MATHEMATICAL MODELLING AT IIASA

Edited by

Roger E. Levien
Xerox Corporation, Stamford, Connecticut, USA
(Director of the International Institute for Applied Systems Analysis, Laxenburg, Austria, 1975–1981)

Andrzej P. Wierzbicki
International Institute for Applied Systems Analysis, Laxenburg, Austria

Brian Arthur
Food Research Institute, Stanford University, Stanford, California, USA
(Research Scholar at the International Institute for Applied Systems Analysis, Laxenburg, Austria, 1977–1982)

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FOREWORD

Much of IIASA's research in the first ten years of its existence has been concerned with the analysis of complex systems; some of these have been global in scale, such as the studies of energy or food and agriculture, while others, like the Lake Balaton study or the work on small open economies, have concentrated on individual, smaller systems. Mathematical modelling has played an important role in all of these analyses. Not only does this approach provide a simplified representation of real-world systems, allowing the modeller to study and sometimes even predict the behavior of the system, but also it can be used for policy analysis and planning purposes. To support these many and varied applications, IIASA has had to make significant advances in modelling methodology, which have allowed the modelling activities to develop in new directions. In view of IIASA's contributions to both the theory and the practice of mathematical modelling, therefore, it is particularly appropriate that a collection of papers by IIASA researchers should have been published as a special issue of *Mathematical Modelling*. These seven papers give some idea of the range of IIASA's modelling activities, but by no means represent the full scope of IIASA's research program. They have been selected for their methodological or practical relevance to the art of mathematical modelling.

ANDRZEJ P. WIERZBICKI
Leader
System and Decision Sciences

CONTENTS

Mathematical Modelling, Vol. 3, pp. 385–389, 1982

INTRODUCTION

ROGER E. LEVIEN,[a] ANDRZEJ P. WIERZBICKI,[b] AND BRIAN ARTHUR[c]

IIASA, an interdisciplinary and international (but nongovernmental) research organization, was founded in October 1972 by the academies of sciences or equivalent scientific organizations of 12 nations, both East and West. Its goal is to bring together scientists of many nationalities and disciplines to work jointly on problems facing either the world as a whole or many nations in common, especially those problems resulting from scientific, technological, demographic, social and economic development. The Institute currently has 17 member organizations (see Table 1).

The basic goal of the Institute is further specified by its broad objectives:

- To increase *international collaboration*, particularly by bringing together scientific approaches emerging from different cultural backgrounds;
- To contribute to the *advancement of scientific methods and systems analysis*;
- To achieve application to *problems of international importance*, either *universal*, common to many nations, or *global*, important to the entire world; in particular, problems that require joint resolution because of the interdependence or possible conflicts among many nations.

Through its first nine years, the Institute has striven to achieve these objectives.

The first objective, *international cooperation*, is actually a prerequisite for the other objectives, and the Institute was able relatively quickly to achieve significant success in promoting detached and rigorous scientific work on problems of international importance by teams of scientists from quite different cultural backgrounds. In the papers contained in this issue, this aspect of IIASA work is represented, for example, by work in stochastic optimization applied to facility location modelling, done by scientists from Italy and the USSR; and by economic work on computable equilibrium models, done by scientists from Hungary and Sweden, examining both the planned and the market economy interpretations.

The second objective, *advancement of science and systems analysis*, particularly in its interdisciplinary aspects, has taken a longer time to be achieved. (It is much easier to bring together specialists from different nations than from different disciplines.) However, all of the papers presented in this issue have interdisciplinary aspects.

The third objective, *application to problems of international importance*, has proven to be the most difficult to achieve. Yet IIASA recently completed a major Energy Systems Program that analyzed from a global perspective the options open to mankind

[a] Roger E. Levien, from the United States of America, a mathematician and systems analyst, served as the Director of IIASA from 1975 to 1981. He is now with the Xerox Corporation, Stamford, CT.

[b] Andrzej Wierzbicki, from Poland, an engineer and mathematician, is the Chairman of the System and Decision Sciences Area at IIASA.

[c] Brian Arthur, from the United Kingdom, an economist and demographer, is a member of the System and Decision Sciences Area staff.

Table 1. National member organizations of IIASA, 1982.

The Academy of Sciences of the USSR	The National Academy of Sciences, United States of America
The Canadian Committee for IIASA	The National Research Council, Italy
The Committee for the IIASA of the Czechoslovak Socialist Republic	The Polish Academy of Sciences
The French Association for the Development of Systems Analysis	The Royal Society, United Kingdom
The Academy of Sciences of the German Democratic Republic	The Austrian Academy of Sciences
The Japan Committee for IIASA	The Hungarian Committee for Applied Systems Analysis
The Max Planck Society for the Advancement of Sciences, Federal Republic of Germany	The Swedish Committee for the IIASA
The National Committee for Applied Systems Analysis and Management, Bulgaria	The Finnish Committee for IIASA
	The Foundation IIASA-Netherlands

in satisfying energy needs for the next 50 years. The results of this extensive study are not presented in this issue, being available elsewhere.* Another problem of international importance—food production and distribution to meet global needs—is addressed in IIASA's Food and Agriculture Program, and a paper in this issue illustrates the questions that arise when trying to model and link national food production and distribution systems. In addition to these two global problems, IIASA's research program has addressed a large number of other issues of international significance; in 1982 the research program of IIASA is organized as follows:

- Energy Systems Group
- Food and Agriculture Program
- Resources and Environment Area
- Human Settlements and Services Area
- Management and Technology Area
- System and Decision Sciences Area
- Regional Development Group
- General Research Group (including questions of industrial development and the craft of systems analysis).

In each of these units, mathematical modelling is used as one of the important tools of analysis. Mathematical models serve various purposes in IIASA research:

(1) They serve as a description of selected aspects of complex reality; as a tool for theory building.

(2) They are also used in the prescriptive sense, by using classical optimization

* See W. Hafele, ed., *Energy in a Finite World*, IIASA–Ballinger Publishing Company, Cambridge, MA (1981).

techniques to show what should be done in a specific situation. However, optimization is now used at IIASA for the following less conventional purposes:

- Models serve as a tool for policy testing and policy analysis, in which some optimizing behavior of the several actors can be taken into account, but the issues of incomplete information and uncertainty are of basic importance.
- Models are used to assist in organizing complexity and assuring consistency of analysis; optimization then serves, not prescriptively, but as a tool for consistently linking various submodels. (This is the way optimization was used in the Energy Program; to spell out an energy supply scenario consistent with energy demand.)
- Models are used as planning tools; however, this planning is not a prescriptive exercise, but rather the generation of efficient alternatives from among which one will be selected by less formal choice processes. (See the paper on "A Mathematical Basis for Satisficing Decision Making" by Andrzej Wierzbicki in this issue.)

These various modes of use do not exhaust the potential role of models. IIASA is also paying increasing attention to using models as a tool for studying process dynamics; in particular to understand adaptivity, stability, and resilience as systems properties. Another emerging issue is the use of models to study the procedures for resolution of conflicts among various actors or objectives.

The experience of its first decade has taught IIASA several lessons concerning the use of mathematical models.

The first lesson is that *models must be built to serve a definite purpose.* Without a proper initial conceptualization of this purpose and its implications for the process of model building and use, modelling becomes a barren intellectual exercise. For example, a common misconception is that a general purpose forecasting model can be built and then be used for a wide range of policy analyses. Although a forecasting model might be used to study the impacts of slight changes in traditional policies, any larger policy change is likely to invalidate the assumptions under which the forecasting model was built. For wide ranging policy analyses, models should usually be less detailed than the forecasting type, and should concentrate on a good representation of the possible uncertainties about the future, rather than concentrating on the best possible forecast of that uncertain future.

The second lesson is that *models are hypotheses about reality*, and should be treated as such. A hypothesis cannot be proven, only disproved by a test or accepted by consensus if various tests fail to disprove it. Thus, *tests for model invalidation* should be a standard part of any model-building process; the nature of the tests depending on the modelling purpose. A model validation process consists of a series of tests and can never be fully finished, only ended by a consensus. Too often, modellers do not fully understand this principle and assign too much credence to their models, failing to provide for and document invalidation tests, or do not use tests consistent with the modelling purpose.

The third lesson is that *computerized mathematical models*, in the same way as other scientific hypotheses or theories, *should be subject to critical review*—including their consistency with their purpose, the credibility of the data used, mathematical modelling methodology used, and other aspects.

These lessons indicate that modelling is still a craft that must be learned by doing and following good examples, even though it has many of the elements of the more rigorous sciences. The knowledge of various tools from applied mathematics, of what can and what cannot be done in computerized modelling, of what types of tools should be used when addressing given modelling issues, is by no means complete yet. IIASA has

contributed to the development of this knowledge, mostly through the work of its System and Decision Sciences Area, which has concentrated on the development in a real problem environment, of techniques such as nondifferentiable and stochastic optimization, multiattribute decision theory and interactive decision support systems, game theoretical approaches to public goods allocation and the general methodology of economic modelling.

Some results of this work are illustrated in this issue by the paper on "A Mathematical Basis for Satisficing Decision Making" by Andrzej Wierzbicki. The paper addresses the question of how to build a rigorous bridge between two distinct traditions in decision theory: utility optimization and satisficing. This leads to a new way of interpreting optimization techniques: instead of being used to obtain normative solutions, they can be employed interactively to produce efficient solutions in response to a user's stated aspiration levels. These solutions become alternative efficient plans for consideration in a planning process.

Another paper in this basic research category, a result of cooperation between the System and Decision Sciences and Human Settlements and Services Areas, is "Some Proposals for Stochastic Facility Location Models," by Yuri Ermoliev and Giorgio Leonardi. Facility location models are usually formulated in a deterministic framework, although such phenomena as demand for facilities (schools, hospitals) and the trip patterns of the customers are clearly stochastic. The authors show that these stochastic aspects can be consistently modelled and optimized, using a stochastic nondifferentiable optimization technique.

The methodology of economic modelling is represented by two papers, both addressing computable general equilibrium models. This class of nonlinear equilibrium models has found application only recently, after the difficulties related to finding their numerical solutions had been overcome. (At IIASA we did not overcome this difficulty by applying general fixed-point algorithms, which are theoretically very powerful but not quite efficient in practice, but we used rather specific model-oriented algorithms of decomposition and coordination of a quasi-Newton type.) The paper, "A System of Computable General Equilibrium Models of a Small Open Economy," by Lars Bergman presents two variants of models of this class describing a small national market economy facing changes in world market conditions: one is an instantaneous equilibrium model, in which the endowments of capital and labor can be reallocated very quickly following world market changes, and the other, more realistic, is a dynamic model, in which a distinction between short run equilibria and long run equilibria is made. The models have been used for analysis of the impacts of energy cost increases in Sweden. The other paper, "Computable General Equilibrium Models: An Optimal Planning Perspective," by Ernö Zalai provides, from a Hungarian perspective, an interpretation and possible adaptation of computable equilibrium models for central planning purposes.

The next paper, "The Basic Linked System of the Food and Agriculture Program," by Günther Fischer and Klaus Frohberg (from the Food and Agriculture Program) describes other aspects of economic modelling, though also of general equilibrium type, but with the emphasis on viewing the world system of demand and supply of food as a closed entity in which the agricultural policies of all countries are interdependent. National agricultural policies can either be analyzed as exogenous decision variables or simulated descriptively, as a process of policy implementation in response to the changing economic situation. The national models form a system linked through a model of international trade.

The last two papers are related to environmental modelling and come from the Resources and Environment Area. The paper, "Identifying Models of Environmental Systems' Behavior," by M. B. Beck turns back, on the basis on some examples of ecologic

modelling, to methodological questions of model building and identification. The author stresses the basic point that a model is actually an hypothesis and proposes recursive model parameter estimation algorithms as an aid in identifying model structure. The paper, "Modelling a Complex Environmental System: The Lake Balaton Study," by Laszlo Somlyódy illustrates some of the points made by Beck and presents a decomposition approach to modelling complex eutrophication issues in a big lake. Smaller, tractable models form a basis for a hierarchical system of models: in a higher startum, a realistic but yet simple model is obtained by aggregating essential features of a number of lower stratum models. The aggregated, simpler model is used for analyzing water quality management strategies.

This short selection of seven papers by no means represents the entire research program nor even the full variety of mathematical modelling applications at IIASA. They have been selected either for their methodological or their applicational relevance to the craft of mathematical modelling. Other modes of scientific enquiry are also strongly represented in IIASA. Empirical data analysis and qualitative conceptual analysis are needed for the examination of social or even political factors that are not easily represented in mathematical models. For poorly defined issues, any attempt to build models must be preceded by exploratory and comparative studies. Often, exploration and conceptual analysis lead to new, challenging questions in modelling methodology. Some of the emerging questions are, for example: How can interactions among groups in negotiations be modelled? How can the equity and efficiency of various policies (as opposed to decisions) under uncertainty be compared in a mathematical modelling framework? What results from modelling adaptive and learning processes in biology and control sciences can be adapted to socioeconomic modelling? These and other questions will define new, challenging directions for the further development of IIASA's research program and its contributions to the craft of mathematical modelling.

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A MATHEMATICAL BASIS FOR SATISFICING DECISION MAKING

ANDRZEJ P. WIERZBICKI

International Institute for Applied Systems Analysis
Laxenburg, Austria

and

The Institute of Automatic Control
Technical University of Warsaw, Poland

Abstract—This paper presents a conceptual and mathematical model of the process of satisficing decision making under multiple objectives in which the information about decision maker's preferences is expressed in the form of aspiration levels. The mathematical concept of a value (utility) function is modified to describe satisficing behavior; the modified value function (achievement scalarizing function) should possess the properties of order preservation and order approximation. It is shown that the mathematical basis formed using aspiration levels and achievement scalarizing functions can be used not only for satisficing decision making but also for Pareto optimization, and thus provides an alternative to approaches based on weighting coefficients or typical value functions. This mathematical basis, which can also be regarded as a generalization of the goal programming approach in multiobjective optimization, suggests pragmatic approaches to many problems in multiobjective analysis.

1. INTRODUCTION

The aim of this paper is to provide a mathematical background for satisficing decision making. Although it is assumed that the reader is acquainted both with the methodology behind the idea of satisficing decision making [1] and with the state of the art of optimization decision making and multiattribute decision analysis [2–8], some of the main features of these two approaches are briefly summarized below.

The basic questions in multiattribute decision analysis or multiobjective optimization may take various forms [9–11], but can be summarized as follows:

> Is the maximization of a value (utility) function an adequate model for typical decision-making processes? If the rationality of a decision is restricted by various institutional aspects, how do we best model the decision-making process mathematically?

The development of the classic apparatus of multicriteria optimization, preference relations, utility, and value theory, beginning with Pareto in 1896 and culminating with Debreu in 1959, has been strongly related to economic theory. However, economic theory is concerned with averages of thousands of decisions; the individual consumer is a mathematical construction in which institutional factors and personal preferences have been averaged out. More recent developments of this theory take restrictions into account by introducing additional constraints and examining the restricted rationality of the decision-making process [12].

On the other hand, most individual decisions are made within an organizational structure. Even when choosing goods in a supermarket, an individual consumer often has a shopping list composed with the help of his family, and his own rationality of choice is modified by this list. When buying new equipment, a factory manager is restricted by various environmental and safety standards. It is possible to express such

restrictions by additional constraints to utility maximization, although this is not necessarily the best way to deal with them.

This fact has been recognized by many economic theorists, including Boulding [13] and March and Simon [1], and an alternative approach to decision making has been developed. In the satisficing approach, the decision is based not on the maximization of a utility or value function, but on reaching certain aspiration levels. Much methodological reflection and analysis support this approach.

More recently, interest in decision analysis has been stimulated by systems analytical problems with wide-ranging economic, technological, sociological, and environmental objectives and constraints. Not all such objectives can be formulated mathematically. Even if this is possible, the resulting utility or value functions do not usually have a straightforward objective meaning but reflect the rather subjective preferences of a decision maker or a group of experts. Although there have been attempts to use the satisficing approach in systems analysis, most of the detailed studies on decision analysis [2, 5, 7] have been concerned largely with preferences and utility theory. There have also been many successful applications of this theory; the results are particularly good if the alternatives being compared are given explicitly, if there are not too many of them, and if the problem is basically to compare the sociological, environmental, and economic consequences of the various options.

However, it has been realized that while evaluating given alternatives is an important task, it is even more important to generate alternatives. For example, the mathematical models used in economic and sociological planning are capable of describing an infinite number of alternatives and their consequences; it is necessary to use these models to generate a restricted number of explicit alternatives of interest to the decision maker. This problem is related to satisficing rather than to optimizing, and many researchers in multiobjective optimization have felt the need for an appropriate mathematical formulation. Sakluvadze [14, 15], Yu and Leitmann [16], and others have considered the use of *utopia* (or *ideal*) *points* (representing unattainable aspiration levels) as reference points for generating alternatives. Charnes and Cooper [3], Dyer [17], Kornbluth [18], Ignizio [19], and others have developed a method called *goal programming*, which involves the use of aspiration levels for objectives in the process of multicriteria optimization. Yet these and related works have not had the impact they deserve, for several reasons.

First, although many partial results have been obtained, the mathematical basis of the satisficing approach and the relationship between satisficing decisions and optimal decisions have not yet been fully developed. Thus, any approach based on the use of reference objectives has been regarded as a somewhat less scientific, *ad hoc* approach. It was not clear whether it would be possible to develop a consistent, basic theory of multiobjective optimization and decision making starting with aspiration levels for objectives rather than with weighting coefficients or value (utility) functions. This would mean that the necessary and sufficient conditions, existence conditions, relations to preference orderings, and so on would have to be formulated in terms of aspiration levels or reference objectives. Some more abstract aspects of this question have been analyzed by the author in earlier work [11, 20–23]; a synthesis of relevant results is presented in this paper.

Second, although many researchers realized that there was a connection between satisficing decision making and such approaches as goal programming (see, e.g., [19]), some basic methodological questions have remained unanswered. *What can be logically assumed about the decision-making process in a simple organization? Whose preferences should be modelled? What is the relationship between satisficing decision making and utility or value maximization?*

The main purpose of this paper is to analyze methodological questions of this type and to develop the underlying mathematical theory.

2. A METHODOLOGICAL HYPOTHESIS

The following conceptual model is assumed to describe the decision-making process in a simple organization. The organization consists of a top decision maker (or a group of decision makers treated here as a single unit) called *the boss*, and technical or professional employees, again considered here as a single unit called *the staff*. The boss formulates a decision problem for the staff, and asks them to prepare one or more courses of action which would attain certain goals; he formulates the goals in terms of aspiration levels for several objectives. The staff then examines the possibilities in detail, checks whether the aspiration levels are attainable, and proposes detailed plans of action. The boss can either accept a proposed plan and decide to execute it, or change his requirements so that the staff must prepare new plans.

It is now necessary to make some assumptions concerning the decision-making process in an ideal organization—these then result in a relatively simple mathematical model of the organization.

