the family of maximal classes of equivalence (by the number of elements) for each ℓ, $\ell \in \overline{1, L}$:

$$Z_\ell = \underset{\tilde{Y} \in Y_{\ell/R}}{\text{Argmax}} |\tilde{Y}|$$

4. The selection of a set of medians:

$M = \prod_{\ell=1}^{L} Z_\ell$, where Z_ℓ is the collection of representatives from the classes of equivalences of family Z_ℓ .

This algorithm selects all medians of a set of marked points Y of morphological space.

Let $\Lambda = \prod_{\ell=1}^{L} \Lambda_\ell$, $Y = \{\overline{y}_j = (y_{j1}, \ldots, y_{jL}),\ j \in \overline{1, N} \subseteq \Lambda\}$

Theorem 4

The relation

$$M \underset{\Delta}{=} \underset{\overline{x} \in \Lambda}{\text{Argmin}} \frac{1}{N} \sum_{j=1}^{N} \rho(\overline{x}, \overline{y}_j) = \prod_{\ell=1}^{L} Z_\ell \qquad (29)$$

is correct.

Proof

Using formula (4) and taking into account properties of maximization operation, we obtain

$$M = \underset{x_\ell}{\text{Argmax}} \sum_{j=1}^{N} \sum_{\ell=1}^{L} \delta_{x_\ell, y_\ell} . \qquad (30)$$

By virtue of non-negativity of summands in (30), the maximum of the expression $\sum_{\ell=1}^{L} \sum_{j=1}^{N} \delta_{x_\ell, y_{j,\ell}}$ is attained on those vectors $\overline{x}$, which l-th coordinates maximise the sum $\sum_{j=1}^{N} \delta_{x_\ell, y_{j,\ell}}$.

Hence,

$$M = \prod_{\ell=1}^{L} \underset{x_\ell}{\mathrm{Argmax}} \sum_{j=1}^{N} \delta_{x_\ell, y_{j,\ell}} \quad . \quad \text{Since } \underset{x_\ell}{\mathrm{Argmax}} \sum_{j=1}^{N} \delta_{x_\ell, y_{j,\ell}}$$

is the set of representatives of maximal classes of equivalences determined on the set $Y_\ell = \{ y_{j,\ell}, j \in \overline{1, N}\}$, then the relation (29) is correct.

Let us give some examples of the medians' determination. All variants marked by points in Figure 6 are variants of the set $Y = \{ \overline{y}_1, \overline{y}_2, \overline{y}_3\}$. Here the set of medians includes m_1, m_2 and m_3 which are marked by circles. The point $\overline{y}_1$ is the median in Figure 7, where $Y = \{\overline{y}_1, \overline{y}_2, \overline{y}_3, \overline{y}_4\}$. Points m_1 and $m_2 = \overline{y}_1$ are medians in Figure 8, where $Y = \{\overline{y}_1, \overline{y}_2, \overline{y}_3, \overline{y}_4, \overline{y}_5\}$ The determination of last medians in accordance with algorithm "Median" is illustrated in Figure 8.

The set of medians for real data is given in Appendix 2.

7.2 Antimedian

Sometimes a researcher is interested in an analysis of morphological variants, the combinations of whose elements are maximally different from some marked (or known) variants. In this case he can use the term antimedian.

<u>Definition 6</u>

The point $\overline{x}_a \in \Lambda = \prod_{\ell=1}^{L} \Lambda_\ell$:

$$\overline{x}_a = \underset{\overline{x} \in \Lambda}{\mathrm{Argmax}} \frac{1}{|Y|} \sum_{\overline{y} \in Y} \rho_r(\overline{x}, \overline{y}) \quad ,$$

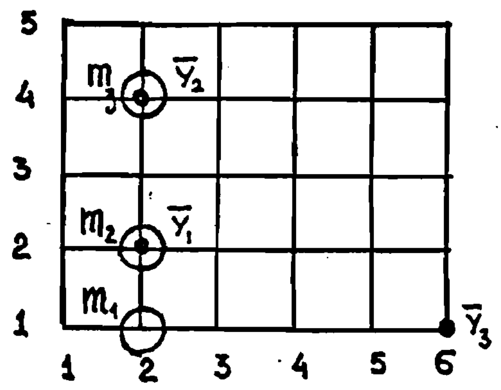


Figure 6.