First, *it is assumed that the goals are clearly and completely perceived.* In other words, the boss and the staff must have the same objectives, including those which might be more important for the staff but less so for the boss, and have a common understanding of what it means to improve each of the objectives. This does not mean that the boss and the staff should have the same preferences on various objectives; they need not agree on details, only on principles. It might also be necessary for the boss to specify aspiration levels for all objectives, even for those which he perceives as less important. In particular, the resources (budget, time, etc.) allocated by the boss to a given problem could usefully be treated as objectives rather than constraints, so that the suggested resource allocations would then become aspiration levels.

Mathematically, this assumption means that the boss and the staff consider the same objective space and have the same notion of a natural inequality in this space (the same partial preordering) but not necessarily the same preference structure (not the same complete preordering). The aspiration levels set by the boss form a reference point in the objective space. To simplify the discussion, it might be agreed that all objectives are improved if their levels are raised; this corresponds to Pareto maximization or to the natural partial ordering generated by the natural positive cone in the objective space. However, it is also possible to analyze more complicated situations.

Second, *it is assumed that the boss is consistent.* This means that he cannot prefer any plan in which one of the objectives has a smaller value than in alternative plans, everything else being equal. Mathematically, this means that his preference mapping (complete preordering) is strictly monotonic in the sense of the natural inequality in objective space (i.e., it preserves the partial preordering of the space). Apart from this requirement, his preferences may be purely arbitrary.

Third, *it is assumed that the staff is dedicated and efficient.* The term "dedication" when applied to the staff has the same meaning as "consistency" when applied to the boss: the preferences of the staff must increase as the objectives of the planned action improve, although the detailed pattern of these preferences might differ from that of the boss. Efficiency means something more: the staff actually maximizes the preferences and proposes only nondominated plans, in which no single objective can be improved without impairing others. (The term "nondominated" is preferred here to "Pareto optimal," which has a more specific meaning, or "efficient," which implies only economic efficiency.) Mathematically, this assumption means that the staff preferences

mapping not only strictly preserves the partial preordering of the objective space, but is also maximized during the preparation of plans.

Fourth, *it is assumed that the staff takes the aspiration levels seriously and strives to attain them.* This assumption is crucial in describing the satisficing behavior in the organization and the limited rationality of choice open to the staff. To illustrate the implications of this assumption, consider the three possible situations that could arise in response to a given problem.

If the aspiration levels given by the boss can be exceeded, the staff is free to use its own preferences to select a plan, although their freedom is restricted to the margin above the aspiration levels. The staff should not bother the boss with too many questions about how to allocate the surplus; one or more detailed plans should be presented for the boss' approval, all of which should be nondominated as required by the third assumption.

If the aspiration levels are unattainable, the staff must choose plans which match these levels as closely as possible. The meaning of "closeness" is left for the staff to decide; again they should not bother the boss too much.

The simplest but most important case is when the aspiration levels may just be attained without any excess. In this case staff rationality is severely modified: the staff must propose at least one plan with outcomes that precisely match the boss' wishes (fourth assumption), although they could also propose alternative plans. Since it is the boss' prerogative to choose and accept plans or to ask for new plans with altered aspiration levels to be prepared, the fourth assumption really implies that he fully controls the organization, and need not be affected by the preferences of the staff.

Any mathematical description of the fourth assumption must reflect this modification of staff rationality. It will be shown later that this assumption can be represented by the following *axiom of order approximation*:

> the set of objective outcomes preferred by the staff to the aspiration levels given by the boss must closely approximate the set of outcomes that are obviously better than the required aspiration levels (in the agreed partial preordering sense).

This axiom has a straightforward interpretation: to avoid a conflict of preferences close to the aspiration levels, the perfect staff should keep to the agreed principles of what is naturally better, and should not guess or bargain about what might be marginally better.

Clearly, all of the assumptions listed above describe an ideal organization which will almost never occur in practice (with one important exception, see below). Staff members *do* bargain with their bosses, bosses are not necessarily consistent in their decisions, and so on. However, this model of an ideal organization could serve as a basis which could then be modified to include deviations from ideal behavior.

It could also be argued that this model is too idealized to describe satisficing decision making in organizations: the main reason for accepting satisficing decisions is usually that there is no time for proper optimization, and therefore the assumption that the staff is efficient could be challenged. However, it is possible to define "conditional efficiency" which takes into account the time allocated for the staff to prepare the plan. Moreover, the staff is not required to optimize a global value function for the entire organization; this task is reserved for the boss, and can be done by changing aspiration levels if he wishes. The assumption of efficiency really means only that the staff should not propose dominated plans of action with outcomes that can clearly be improved.

A useful approach in analyzing an ideal organization of this type is to study a class of functions describing the preferences of the staff under its modified rationality of choice. These are, in a sense, modified value functions. However, these functions must express both the utility of exceeding the aspiration levels and the disutility of not achieving these

levels. Moreover, these functions must reflect the axiom of order approximation implied by the fourth assumption. These functions therefore depend explicitly and nonlinearly on the assumed aspiration levels, and will be called *achievement scalarizing functions*, following the terminology of goal programming and reference point optimization [19, 21]. There are several reasons for studying this class of functions.

First, although the boss can control the ideal organization regardless of the particular achievement scalarizing function characterizing the staff (provided that it fulfills the basic criteria), the shape of this function might influence the ease of interaction between the boss and the staff. This subject requires further theoretical and experimental study; only a few examples are considered in this paper.

Second, a detailed study of the ideal organization might serve as a starting point for the investigation of slightly different problems: hierarchical organizations in which the boss is himself a member of the staff of a higher-level manager; negotiation of aspiration levels between groups of decision makers; inclusion of additional objectives by the staff, uncertainty in either the boss' requirements or the staff's responses, etc.

Third, there is one important special case where the ideal organization described above might exist in practice: *an interactive link between a computer user (the boss) and a computerized multiobjective optimization model (the staff)*. Traditionally the computer user specifies various scenarios as model inputs and analyzes the outcomes. However, the concept of an ideal organization suggests an alternative approach. The user specifies the model outputs that play the role of objectives, the natural inequality in the objective space, and the model inputs (parameters, scenarios, etc.) that may be changed during optimization. An achievement scalarizing function and an optimization procedure which maximizes this function are also chosen. The user then specifies certain model outputs as aspiration levels; the system responds by stating whether these outputs are attainable or not and proposes one or more alternatives. These should be close to the desired result in the unattainable case, better than the desired result in the over-attainable case, and should exactly match the desired result when this is just attainable. By changing his requirements, the user can obtain various proposals from the model.

3. MATHEMATICAL FOUNDATIONS

Let $E_0 \subset E$ be the set of admissible decisions or alternatives to be evaluated. Let G be a (linear topological) space of objectives, performance indices, or outcomes. Let a mapping $Q: E_0 \to G$ be given, which defines numerically the consequences of each alternative. Let $Q_0 = Q(E_0)$ denote the set of attainable objectives. Let a natural inequality (a partial preordering) in G be given; to simplify the presentation, assume that the preordering is transitive and can be expressed by a *positive cone* (any closed, convex, proper cone) $D \subsetneq G$:

$$q_1, q_2 \in G,\ q_1 \leq q_2 \Leftrightarrow q_2 - q_1 \in D. \tag{1}$$

A corresponding strong partial preordering is given by [a]

$$q_1, q_2 \in G,\ q_1 < q_2 \Leftrightarrow q_2 - q_1 \in \tilde{D} \stackrel{\text{df}}{=} D \setminus (D \cap -D). \tag{2}$$

If the cone D has a nonempty interior $\mathring{D}$, it is also possible to introduce a strict partial preordering:

$$q_1, q_2 \in G,\ q_1 \ll q_2 \Leftrightarrow q_2 - q_1 \in \mathring{D}. \tag{3}$$

[a] Here $\stackrel{\text{df}}{=}$ denotes a defining equation.

Suppose that we maximize all objectives (gains, etc.). A generalized Pareto (nondominated) objective $\hat{q}$ is then a *D-maximal element of* Q_0:

$$\hat{q} \in Q_0 \text{ is } D\text{-maximal} \Leftrightarrow Q_0 \cap (\hat{q} + \tilde{D}) = \emptyset. \tag{4}$$

A slightly weaker definition, which includes a few points which are not nondominated, is that of *weak* D-maximal elements[b]:

$$\hat{q} \in Q_0 \text{ is weakly } D\text{-maximal} \Leftrightarrow Q_0 \cap (\hat{q} + \mathring{D}) = \emptyset. \tag{5}$$

For a normed space G, we can also have a stronger definition (D_ϵ-*maximality*) which does not include all nondominated points:

$$\hat{q} \in Q_0 \text{ is } D_\epsilon\text{-maximal} \Leftrightarrow Q_0 \cap (\hat{q} - \tilde{D}_\epsilon) = \emptyset, \tag{6}$$

where D_ϵ is an ϵ-conical neighborhood of D:

$$D_\epsilon \overset{\text{df}}{=} \{q \in G : \operatorname{dist}(q, D) < \epsilon \|q\|\}; \quad \tilde{D}_\epsilon \overset{\text{df}}{=} D_\epsilon \setminus (D_\epsilon \cap -D_\epsilon) \tag{7}$$

with

$$\operatorname{dist}(q, D) = \inf_{\tilde{q} \in D} \|q - \tilde{q}\|$$

implied by the norm of the space G.

If the space G is normed, we can also define an *achievement scalarizing function* (shortened to *achievement function*) $s : G \to R^1$, where s is assumed to satisfy either (8) and (10) below (the order representation case) or (9) and (11) below (the order approximation case). Thus, an achievement function should be

(a) *strictly order-preserving*: for all $\bar{q} \in G$, all $q_1, q_2 \in Q_0$:

$$q_1 \lneqq q_2 \Rightarrow s(q_1 - \bar{q}) < s(q_2 - \bar{q}), \tag{8}$$

or, if possible, *strongly order-preserving*: for all $\bar{q} \in G$, all $q_1, q_2 \in Q_0$:

$$q_1 < q_2 \Rightarrow s(q_1 - \bar{q}) < s(q_2 - \bar{q}), \tag{9}$$

where strong order preservation implies strict order preservation.

(b) *order-representing*:

$$S_0 \overset{\text{df}}{=} \{q \in G : s(q - \bar{q} \geq 0\} = \bar{q} + D; \; s(0) = 0, \tag{10}$$

or, at least, *order-approximating* for some small $\epsilon > 0$,

$$\bar{q} + D \subset S_0 \overset{\text{df}}{=} \{q \in G : s(q - \bar{q}) \geq 0\} \subseteq \bar{q} + D_\epsilon; \; s(0) = 0, \tag{11}$$

where, clearly, order representation implies order approximation.

[b] The concepts of D-maximality, weak D-maximality, and D_ϵ-maximality depend on the definition of the cone D. If this cone is properly chosen these ideas can be used to make many detailed distinctions between different notions of efficiency.

Requirements (a) and (b) both have two formulations; it is easy to show that (9) and (10) are incompatible, and hence we require that either (8) and (10) or (9) and (11) should be satisfied simultaneously.

Observe that the achievement function s is taken to be a function of the difference $q - \bar{q}$, where $q = Q(x)$, $x \in E_0$ is an attainable objective but $\bar{q} \in G$ is an *arbitrary* aspiration level, which is *not constrained* to Q_0, nor otherwise constrained. Moreover, an achievement function is usually constructed such that, if $\bar{q} \notin Q_0 - D$, then the maximization of $s(q - \bar{q})$ over $q \in Q_0$ represents the minimization of the distance between $\bar{q} + D$ and Q_0; if $\bar{q} \in Q_0 - D$, then the maximization of $s(q - \bar{q})$ represents the allocation of the surplus $q - \bar{q} \in D$. However, these comments are only descriptive and the axiomatic definition of an achievement function relies on requirements (a) and (b).

Requirement (a) results directly in a sufficient condition for Pareto-maximality, and the following well-known lemma holds [24] (see also refs. 21 and 25):

Lemma 1. If s is strongly order-preserving, then its maximal points in $q \in Q_0$ are D-maximal:

$$\hat{q} = \arg\max_{q \in Q_0} s(q - \bar{q}) \Rightarrow Q_0 \cap (\hat{q} - \tilde{D}) = \emptyset. \tag{12}$$

If s is strictly order-preserving, then its maximal points are weakly D-maximal.

Requirement (b) results in a necessary condition for Pareto-maximality which is much stronger than the known conditions based on weighting coefficients. The following lemma was first given by the author in a less general formulation [21]:

Lemma 2. If s is both order-preserving ($q_1 \leq q_2 \Rightarrow s(q_1 - \bar{q}) \leq s(q_2 - \bar{q})$ for any q_1, q_2, $\bar{q}$) and order-representing and if $\bar{q} = \hat{q}$ is (weakly) D-maximal, then the maximum of s over $q \in Q_0$ is attained at $\bar{q} = \hat{q}$ and is equal to zero[c]:

$$Q_0 \cap (\hat{g} + \mathring{D}) = \emptyset \Rightarrow \hat{q} \in \operatorname{Arg}\max_{q \in Q_0} [s(q - \hat{q})]; \max_{q \in Q_0} [s(q - \hat{q})] = 0. \tag{13}$$

If s is order-preserving and order-approximating for a given $\epsilon > 0$ and if $\bar{q} = \hat{q}$ is D_ϵ-maximal, then the maximum of s over Q_0 is also attained at $q = \hat{q}$ and is equal to zero, so that (13) holds with $\mathring{D}$ substituted by $\tilde{D}_\epsilon$.

The proof of Lemma 2 for an order-approximating function s is as follows. Suppose $\hat{q} \notin \operatorname{Arg}\max_{q \in Q_0} [s(q - \hat{q})]$; then there is $\tilde{q} \in Q_0$ such that

$$s(\tilde{q} - \hat{q}) > s(\hat{q} - \hat{q}) = 0.$$

In other words, $\tilde{q} \in \tilde{S}_0 = \{q \in G : s(q - \hat{q}) > 0\}$. Clearly, $\tilde{S}_0 \subset \hat{q} + D_\epsilon$ by the assumption of order approximation. However, $\tilde{q} \notin \hat{q} + (D_\epsilon \cap -D_\epsilon)$, since $\tilde{q} \in \hat{q} + (D_\epsilon \cap -D_\epsilon) = (\hat{q} + D_\epsilon) \cap (\hat{q} - D_\epsilon)$ would imply $s(\tilde{q} - \hat{q}) = 0$ by the assumption of order preservation. Thus, $\tilde{q} \in \hat{q} + \tilde{D}_\epsilon$ and $\tilde{q} \in Q_0$, which contradicts the assumption that $Q_0 \cap (\hat{q} + \tilde{D}_\epsilon)$ is empty. The modification of this proof for an order-representing function s is obvious. A strictly or strongly order-preserving function must be order-preserving, and hence the assumptions of Lemma 2 are satisfied for all achievement functions.

[c] We use Arg max or Arg min to indicate the set of maximum or minimum points of a function, and arg max or arg min if this set is a singleton.

Observe that Lemma 2 is a necessary condition for D-maximality (or D_ϵ-maximality) *even for nonconvex sets* Q_0; the geometrical interpretation of this condition is that sets Q_0 and $\hat{q} + \tilde{D}$ are separated at $\hat{q}$ by a cone S_0 (see Fig. 1).

Observe also that it is really requirement (b) that mathematically distinguishes an achievement function from a value function; the latter is usually assumed to satisfy requirement (a). We conclude that the results obtained using requirements (a) and (b) and the resulting Lemmas 1 and 2 are more fundamental than the results derived using weighting coefficients. For example, Lemma 2 can be used to check the attainability and Pareto-optimality of a given $\bar{q} \in G$, as follows: if an order-representing and order-preserving function $s(q - \bar{q})$ is maximized, and $\bar{q}$ is unattainable, then

$$\max_{q \in Q_0} [s(q - \bar{q})] < 0;$$

if $\bar{q}$ is attainable and weakly Pareto-optimal, then

$$\max_{q \in Q_0} [s(q - \bar{q})] = 0;$$

if $\bar{q}$ is attainable but not weakly Pareto-optimal, then

$$\max_{q \in Q_0} [s(q - \bar{q})] > 0.$$

It is not possible to make checks of this type using weighting coefficients or typical value or utility functions.

However, note that weighting coefficients $\hat{\lambda}$ are defined at the maximal points $\hat{q}$ of every order-preserving function (value, utility, or achievement function), provided that it is differentiable:

$$\hat{\lambda} = \frac{\partial s(q - \bar{q})}{\partial q} \Big/ \frac{\left\| \partial s(q - \bar{q}) \right\|}{\partial q}; \hat{\lambda} \in D^* = \{\lambda \in G^* : \langle \lambda, q \rangle \geq 0, \forall q \in D\}. \tag{14}$$

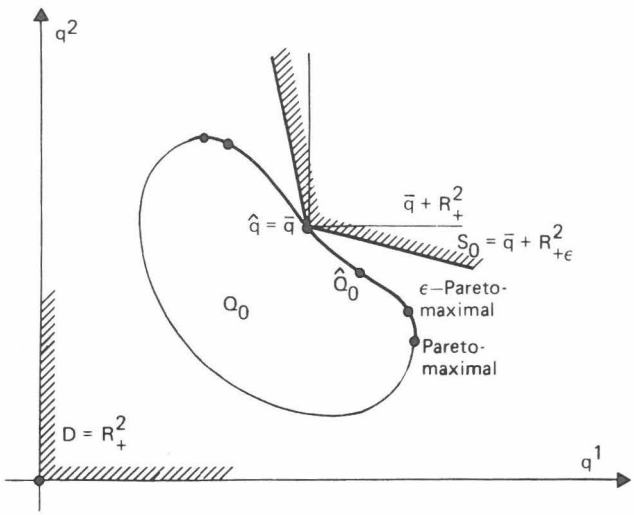

Fig. 1. The separation of Q_0 and $\hat{q} + \tilde{D} = \bar{q} + \tilde{R}^2_+$ by $s_0 = \bar{q} + R^2_{+\epsilon}$.

The norm used in (14) is the norm of the dual space G^* to the objective space, D^* is the dual cone to D, and $\langle \cdot, \cdot \rangle$ denotes the duality relation. If $G = R^n$, then it is usually assumed that the weighting coefficients sum to one, which implies the sum of absolute values norm in (14) and the maximum norm for the objective space. If s is only subdifferentiable, any of its subgradients at $\hat{q}$ can be used to define $\hat{\lambda}$ in a similar way to (14). There are two important corollaries to Lemmas 1 and 2.

Corollary 1. Suppose that an achievement function s is strictly or strongly order-preserving and upper semicontinuous in a topology in G. Suppose there is $\bar{q} \in G$ such that the set $(\bar{q} + D) \cap Q_0$ is compact in the same topology. Then there exist (possibly weakly) D-maximal points of set Q_0.

The proof of this corollary is simple. Weierstrass' theorem implies the existence of a maximum point $\hat{q}$ of $s(q - \bar{q})$ in the set $(\bar{q} + D) \cap Q_0$. By Lemma 1, this point is a (possibly weakly) D-maximal point of $(\bar{q} + D) \cap Q_0$. It is easy to check that it is also a (possibly weakly) D-maximal point of Q_0.

The second corollary establishes the fact that the boss is in full control of the organization if the staff preferences are described by an achievement scalarizing function.

Corollary 2. Suppose that an achievement function s is order-preserving and order-representing. Define the mapping $\hat{\hat{q}}: G \to \hat{Q}_0 = \{\hat{q} \in Q_0 : Q_0 \cap (\hat{q} + \bar{D}) = \emptyset\}$ by

$$\hat{\hat{q}}(\bar{q}) = \arg\min \|\hat{q} - \bar{q}\| \text{ for } \hat{q} \in \operatorname{Arg} \max_{q \in Q_0} [s(q - \bar{q})].$$

The mapping is then onto. If an achievement function s is order-preserving and order-approximating and the mapping $\hat{\hat{q}}$ is defined as above but with $\hat{\hat{q}}: G \to \hat{Q}_{0\epsilon} = \{\hat{q} \in Q_0 : Q_0 \cap (\hat{q} + \bar{D}_\epsilon) = \emptyset\}$, then the mapping is again onto.

The proof is simple: it is only necessary to show that for every $\hat{q} \in \hat{Q}_0$ or $\hat{q} \in \hat{Q}_{0\epsilon}$ there exists a $\bar{q} \in G$ such that $\hat{\hat{q}}(\bar{q}) = \hat{q}$. Lemma 2 implies that it is sufficient to choose $\bar{q} = \hat{q}$ to obtain $\hat{\hat{q}}(\hat{q}) = \hat{q}$. Although simple this result has an important interpretation: any desired nondominated and attainable point $\hat{q} \in \hat{Q}_0$ or $\hat{q} \in \hat{Q}_{0\epsilon}$ can be obtained by moving the reference point (aspiration level) $\bar{q}$, regardless of the other properties of the achievement function (e.g., the type of distance minimization or surplus allocation assumed in this function).

Corollary 2 and the possibility of determining marginal *a posteriori* information $\hat{\lambda}$ from equation (14) also suggest that the boss can alter $\bar{q}$ such that $\hat{q} = \hat{\hat{q}}(\bar{q})$ finally converges to a maximum of his own value or utility function (under certain assumptions concerning the reasonableness of his strategy in changing $\bar{q}$ [11]).

Finally, consider another interpretation of the achievement scalarizing function $s(q - \bar{q})$. Let it represent the value function of a consumer under various institutional restrictions expressed by $\bar{q}$ and let these restrictions have a probability distribution $p(\bar{q})$. After averaging over these restrictions, the average consumer's value or utility function is given by

$$u(q) = \int_G s(q - \bar{q}) p(\bar{q}) \, \mathrm{d}\bar{q}. \tag{15}$$

This function is order-preserving, since it is a generalized convex combination of order-

preserving functions. This represents another possible link between value optimization and satisficing decision making.

4. EXAMPLES OF ACHIEVEMENT SCALARIZING FUNCTIONS

The above theory will only be applicable if functions exist which satisfy the axiomatic requirements (a) and (b) from the previous section. Some examples of suitable functions are therefore presented below.

Assume that $G = R^n$, $G = R^n_+$. Let a utility (value) function $u(q)$ be defined for $q \in R^n_+$; assume that the utility function is (i) nonnegative, $u(g) \geq 0$ for $q \in R^n_+$, (ii) zero on the boundary of R^n_+, $u(q) = 0$ for $q \in \partial R^n_+$, and (iii) strictly order-preserving (not necessarily strongly order-preserving, since this is impossible for $q \in \partial R^n_+$). Now suppose that a threshold $\bar{q} \in R^n$ is defined, and the origin of the space shifted to this threshold; the utility function $u(q - \bar{q})$ is now defined only for $q \in \bar{q} + R^n_+$. The following expression can be used to define the function for $q \notin \bar{q} + R^n_+$:

$$s(q - \bar{q}) = u((q - \bar{q})_+) - \rho\|(\bar{q} - q)_+\|, \tag{16}$$

where $(\cdot)_+$ denotes the positive part of a vector, $\|(\bar{q} - q)_+\| = \operatorname{dist}(q, \bar{q} + R^n_+)$, and $\rho > 0$ is a penalty coefficient. The function $s(q - \bar{q})$ has two interpretations as used here.

First, it is an *extended (beyond) threshold utility function*: it might describe the behavior of an average consumer both above and below a subsistence threshold $\bar{q}$. Above the threshold, the average consumer maximizes his utility u; below the threshold, his disutility corresponds to his distance from the subsistence level.

Second, it is an achievement scalarizing function. It is clearly strictly order-preserving: any norm in R^n is strictly order-preserving for positive components (it cannot be strongly order-preserving if the maximum norm is used). It is also order-representing:

$$S_0 \overset{\text{df}}{=} \{q \in R^n : s(q - \bar{q}) \geq 0\} = \bar{q} + R^n_+,$$

since $u((q - \bar{q})_+)$ will be positive only for $q \in \bar{q} + R^n_+$. (If any component of the vector $(q - \bar{q})$ is negative or zero, then the corresponding component of the vector $(q - \bar{q})_+$ is zero, and $u((q - \bar{q})_+) = 0$ for $(q - \bar{q})_+ \in \partial R^n_+$.) The function also expresses the idea of surplus allocation resulting from utility maximization if $q - \bar{q} \in R^n_+$, and the idea of distance minimization if $q - \bar{q} \notin R^n_+$. In fact,

$$\operatorname{Arg}\min_{q \in Q_0} \|q - \bar{q}\| \subset \operatorname{Arg}\max_{q \in Q_0} [s(q - \bar{q})],$$

if $\bar{q} \notin Q_0 - R^n_+$.

Various norms in R^n and various utility functions can be used to define specific functions with the same general form as (16) [26]. One of the most useful is the following convex, piecewise linear function:

$$s(q - \bar{q}) = \min\left(\rho \min_{1 \leq i \leq n} (q^i - \bar{q}^i), \sum_{i=1}^{n} (q^i - \bar{q}^i)\right); \rho \geq n, \tag{17}$$

where the superscripts denote vector components. Then, provided that the set E_0 of admissible decisions x is described by linear inequalities and that all of the objective functions $q_i = Q_i(x)$ are also linear, the maximization of (17) is equivalent to the

following linear programming problem: maximize y, $q \in Q_0 = Q(E_0)$,

$$y \in Y_0(q - \bar{q}) = \left\{ y \in R^1 : y \leq \rho(q^i - \bar{q}^i),\ i = 1 \ldots n;\ y \leq \sum_{i=1}^{n} (q^i - \bar{q}^i) \right\}. \tag{18}$$

After solving this problem, the weighting coefficients $\hat{\lambda}$ can be determined *a posteriori* from the dual program.

Another class of achievement functions is that of *penalty scalarizing functions*. These are constructed using the following simple reasoning: if $q \in \bar{q} + R^n_+$, we maximize a norm or a component of $q - \bar{q}$; if $q \notin \bar{q} + R^n_+$, we penalize for the distance between q and $\bar{q} + R^n_+$. An example of this class is the following function

$$s(q - \bar{q}) = \|q - \bar{q}\| - \rho\|(\bar{q} - q)_+\|,\ \rho > 1, \tag{19}$$

which is strictly order-preserving (strongly order-preserving for all norms in R^n except the maximum norm) and order approximating with $\epsilon \geq 1/\rho$ [23]. If $q \notin \bar{q} + R^n_+$, this function also expresses a specific idea of distance minimization: if $\bar{q} \notin Q_0 - R^n_{+\epsilon}$ and

$$\operatorname{Arg} \max_{q \in Q_0} [s(q - \bar{q})] \subset \hat{Q}_{0\epsilon},$$

then

$$\operatorname{Arg} \min_{q \in \hat{Q}_{0\epsilon}} \|q - \bar{q}\| \subset \operatorname{Arg} \max_{q \in Q_0} [s(q - \bar{q})].$$

However,

$$\operatorname{Arg} \max_{q \in Q_0} [s(q - \bar{q})]$$

is not always contained in $\hat{Q}_{0\epsilon}$, although it is always contained in (weak) $\hat{Q}_0$, because the function $s(q - \bar{q})$ is R^n_+-order-preserving, not $R^n_{+\epsilon}$-order-preserving. Depending on the norm chosen, this function also possesses various other properties [11, 26].

Another example is the penalty function resulting from maximization of component q^1 under (soft) constraints $q^2 \geq \bar{q}^2 \ldots q^n \geq \bar{q}^n$:

$$s(q - \bar{q}) = q^1 - \bar{q}^1 - \rho\|(\bar{q}^r - q^r)_+\|R^{n-1};\ q^r = (q^2 \ldots q^n) \in R^{n-1}. \tag{20}$$

This function is frequently used to scan the Pareto set in multiobjective optimization; however, it is not generally known that this function is (strictly or strongly, depending on the norm) order-preserving for $\rho > 0$ and order approximating with $\epsilon > 1/\rho$. Thus any maximal point of this function, while not necessarily satisfying the constraints, is Pareto-maximal; and ϵ-Pareto-maximal point $\bar{q} = \hat{q}$ is also maximal for this function.

The penalty function (19) may easily be generalized for the case when G is a Hilbert space—for example, the space containing the trajectories of solutions to a continuous-time dynamic economic model. The corresponding equation is

$$s(q - \bar{q}) = \|q - \bar{q}\| - \rho\|(\bar{q} - q)^{D^*}\|, \tag{21}$$

where $(\cdot)^{D^*}$ denotes the operation of projection on the dual cone $D^* = \{q^* \in G^* : \langle q^*, q \rangle \geq 0,\ \forall q \in D\}$ [22]. This function is strongly order-preserving, if $\rho > 0$ and $D \subseteq D^*$, and order-approximating with $\epsilon \leq 1/\rho$ [21].

Thus, there are many forms of achievement function which fulfill our requirements, two of them (Eqs. 17 and 20) being particularly simple and easy to apply.

Observe, finally, that a maximal point $\hat{q}$ of an achievement function $s(q - \bar{q})$ depends on various factors: on the aspiration objective $\bar{q}$, on the choice of norm, on the choice of penalty coefficient ρ, and on the type of surplus allocation or utility used in extended threshold utility functions. However, as shown in the previous section, the influence of the reference objective $\bar{q}$ is of primary importance, the other factors having only a secondary effect. If a mathematical model is used in the decision-making process, these other factors can be specified by an optimization specialist: he can choose the norm most appropriate to the mathematical model (for example, if the model is linear he might choose the maximum norm, while if the model is nonlinear he might prefer the Euclidean norm); he can choose the penalty coefficient ρ such that the problem is not too badly conditioned, with reasonable violations of soft constraints; he can guess how to allocate a possible surplus $q - \bar{q} \in R^n_+$, etc. These decisions are important in the sense of computational efficiency, but they are clearly not essential to the decision maker, who can choose any $\hat{q} \in \hat{Q}_0$ (or, at least, any $\hat{q} \in \hat{Q}_{0\epsilon}$) by specifying and changing the value of $\bar{q}$.

5. AN INTERACTIVE TECHNIQUE FOR SATISFICING DECISION MAKING VIA MULTIOBJECTIVE OPTIMIZATION

Let us now consider a practical interactive procedure for choosing a Pareto-maximal point. It is assumed that the actual decisions are made by a decision maker, and that the mathematical model and optimization techniques serve only as an aid in identifying the relevant part of the Pareto-maximal set.

The decision maker is first presented with all the information he desires about the model being used to solve his problem. This may include the maximal and minimal levels of objective functions when maximized separately, and the corresponding decisions. He is then asked to specify the vector of aspiration levels for all objective functions, $\bar{q}_0 = (\bar{q}_0^1 \ldots \bar{q}_0^n) \in R^n$. For each vector of aspirations $\bar{q}_i$, the computer responds with the following:

(1) The Pareto-maximal attainable objective vector $\hat{q}_i$, which is obtained by maximizing an achievement function, and the corresponding weighting coefficients and decision variables;

(2) A number (n) of other Pareto-maximal attainable objective vectors $\hat{q}_{i,j}$, $j = 1 \ldots n$, obtained by maximizing the achievement function with perturbed aspiration points:

$$\bar{q}_{i,j} = \bar{q}_i + d_i e_j; \; e_j = (0 \ldots 1_j \ldots 0); \; d_i = \|\bar{q}_i - \hat{q}_i\|, \tag{22}$$

where d_i is the distance between the vector of aspirations $\bar{q}_i$ and the attainable vector $\hat{q}_i$; e_j is the jth unit basis vector. The advantage of perturbation (22) is that if the point $\bar{q}_0$ is distant from the Pareto set, the decision maker obtains a global description of the Pareto set through the points $\hat{q}_{0,j}$; if $\bar{q}$ is close to the Pareto set, then the points $\hat{q}_{i,j}$ provide a detailed description of the Pareto set in a neighborhood of the aspiration point $\bar{q}_i$ (see Fig. 2).

The decision maker can now either choose one of the proposed alternatives, or change his aspiration point to $\bar{q}_{i+1}$. This procedure can be refined by using the differences $\bar{q}_{i+1} - \hat{q}_i$ to identify the utility or value function of the decision maker; these differences can then be constrained so that the procedure converges to a point that maximizes the utility function. However, these refinements are not very important in practice; decision makers usually adopt the satisficing approach and choose one of the computer-generated alternatives relatively quickly.

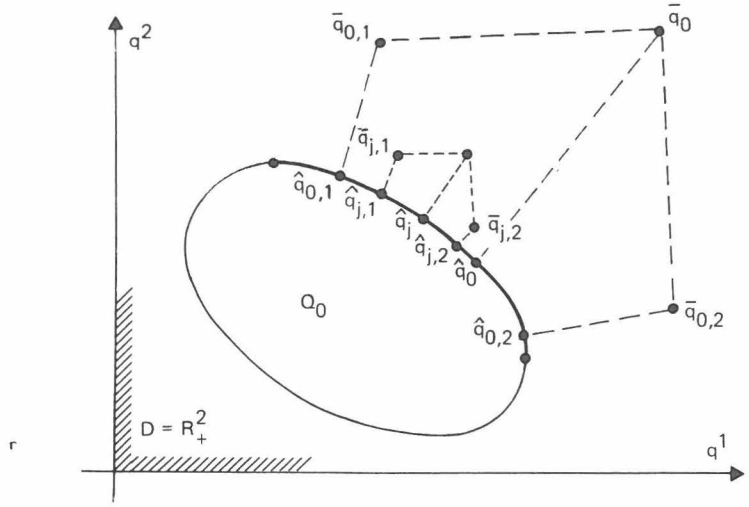

Fig. 2. Interpretation of the interactive procedure.

Similar procedures have been proposed by researchers working on goal programming in multiobjective optimization [17–19], although achievement functions have not been fully investigated in this field; questions such as what to do if the aspiration level can be exceeded and how to choose the best norm have not yet been settled. Thus, the interactive procedure presented here can also be considered as a generalization of the goal programming approach.

6. OTHER APPLICATIONS OF ACHIEVEMENT SCALARIZING FUNCTIONS

Scanning the Pareto set

When building a multiobjective optimization model, the analyst should at least attempt to scan the Pareto set, i.e., to obtain a representation of it. The author has found that an approach based on aspiration points and achievement scalarizing functions is effective in scanning the Pareto set, particularly if the number of objectives is very large, as in trajectory optimization.

Trajectory optimization

In typical dynamic optimization problems, single or multiple objectives are obtained by aggregating dynamic trajectories using integral functionals. However, experienced analysts, economists, and decision makers can often evaluate entire trajectories (functions of time) better than aggregate integral indices. A decision maker experienced in evaluating trajectories can easily state his requirements in terms of an *aspiration trajectory* $\bar{q}(t)$, a scalar- or vector-valued function of time; however, it would be very difficult to identify his preference relation in trajectory space. We should therefore construct an *ad hoc* achievement functional, possibly similar to (21), with $G = L^2[0; T]$ and $D =$

$\{q \in L^2[0; T]: q(t) \geq 0$ a.e. on $[0; T]\}$:

$$s(q - \bar{q}) = \int_0^T ((q(t) - \bar{q}(t))^2 - \rho(\bar{q}(t) - q(t))_+^2)\, dt. \tag{23}$$

If time is assumed to be discrete rather than continuous, the integral should be replaced by a sum and the problem becomes finite-dimensional. However, even in this case it is still more convenient to think in terms of trajectories than to consider separate objectives. This technique can be used (in economic models, for example) to obtain feasible and (generalized) Pareto-optimal trajectories that are either close to or better than any given aspiration trajectories. The concept of trajectory optimization via aspiration trajectories has been used by Kallio *et al.* [27] in a study of alternative policies for the Finnish forestry and forest-based industries.

Semiregularization of solutions of mathematical models

Any model that possesses many solutions or quasisolutions can be *Tikhonov-regularized* [28] by choosing the solution that is closest to a given reference point. Achievement functions actually represent a generalization of this idea: the principle of *semiregularization*. Consider function (20) and suppose that $\bar{q}^r = (\bar{q}^s, \bar{q}^t)$, where $\bar{q}^s$ denotes the components of the reference objective which the solution should either be close to, or, if possible, exceed, and $\bar{q}^t$ denotes the components which the solution should be close to regardless of the sign of $\bar{q}^t - q^t$. The following penalty scalarizing function

$$s(q - \bar{q}) = q^1 - \bar{q}^1 - \rho^s \|(\bar{q}^s - q^s)_+\| - \rho^t \|\bar{q}^t - q^t\|, \tag{24}$$

is both order-preserving and order-approximating, in terms of the partial ordering defined by the cone

$$D = \{q \in R^n : q^1 \geq 0, \bar{q}^{s,i} \geq 0, \bar{q}^{t,j} = 0\}.$$

Thus, scalarizing functions can also be used to represent components that should not move too far from the reference level in either direction.

7. CONCLUSIONS AND POSSIBLE EXTENSIONS

The main idea in constructing a mathematical basis for satisficing decision making is to introduce the wishes of the decision maker as basic *a priori* information in the form of aspiration levels. It is assumed that the decision maker is helped by his staff (or a mathematical model), which proposes attainable nondominated alternatives corresponding to the aspiration levels. Achievement scalarizing functions are then constructed; these are based on the aspiration levels, but also reflect the modified rationality of the staff by fulfilling the conditions of order approximation and order preservation. The first of these properties (order approximation) also results in a necessary condition for Pareto optimality, which is applicable to nonconvex problems and stronger than other known necessary conditions. Thus, the mathematical basis for satisficing decision making may be seen as an alternative approach to multiobjective optimization, in that it generalizes goal programming and utopia point techniques. It is also related to other problems, such as trajectory optimization or the problem of regularizing the solutions of badly defined mathematical models. However, this abstract basis is also eminently

pragmatic: the basic idea of responding to the wishes of a decision maker rather than telling him what his wishes should be results in a practical interactive procedure with institutional implications.

There are many problems still to be investigated: these include the use of aspiration objectives under uncertainty, their use in hierarchical decision-making structures, and so on. Moreover, much still remains to be done simply in testing this approach more widely in many different fields and applications.