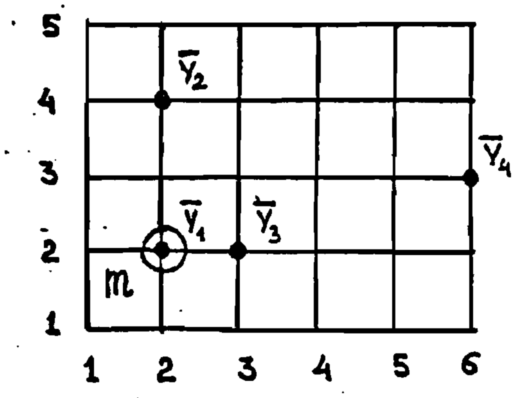
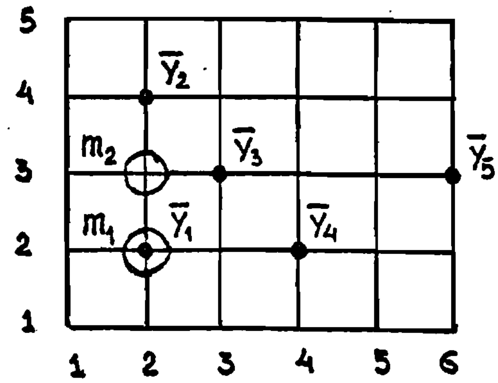


Figure 7.



$j \backslash \ell$	1	2
1	2	2
2	2	4
3	3	3
4	4	2
5	6	3

$$m_1 = (2,2)\ ;\quad \rho\,(m_1, Y) = \frac{6}{5}$$

$$m_2 = (2,3)\ ;\quad \rho\,(m_2, Y) = \frac{6}{5}$$

Figure 8.

is called the antimedian of the set of marked points Y, $Y \subseteq \Lambda$

As we see, the term antimedian is in a certain sense, a dual term with respect to median. As it is shown in (Iakimets, 1980), the algorithm of determination of the set of antimedians in fact coincides with the algorithm "Median." It is required only to interchange symbols "min" and "max" and to make a test connected with the use of all elements of each morphoclass.

8. Conclusion

Apparently, it cannot be an exaggeration to say that the quality of the final result of any applied problem's system analysis depends on the extent of the detail and completeness of the investigation of the generating alternative versions of the system (or decision). In the presence of the increasing complexity of studied objects, the practioner's requirements are exhibited and will be more strengthened by the formalized methods of generating the visible number of preferable variants of systems.

This paper can be considered one of the initial papers, connected with the solution of the forementioned problem. Besides, it should be emphasized that the theoretical results in this paper concern, on the whole, the morphological analysis method which is widely used in the practice of systems analysis and technological forecasting. These results allow one to localize in morphological space some areas contained variants which are preferable for the researcher.

Taking into account that the number of feasible morphological variants of a system of average complexity is usually large, it is possible to say that such an approach will be effective and will provide the systems analyst with methods of structuring and grouping morphological variants in accordance with the structure of analysts' preferences.

In conclusion it should be stressed that the metric morphological space is a new object for study and it's investigation can bring interesting theoretical and applied results. In the next paper the author proposes to describe the lexicographic morphological method for generating variants of the system evaluated by many strict ordered criteria.

APPENDIX I: THE NUMBER OF POINTS OF THE EQUIDISTANT SET

M(R,L)=M(R,L-1)+(K(L)-1*M(R-1,L-1)

K(L)=2 R=1,10; L=1,10

R	0	1	2	3	4	5	6	7	8	9	10
L											
0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
1	1.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
2	1.0	2.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
3	1.0	3.0	3.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
4	1.0	4.0	6.0	4.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0
5	1.0	5.0	10.0	10.0	5.0	1.0	0.0	0.0	0.0	0.0	0.0
6	1.0	6.0	15.0	20.0	15.0	6.0	1.0	0.0	0.0	0.0	0.0
7	1.0	7.0	21.0	35.0	35.0	21.0	7.0	1.0	0.0	0.0	0.0
8	1.0	8.0	28.0	56.0	70.0	56.0	28.0	8.0	1.0	0.0	0.0
9	1.0	9.0	36.0	84.0	126.0	126.0	84.0	36.0	9.0	1.0	0.0
10	1.0	10.0	45.0	120.0	210.0	252.0	210.0	120.0	45.0	10.0	1.0

M(R,L)=M(R,L-1)+(K(L)-1)*M(R-1,L-1)

K(L)=3 R=1,10; L=1,10

R / L	0	1	2	3	4	5	6	7	8	9	10
0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
1	1.0	2.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
2	1.0	4.0	4.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
3	1.0	6.0	12.0	8.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
4	1.0	8.0	24.0	32.0	16.0	0.0	0.0	0.0	0.0	0.0	0.0
5	1.0	10.0	40.0	80.0	80.0	32.0	0.0	0.0	0.0	0.0	0.0
6	1.0	12.0	60.0	160.0	240.0	192.0	64.0	0.0	0.0	0.0	0.0
7	1.0	14.0	84.0	280.0	560.0	672.0	448.0	128.0	0.0	0.0	0.0
8	1.0	16.0	112.0	448.0	1120.0	1792.0	1792.0	1024.0	256.0	0.0	0.0
9	1.0	18.0	144.0	672.0	2016.0	4032.0	5376.0	4608.0	2304.0	512.0	0.0
10	1.0	20.0	180.0	960.0	3360.0	8064.0	13440.0	15360.0	11520.0	5120.0	1024.0