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SOME PROPOSALS FOR STOCHASTIC FACILITY LOCATION MODELS

Y. M. ERMOLIEV

International Institute for Applied Systems Analysis
Laxenburg, Austria
and
Institute of Cybernetics
Kiev, USSR

G. LEONARDI

International Institute for Applied Systems Analysis
Laxenburg, Austria
and
Istituto di Scienza dei Sistemi Architectonica e Territoriali
Turin, Italy

Abstract—The aim of this paper is to introduce stochastic features into a facility location model to describe both the total demand for facilities and the trip pattern of the customers. The usefulness of stochastic programming tools in formulating and solving problems of this type is explored. Numerical stochastic nondifferentiable optimization techniques are outlined, and optimality conditions and practical computations are discussed.

1. INTRODUCTION

The public provision of urban facilities and services often takes the form of a few central supply points serving a large number of spatially dispersed demand points. These facilities include hospitals, schools, libraries, and emergency provisions such as fire and police services. One of the fundamental features of these systems is the spatial interaction between suppliers and consumers. The need to introduce behavioral patterns more realistic than simply assuming that customers use the nearest facility has been recognized by many authors, among them Coelho and Wilson [1], Hodgson [2], Beaumont [3], and Leonardi [4, 5]. Since the proposed spatial interaction ("gravity") models can be justified both theoretically and empirically, their use in location modelling seems promising.

However, the classical spatial interaction models solve only part of the problem. Although they are based on stochastic assumptions [6–8], they use only the expected values of the underlying stochastic processes. A natural further step is therefore to introduce the stochastic behavior explicitly, thus allowing for uncertainty in both customer choice and knowledge of demand.

This paper investigates some of the problems arising when such stochastic features are introduced, and suggests some numerical tools which could be used to solve these problems. Since this paper is of an exploratory nature, the examples are kept as simple as possible. However, it is felt that the approach is much more general than the applications discussed here would suggest; it can easily be extended to more complex problems without involving any major change in theory and tools.

2. STATEMENT OF THE PROBLEM

The simplest formulation of the deterministic facility location problem is as follows: minimize the performance function

$$\sum_{i,j} (x_{ij} \ln(x_{ij}) + c_{ij}x_{ij}) \tag{1}$$

subject to the constraints

$$\sum_{j=1}^{n} x_{ij} = a_i, \; i = \overline{1, r}, \tag{2}$$

$$x_{ij} \geq 0, \; \forall i, j, \tag{3}$$

where x_{ij} is an (unknown) expected flow of users from demand location i to facility location j ($i = \overline{1, r}$, $j = \overline{1, n}$) per unit time; a_i is the total demand (in terms of customers to be served per unit of time) at each demand location i; c_{ij} are the costs of travel between each pair of locations (i, j).

The objective function (1) was first introduced into transport planning evaluation by Bregman [9] and Neuburger [10] and extended to location analysis by Coelho and Wilson [1]. These authors gave this function an economic interpretation, namely the consumer surplus measure associated with the pattern of consumer trips $\{x_{ij}\}$.

Due to the simple form of the problem (1)–(3), the closed-form optimal solution is not hard to find:

$$x_{ij} = a_i P_{ij}, \; x_j = \sum_{i=1}^{r} x_{ij}, \tag{4}$$

where

$$P_{ij} = \frac{\exp(-c_{ij})}{\sum_{j} \exp(-c_{ij})},$$

and x_j is the size of the facility at j. Note that the quantities P_{ij} satisfy the following conditions:

$$\sum_{j=1}^{n} P_{ij} = 1, \; P_{ij} \geq 0, \; i = \overline{1, r}, \; j = \overline{1, n}. \tag{5}$$

Equations (4) and (5) imply that trips from demand locations to facilities are made according to a very simple interaction rule. The quantity P_{ij} can be interpreted as the probability that a customer living at location i will choose the facility at location j. Then x_{ij} is the expected number of customers traveling between i and j.

It is worth noting that the interpretation of the quantities P_{ij} as probabilities is connected with the theory of probabilistic choice behavior [11]. It has also been shown by Bertuglia and Leonardi [8] that these quantities can be considered as a steady-state distribution of a suitably defined Markov process.

It is now possible to use Eq. (4) as the basis from which to make some generalizations concerning stochasticity. The simplest of these are as follows:

(1) The demand a_i of demand location i is not known in advance; it is a random variable. This assumption is reasonable in many long-term planning applications. For instance,

in a high-school location problem the total number of students living in each demand location may change over time and so cannot be known in advance.

(2) Customers living in district i choose their destinations j independently of each other with probability P_{ij}.

These assumptions are embodied in the following model, which assumes that the choices made by the customers are stochastic. Let ϵ_{ij} be the actual (random) numbers of customers traveling from i to j and define τ_j, the total number of customers attracted to j, as follows:

$$\tau_j = \sum_{i=1}^{r} \epsilon_{ij}, \; j = \overline{1, n}.$$

Note also that

$$\sum_{j=1}^{n} \epsilon_{ij} = a_i, \; i = \overline{1, r}. \tag{6}$$

Let $H_j(y)$ denote the distribution function τ_j:

$$H_j(y) = P\{\tau_j \leq y\}.$$

The distribution function $H_j(y)$ cannot easily be given in closed form, but random draws of τ_j can be computed using a simple simulation model based on Eq. (6). If x_j is the planned size of the facility at j, then the actual number τ_j of customers attracted to j may not be equal to x_j. Suppose that a cost

$$\alpha_j^+(x_j - \tau_j)$$

has to be paid when $x_j \geq \tau_j$ and a cost

$$\alpha_j^-(\tau_j - x_j)$$

has to be paid when $x_j < \tau_j$. We therefore have the cost function

$$f_j(x_j, \tau_j) = \begin{cases} \alpha_j^+(x_j - \tau_j), \text{ if } x_j \geq \tau_j, \\ \alpha_j^-(\tau_j - x_j), \text{ if } x_j < \tau_j. \end{cases}$$

The resulting stochastic programming problem is then as follows: determine the sizes x_j of the facilities $j = \overline{1, n}$ that minimize the expected cost

$$F(x_1 \ldots x_n) = \sum_{j=1}^{n} Ef_j(x_j, \tau_j) \tag{7}$$

$$= \sum_{j=1}^{n} \left[\alpha_j^+ \int_0^{x_j} (x_j - y) \, dH_j(y) + \alpha_j^- \int_{x_j}^{\infty} (y - x_j) \, dH_j(y) \right]$$

subject to constraints

$$x_j \geq 0, \; j = \overline{1, n}. \tag{8}$$

Note that the objective function contains no spatial interaction embedding term since the behavior of the customer is included in the structure of the probabilities P_{ij}.

Practical problems that lead to the minimization of a function such as Eq. (7) are common in operations research. For example, we could consider a facility allocation problem or a storage inventory control problem where some capacities have to meet random demand and both surpluses and deficits incur penalty costs.

In the special case where $F(x)$ has continuous derivatives, minimization of $F(x)$ by analytical means would lead to consideration of the partial derivatives

$$\frac{\partial}{\partial x_j} F(x) = \alpha_j^+ \int_0^{x_j} dH_j(y) - \alpha_j^- \int_{x_j}^{\infty} dH_j(y).$$

The solution would then require the determination of $x = (x_1 \dots x_n)$ such that

$$H_j(x_j) = \frac{\alpha_j^-}{\alpha_j^+ + \alpha_j^-}, \; j = \overline{1, n}.$$

In general it may not be possible to solve this equation analytically [for instance, if $H_j(y)$ is unknown, as in problem (7)–(8)].

3. THE STOCHASTIC QUASIGRADIENT METHOD

The solution of problems such as (7)–(8) usually gives rise to two main difficulties. First, it is often difficult or impossible to compute the exact values of the integrals appearing in (7), except for very special and well-behaved forms of the distribution of functions $H_j(y)$. Functions of this type are often defined not by a closed-form equation, but rather by means of a rule for generating random draws from them by Monte-Carlo-type simulation procedures. Thus, to solve such problems it is necessary to develop algorithmic minimization procedures which do not calculate the exact values of the objective function. Second, although the objective function (7) is convex, it is generally nonsmooth. This becomes clear after reformulating problem (7)–(8) as a stochastic minimax problem. It is easy to see that

$$f_j(x_j, \tau_j) = \max\{\alpha_j^+(x_j - \tau_j), \; \alpha_j^-(\tau_j - x_j)\}.$$

The objective function (7) is therefore

$$F(x) = \sum_{j=1}^{n} E \max\{\alpha_j^+(x_j - \tau_j), \; \alpha_j^-(\tau_j - x_j)\}. \tag{9}$$

Function (9) is convex, but in general nonsmooth, since the maximization operator is present under the mathematical expectation sign.

These difficulties can be overcome by using a numerical procedure [16] in which successive approximations $x^s = (x_1^s \dots x_n^s)$, $s = 0, 1 \dots$ are derived as follows.

Let $x^0 = (x_1^0 \dots x_n^0)$ be an arbitrary initial approximation and $x^s = (x_1^s \dots x_n^s)$ be the approximation computed after the sth iteration. A random observation $\tau^s = (\tau_1^s \dots \tau_n^s)$ of the vector $\tau = (\tau_1 \dots \tau_n)$ is obtained by simulation. A new approximation is determined by the rule:

$$x_j^{s+1} = \max\{0, x_j^s - \rho_s \xi_j^s\}, \; j = \overline{1, n}, \; s = 0, 1 \dots \tag{10}$$

where ρ_s is a step multiplier, such that

$$\rho_s \geq 0, \; \sum_{s=0}^{\infty} \rho_s = \infty, \; \sum_{s=0}^{\infty} \rho_s^2 < \infty, \tag{11}$$

and

$$\xi_j^s = \begin{cases} \alpha_j^+, \text{ if } x_j^s \geq \tau_j^s, \\ -\alpha_j^-, \text{ if } x_j^s < \tau_j^s. \end{cases} \tag{12}$$

THEOREM 1. The sequence $\{x^s\}$ generated by (10)–(12) converges with probability 1 to an optimal solution of problem (7)–(8).

Expression (10) represents a stochastic quasigradient procedure [12, 13]. The convergence of the sequence $\{x^s\}$ to an optimal solution of problem (7)–(8) is based on the fact that the random vector $\xi^s = (\xi_1^s \ldots \xi_n^s)$, as defined in (12), is a stochastic estimate of the subgradient of function (9).

It should be briefly recalled [14] that a subgradient $\hat{F}_x(x)$ of a convex function $F(x)$ is a vector such that the inequality

$$F(y) - F(x) \geq \langle \hat{F}_x(x),\ y - x \rangle$$

holds for all y (the angular brackets on the right-hand side denoting the inner product of two vectors). A subgradient of a differentiable function $F(x)$ is equal to the gradient.

It can be proved that the conditional mathematical expectation $E\{\xi^s \mid x^s\}$ of random vector ξ^s is a subgradient of function (9) at $x = x^s$. To show this, consider the more general problem of minimizing the function

$$F(x) = E \max_{1 \leq k < K} \sum_{j=1}^{n} [\alpha_{kj}(\omega) x_j + \beta_k(\omega)] \tag{13}$$

subject to constraints

$$\sum_{j=1}^{n} a_{ij} x_j \geq b_i,\ i = \overline{1, m}, \tag{14}$$

where $\alpha_{kj}(\omega)$ and $\beta_k(\omega)$ are random numbers defined on a probability space (Ω, A, P), $E\alpha_{kj} < \infty$, $E\beta_k < \infty$, $\forall k$, j, and a_{ij}, b_i are deterministic parameters.

A number of stochastic facility location models, and inventory control models can be

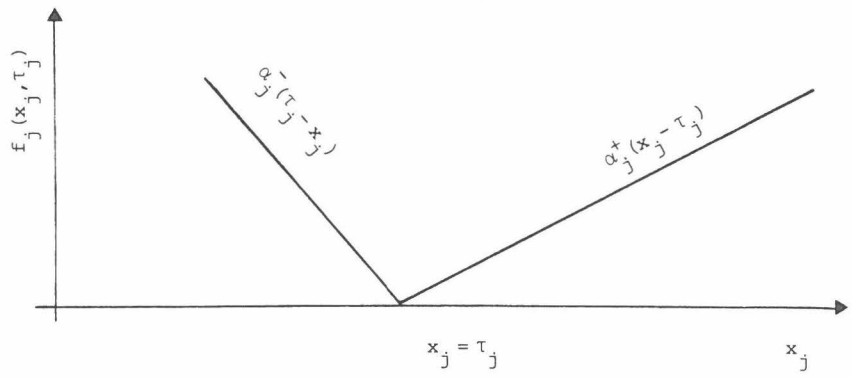

Fig. 1.

reduced to this type of stochastic minimax problem. For instance, the cost function $f_j(x_j, \tau_j)$ connected with overestimating and underestimating the demand τ_j in district j may be a piecewise linear function more general than the function $f_j(x_j, \tau_j)$ discussed above (see Figs. 1 and 2).

In this case, instead of function (9), we would have to minimize a function of the following type:

$$F(x) = \sum_{j=1}^{n} E \max_{1 \le k \le K_j} [\alpha_{jk}(x_j - \tau_j)].$$

The constraints (14) may reflect limits on the size of facilities, the total budget, or relations between different types of facilities to be located at the same time. The stochastic quasigradient procedure for solving (13)–(14) is defined by the following rule.

Let x^0 be an arbitrary initial approximation and x^s be the approximation obtained after the sth iteration. The random parameters α_{kj}^s, β_k^s are observed according to the probability space (Ω, A, P). An index k_s is computed, satisfying the relation

$$\sum_{j=1}^{n} [\alpha_{k_s j}^s x_j^s + \beta_{k_s}^s] = \max_{1 \le k \le K} \sum_{j=1}^{n} [\alpha_{kj}^s x_j + \beta_k^s]$$

and a vector $\xi^s = (\xi_1^s \dots \xi_n^s)$, where

$$\xi_j^s = \alpha_{k_s j}^s, \; j = \overline{1, n} \tag{15}$$

is determined. A new approximation x^{s+1} is then computed for $s = 0, 1 \dots$

$$x^{s+1} = \Pi(x^s - \rho_s \xi^s), \tag{16}$$

where Π is the projection on the feasible set X defined by constraints (14).

The projection of a point y on the set X is the solution to the problem of minimizing $|y - x|^2$ over $x \in X$ for fixed y. Since X is a polyhedral set, the computer can be used to carry out the projection. At each iteration the preceding projection is taken as the initial

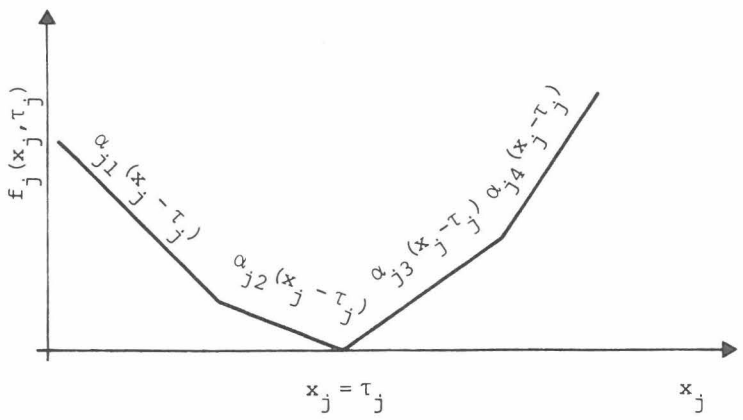

Fig. 2.

approximation for the new projection. In this case the new projection is found in a small number of steps. If $X = \{x \mid \underline{x} \le x \le \bar{x}\}$, then $\Pi(y) = \min\{\bar{x}, \max\{\underline{x}, y\}\}$. Under any reasonably simple choice of k_s, the vector ξ^s becomes random.

THEOREM 2. Assume that $E\alpha_{kj} < \infty$, $\forall k, j$. Then

$$E\{\xi^s \mid x^s\} = \hat{F}_x(x^s)$$

where $\hat{F}_x(x^s)$ is a subgradient of function (13).

Proof. The existence of the conditional mathematical expectation of ξ^s follows from the existence of $E\alpha_{kj}$, $\forall k, j$. We have

$$\begin{aligned}\max_{1\le k\le K} \sum_{j=1}^{n} [\alpha_{kj}x_j + \beta_k] - \sum_{j=1}^{n} [\alpha^s_{k_s j}x^s_j + \beta^s_{k_s}] \\ \ge \sum_{j=1}^{n} [\alpha^s_{k_s j}x_j + \beta^s_{k_s}] - \sum_{j=1}^{n} [\alpha^s_{k_s j}x^s_j + \beta^s_{k_s}] \\ = \sum_{j=1}^{n} \alpha^s_{k_s j}(x_j - x^s_j).\end{aligned}$$

Taking the conditional expectation of both sides, we obtain

$$F(x) - F(x^s) \ge \langle E\{\xi^s \mid x^s\}, x - x^s\rangle,$$

and the theorem is proved.

THEOREM 3. Assume that

(a) $$E \sum_{k,j} |\alpha_{kj}|^2 < C < \infty,$$

(b) $$\rho_s \ge 0, \sum_{s=0}^{\infty} \rho_s = \infty, \sum_{s=0}^{\infty} \rho_s^2 < \infty.$$

Then the sequence $\{x^s\}$ generated by (16) converges with probability 1 to an optimal solution of problem (13)–(14).

This theorem follows from the general result for the convergence of stochastic quasigradient methods [12, 15]. A simplified version of the proof follows.

The properties of the operation Π yield the expression

$$\begin{aligned}\|x^* - x^{s+1}\|^2 \le \|x^* - x^s + \rho_s\xi^s\|^2 = \|x^* - x^s\|^2 \\ + 2\rho_s\langle\xi^s, x^* - x^s\rangle + \rho_s^2\|\xi^s\|^2\end{aligned}$$

for any optimal solution x^*. From this

$$E\{\|x^* - x^{s+1}\|^2 \mid x^s\} \le \|x^* - x^s\|^2 + 2\rho_s[F(x^*) - F(x^s)] + C\rho_s^2. \tag{17}$$

Taking into account that $F(x^*) - F(x^s) \le 0$,

$$E\{\|x^* - x^{s+1}\|^2 \mid x^s\} \le \|x^* - x^s\|^2 + C\rho_s^2.$$

From this inequality, we have

$$E\{Z_{s+1} \mid Z_s\} \leq Z_s, \; s = 0, 1 \ldots$$

where

$$Z_s = \|x^* - x^s\|^2 + C \sum_{k=s}^{\infty} \rho_k^2.$$

The sequence $\{Z_s\}$ is therefore a super martingale and $Z_s \geq 0$. From this, the sequence $\{Z^s\}$ converges with probability 1. The assumption

$$\sum_{s=0}^{\infty} \rho_s^2 < \infty$$

means that $\{\|x^* - x^s\|^2\}$ also converges with probability 1. For two distinct accumulation points x', x'' of the sequence $\{x^s\}$ we have

$$\|x^* - x'\|^2 - \|x^* - x''\|^2 = 0 = 2\langle x^*, x'' - x'\rangle + \|x'\|^2 - \|x''\|^2.$$

Thus, if x', x'' do not belong to the set of optimal solutions X^*, then X^* lies in the hyperplane equidistant from points x', x''. Therefore, if we show that one of the accumulation points of $\{x^s\}$ belongs to X^* with probability 1, it would follow that the sequence $\{x^s\}$ converges with probability 1 to X^*.

$$E\|x^* - x^{s+1}\|^2 \leq E\|x^* - x^0\|^2 + 2E \sum_{k=0}^{s} \rho_k [F(x^*) - F(x^s)] + C \sum_{k=0}^{s} \rho_k^2$$

from which we get

$$E \sum_{s=0}^{\infty} \rho_s [F(x^*) - F(x^s)] > 0.$$

Since

$$\sum_{s=0}^{\infty} \rho_s = \infty \text{ and } F(x^*) - F(x^s) \leq 0,$$

then with probability 1 there exists a subsequence x^{s_k} such that

$$F(x^*) - F(x^{s_k}) \to 0,$$

and this completes the proof.

4. PRACTICAL COMPUTATIONS

Procedure (10) has been used to solve some problems of high-school location in Turin; the computational aspects have been explored by Ermoliev, Leonardi, and Vira [16].

The step-size ρ_s is not determined by the control equations (11); the ρ_s are usually controlled by keeping the step multiplier constant during a number of iterations and then reducing it according to certain rules. In the course of the iterations a succession of

values

$$\bar{f}_s = \frac{1}{s}\sum_{k=0}^{s} f(x^k, \tau^k),$$

where

$$f(x^k, \omega^k) = \sum_{j=1}^{n} \max\{a_j^+(x_j^k - \tau_j^k), a_j^-(\tau_j^k - x_j^k)\},$$
$$\omega^k = (\tau_1^k \ldots \tau_n^k)$$

is observed. The values of the sequence $\{f(x^s, \omega^s)\}$ usually vary over a wide range, while the sequence $\{\bar{f}_s\}$ shows much smoother behavior, as shown in Fig. 3; one rule used in controlling the step-size is based on this fact. The whole method can be summarized as follows:

(1) Choose an initial value ρ_0 for the step multiplier.
(2) Using ρ_0 for the step multiplier, calculate the value of $\bar{f}_s$.
(3) When the values of the sequence $\{\bar{f}_s\}$ remain at approximately the same level, reduce the value of step multiplier by one half.
(4) Go back to step 2 until no improvement in the test function $\bar{f}_s$ is observed.

There are some unanswered questions in the procedure outlined above. First, how should the initial step multiplier be chosen? If it is too large, the sequence $\{\bar{f}_s\}$ will oscillate heavily and no decrease in the objective function will be observed. If the initial step multiplier is too small, the rate of decrease will be very small, perhaps almost imperceptible.

One of the best ways of controlling the procedure would be to use an on-line code, where the program continuously plots the values of the sequence $\{\bar{f}_s\}$ on the screen and the iterations could be manually interrupted to reduce the step multiplier. This is not always possible, which means that the iterations must be performed in small batches, the

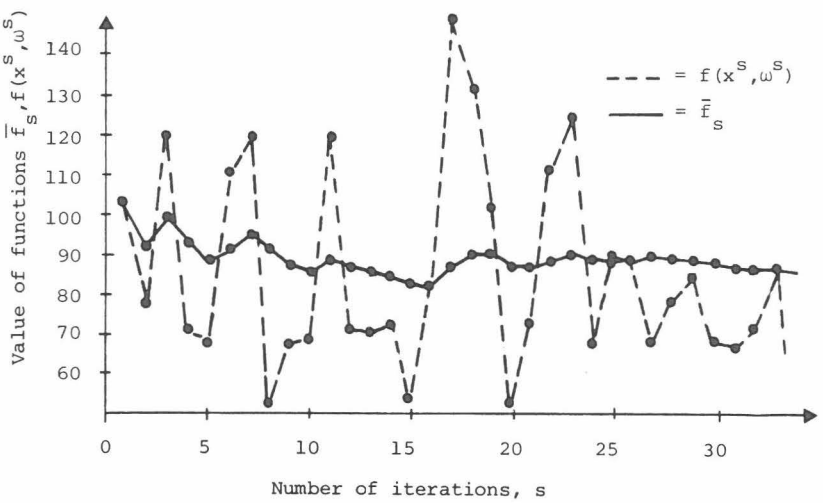

Fig. 3. The behavior of the sequences $\{f(x^s, \omega^s)\}$ and $\{\bar{f}_s\}$ as a function of the iteration number.

values of $\bar{f}_s$ are then plotted and adjustments of the step multiplier can be made. The manual step-size control requires considerable effort from the operator and usually results in a slow computer code.

An automatic version of the manual step-size control would overcome the need for numerous manual operations. Ermoliev, Leonardi, and Vira [16] have developed a simple procedure which, given three parameters, simulates the behavior of the controlling person and reduces the step multiplier as soon as it observes a stationary, or oscillatory, sequence $\bar{f}_s$.

Let the three input parameters be N, γ, and δ. The parameter N fixes the batch size, i.e., the iterations $t = 1, 2 \ldots$ will be performed in batches of N. Define

$$M_t = \{s \mid (t-1)N \le s \le tN\}, \; t = 1, 2 \ldots$$
$$\bar{f}_s^+ = \max\{0, \bar{f}_s - \bar{f}_{s-1}\}.$$

The procedure checks two conditions:

$$\frac{\bar{f}_{(t-1)N} - \bar{f}_{tN}}{\sum_{s \in M_t} \rho_s \|\xi^s\|} \le \gamma,$$

$$\frac{\sum_{s \in M_t} \bar{f}_s^+}{\max_{s \in M_t}(\bar{f}_s) - \max_{s \in M_t}(\bar{f}_s)} \ge \delta.$$

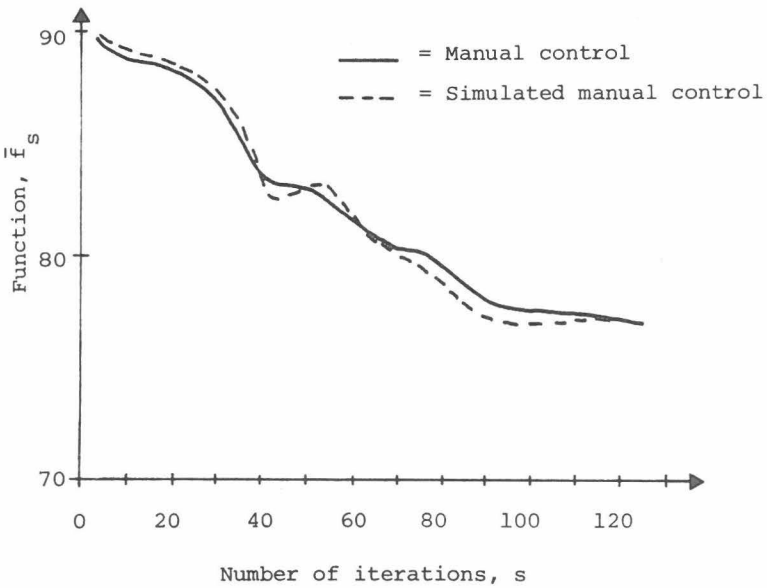

Fig. 4. The convergence behavior of $\{\bar{f}_s\}$ in the manual control and simulated manual control cases.

If either of these conditions holds, the step multiplier will be reduced by one half. The first condition tests whether the decrease in the sequence, proportional to the step-size used, is less than the given limit. The second condition then checks if the sequence is oscillatory. This is done by considering the ratio of the sum of positive jumps of the sequence $\{\bar{f}_s\}$ to the maximum change in the sequence during the batch of iterations.

With $\gamma = 0.01$, $\delta = 0.30$, and $N = 5$ the procedure simulates manual control very closely (Fig. 4). Note that if the initial approximation x^0 is far from the actual solution and a small initial step multiplier is used, then the procedure described above may reduce the step-size too rapidly. This danger can normally be eliminated by selecting an initial step-size that is too big, rather than too small.

5. OPTIMALITY CONDITIONS

The numerical methods outlined in Sec. 3 are quite general and can be used even if the distributions of random parameters are very ill-conditioned.

If, however, these distributions are sufficiently well-behaved, it may be worth trying to develop the exact optimality conditions and trying to find a set of simple equations for the optimal solution. The general optimality conditions for stochastic programming problems have been investigated [12, 17–21], but the special structure of problems (7)–(8) and (13)–(14) can be exploited to obtain the optimality conditions in a more specialized form. These conditions are outlined below.

Let $\hat{F}_x(x)$ be a subgradient of convex function (13). We have a number of well-known conditions necessary and sufficient for optimality [19–21]: there must exist $\hat{F}_x(x)$ and numbers $\lambda_i \geq 0$, $i = \overline{1, m}$ such that

$$\sum_{i=1}^{m} \lambda_i \neq 0, \ \hat{F}_x(x) - \sum_{i=1}^{m} \lambda_i a^i = 0,$$

$$\lambda_i \left(\sum_{j=1}^{n} a_{ij} x_j - b_i \right) = 0, \ i = \overline{1, m},$$

where $a^i = (a_{i1} \ldots a_{in})$. However, it is also known [22] that $\hat{F}_x(x)$ is a subgradient of a function of type (13) if and only if

$$\hat{F}_x(x) = E\hat{f}_x(x, \omega),$$

where $\hat{f}_x(x, \omega)$ is a subgradient of function

$$f(x, \omega) = \max_{1 \leq k \leq K} \sum_{j=1}^{n} [\alpha_{kj}(\omega) x_j + \beta_k(\omega)],$$

for fixed ω. We therefore have the following conditions: a point x is an optimal solution of the problem (13)–(14) if and only if there exist a subgradient $\hat{f}_x(x, \omega)$ and multipliers $\lambda_i \geq 0$,

$$\sum_{i=1}^{m} \lambda_i \neq 0$$

such that

$$E\hat{f}_x(x, \omega) - \sum_{i=1}^{m} \lambda_i a^i = 0, \tag{18}$$

$$\lambda_i\Big(\sum_{j=1}^{n} a_{ij}x_j - b_i\Big) = 0,\ i = \overline{1, m}. \tag{19}$$

It is known (see, for instance, [12, 15, 23]) that for fixed ω

$$\hat{f}_x(x, \omega) \in G(x, \omega) = \mathrm{Co}\{\alpha^\nu(\omega),\ \nu \in K(x, \omega)\}, \tag{20}$$

where Co denotes the set of all linear combinations of the argument vectors

$$\alpha^\nu(\omega) = (\alpha_{\nu 1}(\omega) \ldots \alpha_{\nu n}(\omega)),$$

$$K(x, \omega) = \{\nu \mid \langle \alpha^\nu(\omega), x\rangle + \beta_\nu(\omega) = \max_k[\langle \alpha^k(\omega), x\rangle + \beta_k(\omega)]\}.$$

Let us now come back to the original problem (7)–(8). For this problem

$$\omega = (\tau_1 \ldots \tau_n),$$

$$f(x, \omega) = \sum_{j=1}^{n} \max\{\alpha_j^+(x_j - \tau_j), \alpha_j^-(\tau_j - x_j)\}$$

$$= \max_{(k_1 \ldots k_n)} \sum_{j=1}^{n} \alpha_j^{k_j}(x_j - \tau_j),$$

where $k_j \in \{1, 2\}$, $\alpha_j^1 = \alpha_j^+$, $\alpha_j^2 = -\alpha_j^-$. We also have

$$G(x, \omega) = G_1(x, \omega) \times \cdots \times G_n(x, \omega),$$

$$G_j(x, \omega) = \mathrm{Co}\{\alpha_j^k,\ k \in K(x, \omega)\},$$

$$K(x, \omega) = \begin{cases} \{1\}, \text{ if } x_j > \tau_j, \\ \{2\}, \text{ if } x_j < \tau_j, \\ \{1, 2\}, \text{ if } x_j = \tau_j. \end{cases}$$

Then, from (19)–(20), we obtain the following optimality conditions for problem (7)–(8): multipliers $0 \le \gamma_j \le 1$, $\lambda_j \ge 0$, $j = \overline{1, n}$ exist such that

$$\alpha_j^+ H_j(x_j) - \alpha_j^-[1 - H_j(x_j)] + [\gamma_j\alpha_j^+ - (1 - \gamma_j)\alpha_j^-]\, \mathrm{d}H_j(x_j) + \lambda_j = 0$$
$$\lambda_j x_j = 0,\ j = \overline{1, n}$$

if and only if the point x is an optimal solution. These conditions can be rewritten as follows:

$$\left.\begin{aligned} (\alpha_j^+ + \alpha_j^-)H_j(x_j) + [\gamma_j\alpha_j^+ - (1 - \gamma_j)\alpha_j^-]\, \mathrm{d}H_j(x_j) \le \alpha_j^-, \text{ if } x_j = 0 \\ (\alpha_j^+ + \alpha_j^-)H_j(x_j) + [\gamma_j\alpha_j^+ - (1 - \gamma_j)\alpha_j^-]\, \mathrm{d}H_j(x_j) = \alpha_j^-, \text{ if } x_j > 0 \end{aligned}\right\} \tag{21}$$

In particular, if $\mathrm{d}H_j(x_j) = 0$ at an optimal solution, or if the distributions $H_j(\cdot)$ are continuous, then we obtain

$$\left.\begin{aligned} (\alpha_j^+ + \alpha_j^-)H_j(x_j) \le \alpha_j^-, \text{ if } x_j = 0, \\ (\alpha_j^+ + \alpha_j^-)H_j(x_j) = \alpha_j^-, \text{ if } x_j > 0. \end{aligned}\right\} \tag{22}$$

From Eqs. (21) and (22) it is possible to obtain a closed-form optimal solution for some kinds of distributions H_j. One important peculiarity of problem (7)–(8) should also be noted: the set of subgradients of objective function (7) has a closed form,

$$\partial F(x) = \partial F^{(1)} \times \cdots \times F^{(n)},$$

$$\partial F^{(j)} = \{(\alpha_j^+ + \alpha_j^-)H_j(x_j) + [\gamma_j\alpha_j^+ - (1-\gamma_j)\alpha_j^-]\alpha H_j(x_j) - \alpha_j^- \mid 0 \le \gamma_j \le 1\}.$$

This gives us the opportunity to use both descent and nondescent methods of nondifferentiable optimization [12, 24, 25] if H_j is known and the values of the objective function can be computed (note that nondescent procedures do not require the second of these conditions to be met).

6. CONCLUDING COMMENTS AND ISSUES FOR FURTHER RESEARCH

The examples discussed in the previous sections have been kept as simple as possible, in order to introduce the methods. However, when some of the simplifying assumptions are dropped, new and more realistic models are obtained.

One possible path toward generalization is the introduction of more complex cost functions and constraints. For instance, introducing cost functions which include a fixed charge to be paid when a facility is established, regardless of its size. The optimization problem then assumes a combinatorial aspect. Research on this kind of problem is proceeding and some numerical results have already been obtained [16].

Another generalization is obtained by introducing many types of facilities, all to be located at the same time. For example, one may be concerned with locating schools specializing in different subjects or providing different training. All of the above constraints still hold for each type of school, but some new constraints may be needed due to interactions among the different types of schools. For instance, total demand for each type of school may not be known in advance, and customers may be allowed to choose both the location and the type of schools.

When all the above generalizations are introduced, the resulting model looks much more complicated than those discussed earlier in this paper. However, it still belongs to the general class of stochastic programs of type (13)–(14) with linear constraints for which theoretical results and algorithms are available. Some applications of stochastic programming to location problems of this type are in progress, and will be the subject of a forthcoming publication.

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A SYSTEM OF COMPUTABLE GENERAL EQUILIBRIUM MODELS FOR A SMALL OPEN ECONOMY

LARS BERGMAN
Stockholm School of Economics
Stockholm, Sweden

and

International Institute for Applied Systems Analysis
Laxenburg, Austria

Abstract—In this article a system of computable general equilibrium models for a small open economy is presented. One of the models is intended for analysis of "long-run" resource allocation problems. It is a static model in which the economy's endowments of capital and labor can be reallocated among the production sectors in response to, for instance, changes in world market conditions. The other model is a dynamic model, elucidating certain aspects of the economy's development from a "short-run" equilibrium in the direction of a "long-run" equilibrium as determined by the static model. In the article, the models are also used to analyze the impact on the Swedish economy of a 50% increase in the cost of electricity.

1. INTRODUCTION

After the pathbreaking study by Johansen [1], the development and application of computable general equilibrium models (CGE-models) has become a rapidly growing field in economics.[a] Although there is no unambiguous definition of a CGE-model, this term is usually applied to aggregated models of growth and structural change in a national economy, specified in accordance with the basic notions of Walrasian general equilibrium theory. Thus, in CGE-models, prices and quantities of traded goods and factors are typically determined simultaneously, and goods and factor markets are usually treated as if they were competitive. Moreover, the structural equations are generally derived from assumptions about optimization behavior and explicit representations of technology and preferences.

A number of publications originating from the IMPACT project[b] and from the World Bank[c] are standard references in this field. Other relevant references, which cover a variety of different applications of CGE-models, are Hudson and Jorgenson [6], Kelley and Williamson [7], Lloyd [8], Whalley [9] and Zalai [10].

The purpose of this study is to present a system of CGE-models especially designed for analysis of problems related to national energy policies in a small open economy, i.e., an economy with a relatively large trade-exposed sector, but with limited influence in its terms of trade. The model-set consists of a static model for projections of "long-run" equilibria, and a dynamic model for projection of certain aspects of the economy's evolution from a "short-run" equilibrium in the direction of a "long-run" equilibrium. In addition, there are a number of variants of the dynamic model which, in turn, can be regarded as a variant of the static model.

[a]For a brief survey of the field, see Bergman and Por [2].
[b]See, for instance, Dixon *et al.* [3].
[c]See, for instance, Adelman and Robinson [4] and de Melo *et al.* [5].

In Sec. 2, some common features of the models are discussed, while Sec. 3 gives a relatively detailed description of the long-run static model. Section 4 deals with the dynamic model and gives a brief description of the entire model-set. In Sec. 5, the practical implementation of the models is briefly discussed, and in Sec. 6 the results of an analysis of the impact of higher electricity costs are presented. Finally, in Sec. 7, an alternative specification of the foreign-trade part of the models is discussed.

2. SOME COMMON FEATURES OF THE MODELS

All of the models are "real," i.e., there are no financial assets and the exchange rate, being the numeraire of the price system, is given exogenously. This also applies to world market conditions, domestic technology and preferences, and real public consumption. Moreover, the gross savings ratio is determined outside the models. In all models, labor supply is given exogenously and in the static model this also applies to the supply of capital.

In the solution to each of the models, a system of equilibrium relative prices of goods and the real wage rate are determined, as well as a specific pattern of production, consumption, foreign trade, and employment. The static model determines the sectoral use of capital, while all variants of the dynamic model determine the sectoral allocation of gross investments.

All product and factor markets are treated as if they were competitive, and relative product and factor prices are generally assumed to be flexible enough to clear all markets. In some variants of the dynamic model, however, the real wage is determined exogenously in all or some periods and consequently the labor market is not necessarily cleared.