M(R,L)=M(R,L-1)+(K(L)-1*M(R-1,L-1)

K(L)=4   R=1,10;   L=1,10

R L	0	1	2	3	4	5	6	7	8	9	10
0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
1	1.0	3.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
2	1.0	6.0	9.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
3	1.0	9.0	27.0	27.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
4	1.0	12.0	54.0	108.0	81.0	0.0	0.0	0.0	0.0	0.0	0.0
5	1.0	15.0	90.0	270.0	405.0	243.0	0.0	0.0	0.0	0.0	0.0
6	1.0	18.0	135.0	540.0	1215.0	1458.0	729.0	0.0	0.0	0.0	0.0
7	1.0	21.0	189.0	945.0	2835.0	5103.0	5103.0	2187.0	0.0	0.0	0.0
8	1.0	24.0	252.0	1512.0	5670.0	13608.0	20412.0	17496.0	6561.0	0.0	0.0
9	1.0	27.0	324.0	2268.0	10206.0	30618.0	61236.0	78732.0	59049.0	19683.0	0.0
10	1.0	30.0	405.0	3240.0	17010.0	61236.0	153090.0	262440.0	295245.0	196830.0	59049.0

M(R,L)=M(R,L-1)+(K(L)-1)*M(R-1,L-1)

K(L)=5 R=1,10; L=1,10

R / L	0	1	2	3	4	5	6	7	8	9	10
0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
1	1.0	4.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
2	1.0	8.0	16.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
3	1.0	12.0	48.0	64.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
4	1.0	16.0	96.0	256.0	256.0	0.0	0.0	0.0	0.0	0.0	0.0
5	1.0	20.0	160.0	640.0	1280.0	1024.0	0.0	0.0	0.0	0.0	0.0
6	1.0	24.0	240.0	1280.0	3840.0	6144.0	4096.0	0.0	0.0	0.0	0.0
7	1.0	28.0	336.0	2240.0	8960.0	21504.0	28672.0	16384.0	0.0	0.0	0.0
8	1.0	32.0	448.0	3584.0	17920.0	57344.0	114688.0	131072.0	65536.0	0.0	0.0
9	1.0	36.0	576.0	5376.0	32256.0	129024.0	344064.0	589824.0	589824.0	262144.0	0.0
10	1.0	40.0	720.0	7680.0	53760.0	258048.0	860160.0	1966080.0	2949120.0	2621440.0	1048576.0

APPENDIX 2: THE EXAMPLE OF MEDIANS DETERMINATION.

THE SET OF MARKED POINTS OF THE TEN DIMENSIONAL MORPHOLOGICAL SPACE

j is the index of the marked points, $j \in \overline{1,15}$

ℓ is the index of subsystems, $\ell \in \overline{1,10}$

j \ ℓ	1	2	3	4	5	6	7	8	9	10
1	1	1	3	2	1	1	2	1	2	3
2	2	2	6	4	1	1	2	1	2	4
3	3	3	2	1	1	1	2	1	3	4
4	4	4	8	7	1	1	2	1	3	6
5	5	5	6	5	1	1	2	1	3	7
6	6	6	5	9	1	1	2	1	3	5
7	7	7	2	7	1	1	3	1	3	6
8	8	8	5	5	1	1	4	1	3	6
9	9	6	4	3	1	1	1	1	3	6
10	1	5	3	4	1	1	6	1	3	6
11	2	4	4	5	1	1	4	1	3	6
12	3	3	6	6	1	1	5	1	3	6
13	4	2	7	7	1	1	7	1	3	6
14	5	1	8	8	1	1	8	1	3	6
15	6	2	9	3	1	1	9	1	3	6

THE SET OF MEDIANS

m is the index of medians, $m \in \overline{1,12}$

m \ ℓ	1	2	3	4	5	6	7	8	9	10
1	1	2	6	5	1	1	2	1	3	6
2	2	2	6	5	1	1	2	1	3	6
3	3	2	6	5	1	1	2	1	3	6
4	4	2	6	5	1	1	2	1	3	6
5	5	2	6	5	1	1	2	1	3	6
6	6	2	6	5	1	1	2	1	3	6
7	1	2	6	7	1	1	2	1	3	6
8	2	2	6	7	1	1	2	1	3	6
9	3	2	6	7	1	1	2	1	3	6
10	4	2	6	7	1	1	2	1	3	6
11	5	2	6	7	1	1	2	1	3	6
12	6	2	6	7	1	1	2	1	3	6

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