In the models a distinction is made between the *ex ante* production function and the *ex post* production function. The *ex ante* production function is, in principle, a planning concept; it represents the technological constraints which apply in the planning stage when new production units are designed. The *ex post* production function, on the other hand, represents the technological constraints on the operation of existing production units.

The *ex ante* technology is assumed to exhibit constant returns to scale, and in each sector capital, labor, fuels, and electricity are assumed to be substitutable factors of production. The use of manufactured nonenergy inputs, however, is taken to be proportional to the output of the sector in which the inputs are used.

The *ex post* production functions may be derived directly from the *ex ante* production functions if two assumptions are made. The first is that once capital has been invested in a given sector it cannot be reallocated to some other sector. The second assumption is that once the design (in terms of the use of fuels and electricity per unit of output) of a new production unit has been determined, the energy input coefficients are fixed. Thus, *ex post*, the use of energy and of nonenergy produced inputs are determined in the same way. It should be noted that production units designed in period t can be put into operation in period $t + 1$.

From the derivation of the *ex post* production functions it is clear that they exhibit decreasing returns to scale. However, because the *ex ante* production function is assumed to shift over time due to technical progress, production units of different "vintages" must be distinguished in each sector. Consequently, there will be a number of *ex post* production functions in each sector.

As it is assumed throughout that the producers' aim is to maximize profits, a dual representation of technology is more convenient than the traditional representation in terms of production functions. Thus, the *ex ante* technology with constant returns to

scale can be represented by an *ex ante* unit cost function for each sector, and the *ex post* technology can be represented by an *ex post* profit function for production units of each vintage in each sector.[d] In accordance with Shephard's lemma, the *ex ante* factor proportions that minimize the cost are given by the partial derivatives of the *ex ante* unit cost functions, while Hotelling's lemma (see Varian [11]) suggests that product supply and labor demand that maximize the profit in existing production units are given by the partial derivatives of the profit functions.

All of the models describe a small open economy, i.e., an economy facing a completely elastic supply of imports at given world market prices as well as exogenously given export prices. If such an economy consumes, and can produce, n different tradeable goods by means of m factors of production at constant returns to scale, and $n > m$, equilibrium implies that at most m goods will be produced and possibly exported, while the difference between domestic consumption and production of nonexport tradeable goods will be imported (see Samuelson [12]). Thus, tradeable goods will be either imported or exported, but not both. In particular, if there are only two factors of production and several tradeable goods, only two goods will be produced at equilibrium.

It can easily be shown (see Bergman and Por [2], Chap. 2) that if the equilibrium solutions to a model with many goods, many factors and technology with returns to scale are aggregated to a relatively small number of sectors, the aggregated indices for import volumes, import prices, domestic production, and domestic production costs are correlated as if there was a relative-price dependent import function in each aggregated sector producing tradeable goods. Thus, even if all types of goods could be produced in all countries, each country might, at equilibrium, produce a unique aggregate of goods with a given statistical classification. In other words, in a model dealing with large aggregates of goods rather than individual products, similar "goods" with different countries of origin can be regarded as less than perfect substitutes.

This observation provides a rationale for incorporating the so-called Armington assumption in a CGE-model of a small open economy. According to Armington [13], similar goods with different countries of origin are less than perfect substitutes, and domestic users of commodities with a given statistical classification actually use a mixture (composite) of imported and domestically produced goods with that classification.

The Armington assumption, which is incorporated in most CGE-models of open economies, implies that the price indices of domestically consumed composite goods are given by the unit cost functions corresponding to the "production" functions defining the composite goods. By Shephard's lemma, the "input" of domestically produced and imported goods, respectively, per unit of composite goods is given by the partial derivatives of the unit cost function for composite goods, with respect to the price of goods from the two sources of supply. Thus the import functions are given by the product of the domestic demand for composite goods and the "input" of imports per unit of composite goods.

The goods exported from the small country are, of course, the goods imported by the rest of the world. Therefore, by applying the Armington assumption to the rest of the world, it is possible to obtain relative-price dependent export functions for the small country. From the assumption that the economy modelled has a very limited influence on export prices, it follows that the absolute values of the export price elasticities implied by the Armington export functions should be high. The models presented here all contain import and export functions based on the Armington assumption.

[d] See Varian [11] for an exposition of the relation between primal and dual representations of technology when producers maximize profits.

Each model describes an economy with $n+3$ production sectors producing $n+3$ goods of which n are tradeables. There is no joint production, and each good is produced in one sector only. Thus there is no real distinction between domestically produced goods and domestic production sectors. In the following exposition, goods and sectors will be denoted interchangeably by i and j. The production sectors are numbered from 0 to $n+3$, 0 being the sector producing fuels and 1 the electricity-generating sector, while $n+1$ is a private sector producing nontradeable goods and $n+2$ is the public sector. There is also a "bookkeeping" sector, $n+3$ in which different goods are aggregated into one single capital good. On the demand side all households are represented by an aggregated household sector.

3. THE LONG-RUN STATIC MODEL[e]

This model is intended to be a tool for analysis of long-run resource allocation problems. Here, "long-run" simply means that the time horizon is extended far enough to make it reasonable to let the *ex ante* unit cost function represent the technological constraints. The equilibrium condition for producers is then that the prices of domestically produced goods should be equal to the unit production costs of these goods.

As a consequence of the assumptions about the technology, the *ex ante* unit cost function can be divided into two parts. The first represents the minimum cost of fuels, electricity, capital, and labor per unit of output, while the second represents the corresponding cost of nonsubstitutable inputs. In the following, the first part is called "the net unit cost function." The producers' equilibrium condition can now be written

$$P_j = \kappa_j^*(P_0^D, P_1^D, W_j, R_j; t) + \sum_{i=2}^{n} P_i^D a_{ij} + Q_j b_j; \quad j = 0, 1 \ldots n+2, \tag{1}$$

where $\kappa_j^*(\cdot)$ is the *ex ante* net unit cost function; P_j is the price of output j; P_i^D the price of composite good i; Q_j the price of complementary imports used as inputs in sector j; W_j the wage rate in sector j; R_j the user cost of capital in sector j; and t an exogenous shift parameter. The constants a_{ij} represent the input of composite good i per unit of output in sector j, and b_j is the corresponding parameter for input of complementary imports in sector j.[f]

The heterogeneity of labor is roughly accounted for by an exogenous wage structure, i.e.,

$$W_j = \omega_j W; \quad j = 0, 1 \ldots n+2, \tag{2}$$

where W is a general wage index and the ω_j's are constants. The user cost of capital is defined by

$$R_j = P_{n+3}(\delta_j + R); \quad j = 0, 1 \ldots n+2, \tag{3}$$

where P_{n+3} is the price of the aggregated capital good; δ_j the rate of depreciation in

[e] As the static model describes the situation in one single period, the variables are written without a time-index. However, when the exact specifications of a function depend on which particular period is to be analyzed, a time-dependent shift parameter is included.

[f] Complementary imports are only used in the energy sector, i.e., when $j = 0, 1$.

sector j and R the real rate of interest. The price index of capital goods is defined by

$$P_{n+3} = \sum_{i=1}^{n} P_i^D a_{i,n+3}; \tag{4}$$

where the coefficients $a_{i,n+3}$ sum to unity. The equilibrium prices of composite goods are given by the unit cost functions of the composites, i.e., by

$$P_i^D = \phi_i(P_i, P_i^M); \quad i = 0, 1 \ldots n, \tag{5}$$

where $\phi_i(\cdot)$ is the unit cost function corresponding to the "production" function defining composite good i, and P_i^M is the exogenously given world market price, in domestic currency, of goods with classification i.

Having now defined all prices and unit cost functions, the derivation of the static model is quite straightforward. As the *ex ante* technology exhibits constant returns to scale, the sectoral production levels are determined from the demand side, where three types of demand should be distinguished. There are two types of demand for composite goods: intermediate demand and final demand by the household sector. The third type of demand is export demand for the production sector outputs.

By Shephard's lemma and the assumptions about technology, the intermediate demand is given by

$$X_{ij} = \begin{cases} \dfrac{\partial \kappa_i^*(\cdot)}{\partial P_i^D} X_j, & \text{when } i = 0, 1 \\ a_{ij} X_j, & \text{when } i = 2, 3 \ldots n \end{cases} \quad j = 0, 1 \ldots n+2 \tag{6}$$

where X_{ij} is the use of composite good i in sector j, and X_j is the gross output in sector j. Household demand is given by a function of the following type:

$$C_i = C_i(P_0^D \ldots P_i^D \ldots P_n^D, P_{n+1}, E); \quad i = 0, 1 \ldots n+1, \tag{7}$$

where C_i is household demand for good i, E is total household consumption expenditure, and the functions $C_i(\cdot)$ are derived from the assumption that the household sector will maximize the utility budget constraint.

By Shephard's lemma, the demand for competitive imports is given by

$$M_i = \frac{\partial \phi_i(\cdot)}{\partial P_i^M} \left\{ \sum_{j=0}^{n+3} X_{ij} + C_i \right\}; \quad i = 0, 1 \ldots n, \tag{8}$$

i.e., import demand is a function of the prices P_i and P_i^M, and the domestic demand for composite goods. Applying the same assumptions for "the rest of the world" thus means that export demand is given by functions of the type

$$Z_i = Z_i(P_i, P_i^W; t); \quad i = 1, 2 \ldots n, \tag{9}$$

where Z_i is export demand for domestically produced goods with the classification i and P_i^W is the world market price for such goods produced elsewhere. The distinction between P_i^W and P_i^M is due to the fact that P_i^W is normally a f.o.b. price while P_i^M is normally a c.i.f. price. As the home economy is assumed to be small, the use of composite goods in the rest of the world is approximately equal to the production in the rest of the world. Thus the size of the world market can be represented by the exogenous shift parameter t.

Given the different demand equations, the equilibrium conditions for the markets for domestically produced goods are given by

$$X_i = \frac{\partial \phi_i(\cdot)}{\partial P_i}\left\{\sum_{j=0}^{n+3} X_{ij} + C_i\right\} + Z_i; \quad i = 0, 1 \ldots n \tag{10}$$

$$X_i = C_i; \quad i = n+1, n+2 \tag{11}$$

$$X_{n+3} = I + \sum_{j=0}^{n+2} \delta_j \frac{\partial \kappa_j^*(\cdot)}{\partial R_j} X_j; \tag{12}$$

where C_{n+2} is the exogenously given public consumption and I is total net investments.

At equilibrium, household consumption expenditure, E, must be equal to the factor incomes of the household sector less net taxes and household savings. Instead of specifying such an inequality explicitly, it is determined implicitly by a current account constraint. Thus, at equilibrium, the following expression holds:

$$\sum_{i=1}^{n} P_i Z_i = \sum_{i=0}^{n} P_i^M M_i + \sum_{j=0}^{1} Q_j \bar{M}_j + D; \tag{13}$$

where $\bar{M}_j$ is the demand for complementary imports and D is an exogenous variable representing net foreign transfers and net interest payments on foreign debt, expressed in domestic currency.

Finally, as capital and labor are supplied inelastically, the equilibrium conditions for the factor markets become

$$K = \sum_{j=0}^{n+2} \frac{\partial \kappa_j^*(\cdot)}{\partial R_j} X_j; \tag{14}$$

$$L = \sum_{j=0}^{n+2} \frac{\partial \kappa_j^*(\cdot)}{\partial W_j} X_j; \tag{15}$$

where K is total capital supply and L is total labor supply. Altogether these expressions, after appropriate substitutions, yield $6n + 15$ equations in the $6n + 15$ unknowns: $X_0 \ldots X_{n+3}$, $C_0 \ldots C_{n+1}$, $Z_1 \ldots Z_n$; $M_0 \ldots M_n$; $P_0 \ldots P_{n+3}$; $P_0^D \ldots P_n^D$; E; W; and R.

4. THE DYNAMIC MODEL

In the dynamic model there is, at some point in time, a "history" of technology and investment decisions in the form of production units of different "vintages" in each sector. In each production unit the technological constraints are given by the *ex post* production function derived from the *ex ante* production function that existed at the time of investment. The profit function of production units of a particular vintage, in each sector, represents the relevant technological constraints as well as the behavior of producers. Thus, gross profits in production units of vintage ν in sector j can be written

$$\Pi_{\nu j}(t) = \Pi_{\nu j}(P_{\nu j}^*(t), W_j(t); t) \quad \nu = 0, 1 \ldots t; \quad j = 0, 1 \ldots n+2 \tag{16}$$

where $\Pi_{\nu j}(\cdot)$ is the profit function of production units of vintage ν in sector j, and where[8]

$$P_{\nu j}^*(t) = P_j(t) - \sum_{i=0}^{1} P_i^D(t) a_{\nu ij} - \sum_{i=2}^{n} P_i^D(t) a_{ij} - Q_j(t) b_j; \quad \nu = 0, 1 \ldots t; \; j = 0, 1 \ldots n+2. \tag{17}$$

[8] If Eq. (17) leads to $P_{\nu j}^*(t) < 0$ for some vintage, that equation is replaced by $P_{\nu j}^* = 0$ for the vintage in question.

Except for the dating, the symbols have the same meaning as in the preceding section. Observe that the profit functions shift over time as a result of exogenously determined depreciation of the initially invested capital. However, not all depreciation is determined exogenously; employment reductions in excess of those corresponding to the exogenously determined depreciation of old vintages can be interpreted as endogenously determined scrapping of inefficient production units.

By Hotelling's lemma the profit-maximizing supply of domestically produced goods in period t and sector j is given by

$$X_j(t) = \sum_{\nu=0}^{t} \frac{\partial \Pi_{\nu j}(\cdot)}{\partial P^*_{\nu j}}; \quad j = 0, 1 \ldots n+2 \tag{18}$$

and by the same lemma the profit-maximizing demand for labor in period t and sector j can be written

$$L_j(t) = \sum_{\nu=0}^{t} -\frac{\partial \Pi_{\nu j}(\cdot)}{\partial W_j}. \tag{19}$$

Looking at the allocation of resources at a given point in time, these are the only essential differences between the static and the dynamic model. Observe that the specification of the dynamic model implies that the number of vintages increases over time. In the initial period there is only one vintage in each sector, but then a new vintage is introduced in each period. In the case where t is set equal to zero, the dynamic model simply becomes another "snapshot" model, differing from the static model by the fixed sectoral capital stocks and energy input coefficients. In the following, this version of the dynamic model will be called the "Short-Run Static Model." Table 1 gives a summary description of the three basic variants of the resource allocation model for a single period, say t. Where the models differ, the specification which applies to the Long-Run Static Model is indicated by LRS, while SRS and DYN indicate the specification adopted in the Short-Run Static Model and the Dynamic Model, respectively. A complete set of definitions of the symbols can be found in the Appendix. It should also be noted that in

Table 1. Three alternative models of resource allocation in period t (see Appendix for definition of symbols).[a]

Output supply	
SRS[b]	$X_j = \frac{\partial \Pi_j(\cdot)}{\partial P^*_j}; \quad j = 0, 1 \ldots n+2$
DYN	$X_j = \sum_{\nu=0}^{t} \frac{\partial \Pi_{\nu j}(\cdot)}{\partial P^*_{\nu j}} \equiv \sum_{\nu=0}^{t} X_{\nu j}; \quad j = 0, 1 \ldots n+2$
LRS[c]	$P_j = \kappa^*_j(P^D_0, P^D_1, W_j, R_j; t) + \sum_{i=2}^{n} P^D_i a_{ij} + Q_j b_j; \quad j = 0, 1 \ldots n+2$
Input demand	
(a) Intermediate inputs	
SRS	$X_{ij} = a_{ij} X_j; \quad i = 0, 1 \ldots n; \quad j = 0, 1 \ldots n+3$
DYN	$X_{ij} = \begin{cases} \sum_{\nu=0}^{t} a_{\nu ij} X_{\nu j}; & i = 0, 1 \\ a_{ij} X_j; & i = 2, 3 \ldots n \end{cases} \quad j = 0, 1 \ldots n+3$
LRS	$X_{ij} = \begin{cases} \frac{\partial \kappa^*_j(\cdot)}{\partial P^D_i} X_j; & i = 0, 1 \\ a_{ij} X_j & i = 2, 3 \ldots n \end{cases} \quad j = 0, 1 \ldots n+3$

Table 1. *Continued*

(b) Labor

SRS $L_j = -\frac{\partial \Pi_j(\cdot)}{\partial W_j}; \quad j = 0, 1 \ldots n+2$

DYN $L_j = \sum_{\nu=0}^{t} -\frac{\partial \Pi_{\nu j}(\cdot)}{W_j} \equiv \sum_{\nu=0}^{t} L_{\nu j}; \quad j = 0, 1 \ldots n+2$

LRS $L_j = \frac{\partial \kappa_j^*(\cdot)}{\partial W_j} X_j; \quad j = 0, 1 \ldots n+2$

Household demand for composite goods

$C_i = C_i(P_0^D, P_1^D \ldots P_n^D, P_{n+1}, E); \quad i = 0, 1 \ldots n+1$

Export demand[d]

$Z_i = Z_i(P_i, P_i^W; t); \quad i = 1, 2 \ldots n$

Gross investments

SRS I^G exogenous

DYN $sY = P_{n+3} I^G + D$

LRS $I^G = I + \sum_{j=0}^{n+2} \frac{\partial \kappa_j(\cdot)}{\partial R_j} X_j$

Definitions

$P_{vj}^* = P_j - \sum_{i=0}^{1} P_i^D a_{vij} - \sum_{i=2}^{n} P_i^D a_{ij} - Q_j b_j; \quad j = 0, 1 \ldots n+2$

$P_i^D = \phi_i(P_i, P_i^M); \quad i = 0, 1 \ldots n$

$P_{n+3} = \sum_{i=2}^{n} P_i^D a_{i, n+3}$

$R_j = P_{n+3}(\delta_j + R); \quad j = 0, 1 \ldots n+2$

$W_j = \omega_j W; \quad j = 0, 1 \ldots n+2$

[a]As all variables apply for period t, the time indices have been left out. The models are SRS = The Short-Run Static Model, DYN = The Dynamic Model, and LRS = The Long-Run Static Model.

[b]As SRS is defined for the initial period only, production units can have only one vintage and consequently the vintage index is left out.

[c]Note that $b_j \neq 0$ for $j = 0, 1$ only.

[d]Note that $Z_0 = 0$.

alternative versions of SRS and DYN, the wage rate is assumed to be given exogenously. This means that the labor market equilibrium condition becomes an accounting relation, indicating the total demand for labor at a given wage level.

In LRS, net investments (I) in the economy as a whole are determined exogenously, whereas this applies to gross investments (I^G) in SRS. In DYN, however, the level of gross investments is determined by an exogenously given gross savings ratio, $s(t)$, in accordance with the equation

$$s(t)Y(t) = P_{n+3}(t)I^G(t) + D(t), \tag{20}$$

where $Y(t)$ is the gross national income at current (relative) prices.

The creation of new vintages, however, is an important part of the dynamic model. The approach adopted in this part represents a quite significant simplication of what one

might consider a "realistic" approach. It is assumed that producers have expectations about future prices and that all producers have the same expectations. Thus, denoting expected prices by a tilde (˜), the following expressions hold:

$$\tilde{P}_j(t) = \tilde{P}_j(P_j(t), \hat{P}_j(t)); \qquad j = 0, 1 \ldots n+2 \tag{21}$$

$$\tilde{P}_i^D(t) = \tilde{P}_i^D(P_i^D(t), \hat{P}_i^D(t)); \qquad i = 0, 1 \ldots n \tag{22}$$

$$\tilde{Q}_j(t) = \tilde{Q}_j(Q_j(t), \hat{Q}_j(t)); \qquad j = 0, 1 \tag{23}$$

$$\tilde{W}_j(t) = \tilde{W}_j(W_j(t), \hat{W}_j(t)); \qquad j = 0, 1 \ldots n+2 \tag{24}$$

where the caret (ˆ) denotes exogenous variables. These are the price expectations held during period t, and will influence the design of production units put into operation in period $t+1$. If the exogenous variables do not affect the expected prices, expectations are said to be static; an assumption about rational expectations can be modelled by a suitable choice of exogenous variables. Producers are likely to invest only if the expected unit cost does not exceed the expected unit price of the output. This rule is incorporated in the model in two stages. In the first step, a set of sectoral interest rates, $r_j(t)$, which satisfy the investment rule in each sector is determined by means of the *ex ante* unit cost function and the expected prices. Thus, the $r_j(t)$'s are determined by the following equations:

$$\tilde{P}_j(t) = \kappa_j^*(\tilde{P}_0^D(t), \tilde{P}_1^D(t), \tilde{W}_j(t), \tilde{R}_j(t); t) + \sum_{i=2}^{n} \tilde{P}_i^D(t) a_{ij} + \tilde{Q}_j b_j; \quad j = 0, 1 \ldots n+2 \tag{25}$$

where

$$\tilde{R}_j(t) = P_{n+3}(t)(\delta_j + r_j(t)); \quad j = 0, 1 \ldots n+2. \tag{26}$$

By Shephard's lemma, the *ex ante* coefficients for labor, capital, fuels, and electricity are given by the partial derivatives of the *ex ante* cost functions evaluated at the expected prices. Thus, the energy input coefficients in production units designed in period t and put into operation in period $t+1$ are determined by

$$\frac{\partial \kappa_j^*(\cdot)}{\partial \tilde{P}_i^D} = a_{t+1,ij}; \quad i = 0, 1; \; j = 0, 1 \ldots n+2 \tag{27}$$

The desired capital–output ratios are determined in a similar way, i.e., by the partial derivative of κ_j^* with respect to R_j, evaluated at the expected prices.

The second stage is to allocate total investment over those sectors for which $r_j(t)$ is not lower than the market rate of interest, $r(t)$. This is done by means of the equation

$$I_j(t) = \begin{cases} \dfrac{\partial \kappa_j^*(\cdot)}{\partial \tilde{R}_j} \delta_j X_j(t) \left(\dfrac{r_j(t)}{r(t)}\right)^{\psi_j} & \text{if } r_j(t) \geq r(t) \\ 0 & \text{if } r_j(t) < r(t) \end{cases} \qquad j = 0, 1 \ldots n+1 \tag{28}$$

where $I_j(t)$ is the total investment in sector j in period t. Public investments, $I_{n+2}(t)$, however, are determined exogenously. Observe that when $r_j(t) = r(t)$, the existing capacity is maintained by replacement of depreciated capacity by new production units, and if $r_j(t) > r(t)$, the capacity in sector j is increased. The market rate of interest is determined in such a way that the market for investable funds is cleared. Thus, the

Table 2. The production sectors.

Number[a]	Sector
0	Fuel production
1	Electricity
2	Import-competing industries (food, textiles, etc.)
3	Export-oriented energy-intensive industries (paper and pulp, steel, etc.)
4	Other export-oriented industries (mainly manufacturing)
5	Sheltered industries, trade and private services
6	Public sector
7	Capital goods sector

[a]The number of sectors adopted means that $n = 4$.

following expression holds:

$$I(t) = \sum_{j=0}^{n+2} I_j(t). \tag{29}$$

With this, the description of the models is complete.

5. IMPLEMENTATION

In order to implement these models, it is necessary to specify the functions $\kappa_j^*(\cdot)$, $\phi_i(\cdot)$, $C_i(\cdot)$ and $Z_i(\cdot)$. It is beyond the scope of this article to go into details of the choices actually made. Only a brief account can be given here, and the interested reader is referred to Bergman and Por [2].

The *ex ante* unit cost functions $\kappa_j^*(\cdot)$ are derived from a nested Cobb–Douglas-CES production function. Thus there is a constant elasticity of substitution between a composite capital–labor input and a composite fuels–electricity input, defined by a CES function. The unit cost functions for composite goods, ϕ_i, are derived from a CES function defining composite goods. Finally, the household demand equations are derived from a linear expenditure system estimation on ten consumer commodity groups and a matrix defining the consumer commodity groups in terms of the composite goods and the domestically produced good $n + 1$. Given that the functional forms are specified, it remains to estimate the numerical values of all the parameters in the model. For a large number of these, primarily the input–output coefficients a_{ij} and b_j, the estimation has to be based on one or a small number of observations. However, time series are in general available for the estimation of the remaining parameters. The simulations discussed in the following section were based on Swedish data. The basic data source was an input–output table for 1975, aggregated to 8 sectors. The sectors thus defined are presented in Table 2. In addition, estimations of import price-elasticities and household demand equations were available. The assumption about the *ex ante* elasticities of substitution between fuels and electricity and between composite capital–labor and composite fuels–electricity could not be directly based on econometric evidence. Lacking better information, all of these parameters were assumed to have the values 0.75, an assumption which seems reasonable in view of the econometric results presented in Pindyck [14].

6. THE IMPACT OF HIGHER ELECTRICITY COSTS: SOME SIMULATION RESULTS

This section presents some results obtained from simulations using the models. The purpose of the discussion is primarily to illustrate possible applications of the models,

but the simulations are also intended to clarify some issues of more general interest. The analysis concerns the impact of a 50% increase in electricity production costs, caused by a change in the values of the parameter b_1.

First, the immediate impact of this change is investigated under various assumptions about the functioning of the economy, i.e., by means of various models of the system. Second, the impact of higher electricity costs on the economy 10 years after the initial shock is investigated. To summarize, the simulations are focused on two issues. First, to what extent do technological and wage formation rigidities magnify the short-run impact of a sudden increase in electricity costs? Second, to what extent can short-run adjustment problems be disregarded in an analysis where the time horizon is extended over a decade?

The initial situation is one of full equilibrium, determined by means of the static model on the basis of the 1975 Swedish input–output data. Then, still in the initial year, there is a sudden increase in real electricity costs of 50%. This could happen if the existing nuclear power stations were taken out of operation, and replaced by oil-fired power plants previously kept as reserve capacity in the power system. The immediate impact on the economy of this change in electricity costs is estimated under three different sets of assumptions.

Case A. Sectoral capital stocks can be reallocated; capital, labor, fuels, and electricity are substitutable in accordance with the *ex ante* production functions; product and factor prices are flexible enough to clear all product and factor markets.

Case B. Sectoral capital stocks and energy input coefficients are fixed in the short run, but the use of labor can be adjusted in accordance with the *ex post* production functions; prices and wages are flexible enough to clear all product markets and the labor market.

Case C. The same as B, except that the real wage rates are kept at the levels established prior to the electricity cost increase.

It should be obvious that Case A is analyzed by means of LRS, Case B by SRS with market-clearing wages, and Case C by SRS with exogenously given wages. The main results are summarized in Table 3.

The results suggest that rigidities due to immobile capital and fixed energy input coefficients do not magnify the impact of a 50% electricity cost increase significantly, as long as wages are flexible enough to clear the labor market. However, when the real wage rates are completely rigid, such a change in electricity supply conditions leads to unemployment and quite a substantial reduction in the GNP. The main reason for this is that higher electricity prices can only be transferred to export prices to a very small degree. Thus, in Case C, the increase in electricity costs has to be counterbalanced by an increase in labor productivity. However, given the assumptions about the *ex post*

Table 3. The calculated short-run impact of a 50% electricity cost increase (indices).

	Initial	A	B	C
GNP	100	98.5	98.4	94.8
Electricity consumption	100	71.9	100.0	96.9
Electricity consumption/GNP	100	73.0	101.6	102.2
Wage level	100	97.9	95.4	100.0
Employment level	100	100.0	100.0	94.9

technology, productivity increases in a given sector can only be achieved through reducing the number employed, i.e., by closing down the least efficient production units. On the other hand, in Case A and Case B the electricity cost increase is counterbalanced by a reduction in wages and profits sufficiently large to maintain full employment. Of course, there are also other alternatives. For instance, a deterioration in the current account balance can be temporarily accepted, marginal production units can be subsidized, etc., but these alternatives are not considered here.

In the next step, the impact of the electricity cost increase after ten years of adjustment is analyzed. The overall rate of capital formation is the same in all cases, and some technical progress is assumed to take place. However, in order to ensure compatability between the models, the technical progress is assumed to be entirely labor augmenting and apply to the whole labor force. Consequently, there is no distinction between vintages of capital in the dynamic model, only between vintages of production units, which differ in terms of the energy input coefficients. The rate of labor-augmenting technical progress is assumed to be 2.5% per annum, which leads to a "reference case" compatible with a recent long-term forecast made by the Ministry of Economic Affairs in Sweden.

In the "long-run" analysis, Case A is analyzed using LRS, while Cases B and C are analyzed by means of DYN. Case C is divided into two cases, in both of which the real wage rigidity is assumed to persist for only two years followed by full flexibility of real wages thereafter. The difference between the cases is that in C′ the production units which could not cover their operating costs at the initial real wage rate and the higher electricity price are somehow kept in business until the real wages are adjusted downwards. Thus, in this case, real wage rate reductions induce higher employment and production. In Case C″, however, the unprofitable production units are assumed to close down immediately after the electricity cost increases. In the model analysis this is specified as a "floor" for real wages. It is assumed that the real wage rate can never fall below the initial level, which means that workers can only be employed in production units efficient enough to be able to cover operating costs at the new electricity prices and real wages at least as high as the initial real wages.

The main results are displayed in Table 4. In terms of GNP figures, A, B, and C′ clearly differ from C″. Thus, if an initial rigidity in the wage formation process leads to excessive reductions in capacity, an unexpected electricity cost increase can have a lasting impact on the economy. If, on the other hand, such short-run rigidities only result in temporary excess capacity, their effects lose much of their significance if the time horizon of the analysis is extended to a decade.

These findings are potentially quite important from the point of view of building economic models. They suggest that, provided there are no "shocks" displacing the economy significantly from equilibrium and that the time horizon of the analysis is at least ten years, the payoff for the resources put into the development of a complicated model, such as DYN, might be rather limited. A relatively simple model, such as LRS, might be sufficiently good for analyses of resource allocation in such cases. To analyze

Table 4. The calculated impact of a 50% electricity cost increase 10 years after it has taken place (indices).

	Reference case	A	B	C′	C″
GNP	100	98.5	98.5	98.5	96.8
Electricity consumption	100	71.5	88.6	89.4	90.2
Electricity consumption/GNP	100	72.6	89.9	90.8	93.2

the immediate or short-run impact of changes in exogenous conditions, however, models which take into account rigidities affecting the short-run functioning of the economy seem to be necessary.

At first sight, it appears that slightly different conclusions should be drawn from the results for electricity consumption: here, Case A differs significantly from the other cases. However, additional simulations with the LRS model showed that an appropriate choice of elasticities of substitution in the *ex ante* production functions brought the calculated electricity consumption in Case A close to the corresponding results in the other cases, without noticeably affecting the other results. It therefore appears that the technological rigidities limiting energy substitutability can be modelled by a suitable choice of elasticity-of-substitution parameters in the *ex ante* production functions.

7. CONCLUDING REMARKS

It should be noted that the specification of the export functions and the current account constraint implies that the home country has some autonomy in the pricing of its exports. Whether or not a significant deviation between domestic production costs and world market prices, i.e., between P_i and P_i^W, tends to emerge, depends on the parameters of the export functions and the type of simulations that are carried out. As such deviations are not consistent with the notion of a "small open economy," some combinations of parameter values would necessitate a respecification of the foreign trade part of the model. If this were not done, the model would indicate terms-of-trade gains of the optimum tariff type from, for instance, domestic energy cost increases. Such effects did not, however, appear in the analysis presented here.

However, if a respecification of the model is regarded as necessary, it can be carried out along the following lines. The total output in sector j is assumed to consist of an aggregate of a large number of goods. The price P_j is taken to be the price index of this aggregate. Some of the goods in the aggregate are exported at given world market prices. The price index P_j^W is taken to be the price index of the aggregate of goods exported from sector j. Using the same symbols as before, the value of total output from sector j can now be written

$$P_j X_j = P_j^W Z_j + P_j^N \{X_j - Z_j\},$$

where P_j^N is the price index of nonexported goods produced in sector j. The current account constraint should be written

$$\sum_{i=1}^{n} P_i^W Z_i = \sum_{i=0}^{n} P_i^M M_i + \sum_{j=0}^{1} Q_j \bar{M}_j + D,$$

i.e., exports are valued at world market prices rather than domestic producer prices. Moreover, once the price P_i^N has been defined, the price of composite goods used within the country becomes

$$P_i^D = \phi_i(P_i^N, P_i^M).$$

Thus, the model can be made perfectly consistent with the usual "small economy" assumptions, i.e., optimum tariff effects can be completely avoided. Moreover, the two specifications can be combined, i.e., the standard specification can be adopted for some sectors and the alternative specification for others.

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APPENDIX

Definition of symbols used in Table 1

A. Endogenous variables:

X_j gross output in sector $j = 0, 1 \ldots n+3$;
X_{ij} use of composite good $i = 0, 1 \ldots n$ in sector $j = 0, 1 \ldots n+3$;
L_j use of labor in sector $j = 0, 1 \ldots n+2$;
I^G total gross investments;
C_i household consumption of composite good $i = 0, 1 \ldots n$ or the domestically produced good $i = n+1$;
E total household consumption expenditures;
Z_i export of domestically produced good $i = 1, 2 \ldots n$;
M_i competing imports of good $i = 0, 1 \ldots n$;
$\bar{M}_j$ complementary imports to sector $j = 0, 1$;
P_i price of domestically produced good $i = 0, 1 \ldots n+3$;
P^*_{vj} value added per unit of output in vintage $v = 0, 1 \ldots t$ in sector $j = 0, 1 \ldots n+2$;
P^D_i price of composite good $i = 0, 1 \ldots n$;
W_j wage rate in sector $j = 0, 1 \ldots n+2$;
W index of the level of wages in the economy as a whole;
R_j user cost of capital in sector $j = 0, 1 \ldots n+2$;
R real rate of return on capital;
Y gross national income.

B. Exogenous variables

L supply of labor;
K supply of capital;
I net investments in the economy as a whole;
C_{n+2} public consumption;

P_i^M, P_i^W world market price in the domestic currency unit, c.i.f. and f.o.b., respectively, of good $i = 0, 1 \ldots n$;
Q_j world market price, in the domestic currency unit, of complementary imports to sector $j = 0, 1$;
D sum of net foreign transfers and net interest payments on foreign debt.

C. Parameters

$a_{ij}(a_{vij})$ input of composite good $i = 0, 1 \ldots n$ per unit of output (production units of vintage $v = 0, 1 \ldots t$) in sector $j = 0, 1 \ldots n + 3$;
b_j input of complementary imports per unit of output in sector $j = 0, 1 \ldots n + 3$;
ω_j wage rate in sector $j = 0, 1 \ldots n + 2$ deflated by the index of the general wage level;
δ annual rate of depreciation of capital in sector $j = 0, 1 \ldots n + 2$;
s the gross savings ratio in the economy as a whole.

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COMPUTABLE GENERAL EQUILIBRIUM MODELS: AN OPTIMAL PLANNING PERSPECTIVE

ERNÖ ZALAI

International Institute for Applied Systems Analysis
Laxenburg, Austria

and

Institute for National Economic Planning
Karl Marx University of Economics
Budapest, Hungary

Abstract—Linear multisectoral models have been applied to development planning in countries with different socioeconomic systems. The relative simplicity of the underlying technique has concealed much of the conceptual differences between modelling in East and West. However, the recent development of more sophisticated models, under the general title of computable general equilibrium models, has apparently enhanced these differences, and concealed the possibility that these models could also be used in both East and West. This paper investigates issues related to the possible adaptation of equilibrium modelling techniques for central planning purposes.

INTRODUCTION

Linear multisectoral input–output and programming models have become more or less integrated into the complex process of planning in many socialist (centrally planned) economies. These models concentrate on the production and use of economic resources and commodities at some level of aggregation. Similar models are also used in both western and developing countries, the differences in the economic environment and data sources being reflected in the specification and purpose of the models. The use of linear models has been paralleled by the development of more complex, nonlinear models, most of which come under the general heading of computable or applied general equilibrium models.

The basic ideas of a multisectoral general equilibrium growth model were first suggested by Johansen [2] in 1959, although full-scale implementation of large, nonlinear models has become computationally feasible only lately. Recent applications are described in Refs. 3–7; models of this type developed at the International Institute for Applied Systems Analysis (IIASA) are discussed in Refs. 8–11 (retrospective analysis) and Refs. 12 and 13 (projections for small open economies). Some of these models have been designed to capture the interrelationships between economic, spatial, and demographic processes (for a critical review of these models see Ref. 14).

The structure of general equilibrium models, the estimation procedures applied, and the theoretical explanations associated with them generally follow the neoclassical tradition quite closely. The neoclassical approach has often been criticized and even rejected in both East and West (see, for example, Ref. 15), and this partly explains the apparent lack of interest of central planning modellers in these models. It is not at all

This paper is based on an IIASA Working Paper [1] which contains a more detailed discussion of most of the issues only touched upon here. The Working Paper also describes a complex equilibrium model currently being implemented in Hungary.

obvious whether the models, or some of the techniques of applied general equilibrium modelling, could be adapted for central planning processes.

The main purpose of this paper is to highlight the possibility and expected benefits of incorporating nonlinear multisectoral models of the general equilibrium type into the planning methodology of socialist (centrally planned) economies. This is done by means of a comparative modelling exercise, which is essentially a reformulation of the old idea that there is a fundamental equivalence between equilibrium solutions obtained through a competitive mechanism and the optimal solutions of a centrally planned resource allocation problem. This idea has been formulated in many ways (e.g., in terms of welfare economics or as a simple linear programming model). Here it will be put into a specific economic context and used to gain more insight into the central problem studied in this paper: why and how the analytical techniques used in multisectoral general equilibrium models could be fitted into the current planning modelling methodology of centrally planned economies.

Although this paper is mainly addressed to planning modellers in socialist countries who are not very familiar with general equilibrium modelling, it is hoped that some of the conclusions of this comparative exercise will also be of some value to economic modellers elsewhere.

Two simple models of the same resource allocation problem will be developed as a basis for discussion and comparison: first, a stylized competitive equilibrium model (Sec. 1); second, an equally stylized programming model of the type used in central planning (Sec. 2). Some crucial conceptual differences between the models are discussed briefly in Sec. 3, where it will also be shown that the planning model can be reformulated as a general equilibrium model without losing any of its fundamental characteristics. Finally, the possible advantages of such a transformation are considered in Sec. 4.

1. A GENERAL EQUILIBRIUM MODEL OF THE RESOURCE ALLOCATION PROBLEM

General competitive equilibrium theory[a] provides an abstract partial model of the economic systems centered around the law of supply and demand, and rational economic behavior. The abstract economic theory of general equilibrium takes many important elements of the economy as *data* and sets out to define and determine the equilibrium within this postulated environment in which only prices control economic decisions. This theory has more empirical relevance when explaining relative changes than when dealing with absolute magnitudes.

Applied general equilibrium models adopt a relative point of view and try to estimate the likely consequences of various changes in the economic environment by comparing the "base equilibrium solution" with the solutions computed on the basis of these changes. A typical approach may be summarized as follows. A formal model of the necessary and sufficient conditions for general equilibrium is developed. The observed state of the economy is considered to be in equilibrium (base solution), and many of the parameters of the model are estimated on the basis of this assumption. Next, by classifying the economic variables as endogenous or exogenous, the impact of assumed changes in the exogenous variables is analyzed in terms of the model solution. Thus, the equilibrium framework is used to evaluate, consistently and in quantitative terms, the direction of change of certain crucial interdependent economic variables.

The underlying logic of multisectoral general equilibrium models and their relation to some structurally similar optimal planning models may be understood more readily if the

[a] We will confine our attention to competitive or Walrasian general equilibrium models in this paper.

resource allocation problem is stripped to its bare essentials. The model presented below is a simplified, static model of competitive equilibrium, which perhaps resembles most closely the models of the Scandinavian school.

First we define the variables and parameters that are considered in the model.

Variables:

X_j gross output in sector $j = 1, 2 \ldots n$;[b]
X_{n+1} total gross investment;
k_j, K_j capital input coefficient and total capital used in sector $j = 1, 2 \ldots n$;
n_j, N_j labor input coefficient and total employment in sector $j = 1, 2 \ldots n$;
C_i consumption of commodity $i = 1, 2 \ldots n$;
P_i price of commodity $i = 1, 2 \ldots n$;
P_{n+1} price of the composite capital good;
P_i^* "net price" (value added per unit) of commodity $i = 1, 2 \ldots n$;
W general index of the level of wages;
W_j level of wages in sector $j = 1, 2 \ldots n$;
R general index of net return on capital;
R_j rate of return on capital in sector $j = 1, 2 \ldots n$;
Q_j user cost of capital in sector $j = 1, 2 \ldots n$;
E consumption expenditures;
Π_j net income (profit) in sector $j = 1, 2 \ldots n$.

Parameters (*exogenous variables*):

N total labor force;
K total capital stock;
I total net investment;
a_{ij} input of commodity $i = 1, 2 \ldots n$ per unit of output in sector $j = 1, 2 \ldots n$;
$a_{i,n+1}$ input of commodity $i = 1, 2 \ldots n$ per unit of gross investment;
δ_j annual rate of depreciation in sector $j = 1, 2 \ldots n$;
ω_j index of the relative wage rate in sector $j = 1, 2 \ldots n$;
β_j index of the relative rate of return on capital in sector $j = 1, 2 \ldots n$;
b_i, c_i parameters in the consumer's demand function for commodity $i = 1, 2 \ldots n$.

We can now summarize the basic features of a general equilibrium model.

There are n produced commodities in the model available for both intermediate and final use, one composite new capital (investment) good, and two primary commodities (capital and labor).

The production technology for the sectoral commodities is given by a combined Leontief-neoclassical formulation. The use of intermediate inputs is assumed to be proportional to the output level of the produced commodity, i.e.,

$$a_{ij}X_j \qquad i = 1, 2 \ldots n; j = 1, 2 \ldots n$$

whereas in the case of primary inputs we allow for the possibility of substitution. The Leontief character of the production technology is maintained, however, by assuming this substitutability to be independent of the level of output. Namely, once the capital/labor ratio is determined, the required amount of the two factors is proportional to the level of output. Thus, the relation between the output and the primary inputs can be

[b] In the model, each sector produces only one kind of commodity and each commodity is produced by only one sector. Thus there is a one-to-one correspondence between the sectors and the commodities produced.

described by a linear homogeneous (production) function[c]:

$$X_j = F_j(N_j, K_j) \qquad j = 1, 2 \ldots n.$$

The production of the composite investment good is assumed to require only intermediate commodities in amounts proportional to the level of gross investment (capital formation):

$$a_{i,n+1}X_{n+1} \qquad i = 1, 2 \ldots n.$$

The technology defined above exhibits constant returns to scale; therefore, in competitive equilibrium, the nonprofit condition must hold for each producing sector, as will be shown below.

In the spirit of the neoclassical tradition, producers are assumed to maximize their net income, the difference between the value of the commodities bought (rented) and sold. In the case of the investment good sector this simply means that, at equilibrium, prices have to satisfy the following condition:

$$P_{n+1} = \sum_{i=1}^{n} P_i a_{i,n+1}. \tag{1}$$

We will define the net income function (Π_j) for the first n sectors and derive analytically the necessary conditions for a maximum amount of net income.

The cost of labor in sector j is simply $W_j = \omega_j W$. The user's cost of capital is defined as the sum of depreciation and net return requirement (rent). At the same time, capital stock is assumed to be reevaluated at current prices, given by P_{n+1}. Therefore, the cost of using capital (evaluated at some *base* price) in sector j is given by

$$Q_j = (\delta_j + R_j)P_{n+1} = (\delta_j + \beta_j R)P_{n+1}. \tag{2}$$

For future reference it is worth noting here that the introduction of different rate-of-return requirements on capital in different sectors can be interpreted, for instance, as a reflection of lasting market imperfections. It will be shown that this solution has effects similar to individual bounds on sectoral capital inputs, which, in turn, can be interpreted as limited intersectoral mobility of capital.

The net income earned by producing X_j is thus defined by the following expression:

$$\Pi_j = P_j X_j - \sum_{i=1}^{n} P_i a_{ij} X_j - W_j N_j - Q_j K_j \tag{2a}$$

which is to be maximized subject to the constraint

$$X_j = F_j(N_j, K_j).$$

Substituting $F_j(N_j, K_j)$ for X_j in Eq. (2a) and differentiating the net income function with respect to N_j and K_j yields the following necessary first-order conditions for an optimal

[c] The term "production function" is not quite appropriate here since we use it only to define a composite primary factor. The function is assumed to be "well behaved" (i.e., concave, monotone increasing, and differentiable).

solution:

$$P_j^* \frac{\partial F_j}{\partial N_j} = W_j = \omega_j W \tag{3}$$

$$P_j^* \frac{\partial F_j}{\partial K_j} = Q_j, \tag{4}$$

where P_j^* is the value added per unit of output j:[d]

$$P_j^* = P_j - \sum_{i=1}^{n} P_i a_{ij}. \tag{5}$$

It can easily be seen that if we multiply Eqs. (3) and (4) by N_j and K_j, respectively, and add them together, then, because of the assumed linear homogeneity of the production functions, we will have

$$P_j^* = W_j \frac{N_j}{X_j} + Q_j \frac{K_j}{X_j} = W_j n_j + Q_j k_j. \tag{5a}$$

This shows that the net income will be zero at equilibrium (the nonprofit or, more accurately, "no extra profit" condition).

It is worth mentioning here that if we insert Eq. (5a) into Eq. (5), after rearrangement we obtain

$$P_j = \sum_{i=1}^{n} P_i a_{ij} + W_j n_j + Q_j k_j. \tag{5b}$$

The resulting price formation rule is very similar to that used to determine the so-called "two-channel price system" employed in socialist price planning theory and practice. Here, however, n_j and k_j as well as W_j and Q_j are endogenous to the equilibrium model.

The next behavioral assumption concerns the consumers. The consumers' demand for goods and services is represented by a set of demand functions:

$$C_i = C_i(P_1, P_2 \ldots P_n, E) \qquad i = 1, 2 \ldots n,$$

where E is the total consumption expenditure, an endogenous variable.[e] As in most applied models, we will adopt a Linear Expenditure System (LES):

$$C_i = b_i + \frac{c_i}{P_i}\left(E - \sum_{j=1}^{n} P_j b_j\right) \qquad i = 1, 2 \ldots n, \tag{6}$$

where b_i is often interpreted as the minimum (subsistence) consumption of commodity i, which must be dealt with first. The remaining income is then allocated between the various commodities according to their relative prices and to the marginal propensities to

[d] Notice that if instead of substituting F_j for X_j in the net income function we utilized a Lagrange multiplier, then P_j^* could be taken as the value of this multiplier.

[e] The analysis of the programming model will shed some light on the endogenous determination of consumption expenditure in the equilibrium model.

consume different commodities

$$c_i > 0 \text{ and } \sum_{i=1}^{n} c_i = 1.$$

In this simplified model, the state of the economy can be described fully by the values of the endogenous variables: P_j, P_{n+1}, Q_j, R, W, P_j^*, E, X_i, C_i, K_j, N_j. Of these, variables X_i, C_i, K_j, and N_j (the *real* variables) describe the production and use of different commodities. Whether the economy considered is centrally planned or market based (or a mixture of the two), the above variables must fulfill certain "physical" conditions. These conditions include commodity and resource balances (market clearing constraints) and technological restrictions. The balance and capacity constraints are generally given in the form of inequalities. However, it is well known that if the equilibrium price of a commodity is positive, then the corresponding balance inequality must be fulfilled as an equality. The special assumptions of our model guarantee that the equilibrium price of each commodity and resource will be positive, and therefore we may use equalities straight away:

$$\sum_{j=1}^{n+1} a_{ij}X_j + C_i = X_i \qquad i = 1, 2 \ldots n \tag{7}$$

$$\sum_{j=1}^{n} \delta_j K_j + I = X_{n+1} \tag{8}$$

$$\sum_{j=1}^{n} K_j = K \tag{9}$$

$$\sum_{j=1}^{n} N_j = N \tag{10}$$

$$F_j(N_j, K_j) = X_j \qquad j = 1, 2 \ldots n. \tag{11}$$

Equations (7)–(11), together with behavioral and pricing equations (1)–(6), define a simultaneous system of equations that must be fulfilled by equilibrium solutions. It can easily be checked that all of these equations are homogeneous in all prices (both gross and net), wage rate (W), and total consumption expenditure (E). Therefore, the general level of these variables is indeterminate, and can be chosen arbitrarily. This can also be checked by comparing the numbers of equations and variables ($7n + 4$ equations, $7n + 5$ variables).

2. AN OPTIMAL PLANNING MODEL VERSION OF THE PROBLEM

Now we shall describe a planning model that could be used to determine the optimal allocation of resources in the same model economy. An economy-wide planning model, built into and upon the traditional planning methodology of a socialist country, would differ from the above general equilibrium model in several respects. First, it would almost exclusively contain "real" variables and relations reflecting physical constraints on allocation. Second, because the prices used in a planning model are either constant or planned, being predicted more or less independently from "real" processes, the interdependence of the real and value (prices, taxes, rate-of-return requirements, etc.)

variables would not be considered explicitly in the model. Third, most mathematical planning models are closely related to and rely upon traditional or nonmathematical planning. This means, among other things, that the values of the exogenous variables and parameters and also certain upper and/or lower target values for some of the endogenous variables would not be derived directly from statistical observations, but would be based on figures given by traditional planners.[f] (This is not to say, however, that more or less sophisticated statistical estimation techniques would not be combined with experts' "guesstimates" in traditional planning.) And, finally, planning modellers in socialist countries tend to concentrate more on the problems of how to fit their models into the actual process of planning and make them practically applicable and useful. Therefore, applied planning models tend to be both theoretically and methodologically simpler than those in the development planning literature. The above list is, of course, far from complete, but nevertheless includes most of the major characteristics of socialist planning models.

We now proceed to describe a simplified model, which demonstrates how the resource allocation problem could be modelled in a centrally planned economy.

Suppose that at some stage in the planning process the coordinating unit decides to summarize the calculations made so far, and as a result some provisional values of the sectoral outputs, inputs, consumption, etc., are therefore made available. The coordinating unit wishes to know whether these more or less separately planned figures represent a consistent and balanced picture, and if not, how this could be rectified. The unit also wishes to check whether the efficiency of the provisional plan could be increased by reallocating resources among the various sectors. To get some rough answers to these questions, it is necessary to have some idea of how certain changes in one part of the plan would affect other parts of the plan, and what changes (plan variants) seem to be feasible. This information could then be put into a formal model to support the process of checking the consistency and efficiency of a draft plan (coordination process).

We shall consider a rather simple model in which the plan variables are the output levels of various sectoral commodities (X_j; $j = 1, 2 \ldots n + 1$), their levels of consumption (C_i; $i = 1, 2 \ldots n$), and the amounts of labor and capital (fixed assets) allocated for their production (N_j, K_j; $j = 1, 2 \ldots n$).[g] All feasible resource allocation programs must satisfy commodity (resource) balance requirements and technological constraints analogous to equalities (7)–(11), although in this case the constraints will be inequalities.

Beyond that, as mentioned earlier, the planning model would reflect certain requirements derived from traditional planning calculations. We will consider here only a few representative examples. For example, consumption of different commodities may be limited from below by their planned target levels. The model builders may also take into consideration certain limitations concerning the possible intersectoral allocation of given primary resources. In the case of capital, for example, the existing sectoral capacities may be taken as lower limits, while calculations concerning the capital absorptive capacities of the various sectors may indicate upper limits to the amount of capital that should be allocated to each sector. Lower and upper limits to the number of workers employed in different sectors can be established in a similar way.

Suppose now that all options satisfying the above conditions are regarded as feasible plan variants (at least at the stage and level of planning considered). The efficiency of the

[f] This is especially true for the national programming models used in Hungary, where one of the basic aims of the modellers is to check the feasibility and improve upon the efficiency of the plans developed by traditional planners [16].

[g] For simplicity we use the same symbols as before. Notice, however, that there may be important differences in their interpretation in the planning context.

different variants can be measured in a number of ways, and in practice various types of objective function are used to generate a whole range of efficient plan variants. For the sake of simplicity, however, we assume here that the efficiency of the plan variants is measured by a geometric weighted average of the surplus (incremental) consumption of the different commodities. Denoting the surplus consumption of commodity i by C_i^+ and its weight by s_i, where

$$\sum_{i=1}^{n} s_i = 1,$$

we can then formulate the following objective function:

$$g(C^+) = (C_1^+)^{s_1}(C_2^+)^{s_2} \dots (C_n^+)^{s_n}.$$

Using the above specifications, the optimal plan is then determined as the solution of a nonlinear programming problem in which we maximize function $g(C^+)$ subject to the following constraints:

$$(P_i) \qquad \sum_{j=1}^{n+1} a_{ij}X_j + C_i^- + C_i^+ \leqq X_i \qquad i = 1, 2 \dots n \tag{7'}$$

$$(P_{n+1}) \qquad \sum_{j=1}^{n} \delta_j K_j + I \leqq X_{n+1} \tag{8'}$$

$$(S) \qquad \sum_{j=1}^{n} K_j \leqq K \tag{9'}$$

$$(W) \qquad \sum_{j=1}^{n} N_j \leqq N \tag{10'}$$

$$(P_j^*) \qquad X_j - F_j(N_j, K_j) \leqq 0 \qquad j = 1, 2 \dots n \tag{11'}$$

$$(S_j^-, S_j^+) \qquad K_j^- \leqq K_j \leqq K_j^+ \qquad j = 1, 2 \dots n$$

$$(W_j^-, W_j^+) \qquad N_j^- \leqq N_j \leqq N_j^+ \qquad j = 1, 2 \dots n$$

$$X_j, C_j^+, K_j, N_j \geqq 0,$$

where the meaning of the variables and parameters is essentially the same as before. In addition, C_i^- is the lower bound on consumption of commodity i, while K_j^-, K_j^+ and N_j^-, N_j^+ represent, respectively, the lower (−) and upper (+) bounds of K_j and N_j. The symbols in parentheses denote the dual variables[h] associated with each constraint.

Apart from structural and conceptual simplifications, this model differs from traditional programming planning models in that it is not completely linear (the production functions and the objective function are both nonlinear). We will discuss the nonlinearity of the objective function later.

Before going any further, however, we should comment on the use of smooth ("neoclassical") macroproduction functions in our planning model. The use of functions of this type in centrally planned economies is often criticized on both theoretical and

[h] Some dual variables (shadow prices) are denoted by the same symbols as the equilibrium prices, for reasons that become obvious in the next section. See also the previous footnote.

empirical grounds. In addition to the known statistical estimation problems, the estimated parameters of macroproduction functions are often biased by neoclassical assumptions concerning income distribution. We believe, however, that it is possible to design estimation procedures that fit the needs and conditions of centrally planned economies.

To illustrate the point above consider first of all that the production functions in our model serve only one purpose: to allow for alternative sectoral production technologies that may be more or less labor/capital-intensive than current techniques. Linear planning (programming) models usually allow for alternative technologies but in a different way. The range of technological choice in aggregated macromodels is rather limited, and for a realistic description of sectoral production possibilities in a linear model it is necessary to give up the "macro" character of the model, to break down each sector into subsectors, and to introduce individual bounds to limit their levels. This not only increases the size of the model significantly, thus making it less flexible and transparent, but also distorts the dual solution. It therefore seems reasonable to use smooth macroproduction functions to avoid such a situation.

We suggest that the parameters of macroproduction functions could be estimated using the following or a similar procedure.[i] First, a set of subsectoral activities is defined for a given sector. These may be the activities taken into consideration in a large disaggregate model. This information could then be used to generate a number of alternative *intra*sectoral production structures with different output and input levels. Finally, by an appropriate production function form, this information could be condensed into a few (three or less) parameters and this function used in the aggregated macromodel. This procedure, thus, can be viewed as a special way of decomposing the large-scale problem. Production functions defined on the basis of detailed subsectoral activities could be used in both macroprogramming and general equilibrium models. The technological information may be *ex post* or *ex ante* (planned) or some combination of the two.

3. COMPARISON OF THE TWO APPROACHES

We have presented two specific models: a general equilibrium model, intended for *counter-factual* simulations in a market economy, and a mathematical programming model, designed for *counter-plan-proposal* simulations in a centrally planned economy. The first was based on neoclassical assumptions of competitive pricing and individual optimization behavior. Not only the structure of this model, but also the estimation of its parameters (e.g., those in the production and demand functions) should be consistent with the theoretical assumptions of a competitive market mechanism. The planning model contained *no* behavioral assumptions or pricing rules; its purpose was to check the overall consistency and efficiency of proposed resource allocations. The parameters of the model reflected planned technological conditions and targets (*ex ante* statistics). The general equilibrium model had the form of a *system of equations* (preferably with a unique solution), whereas the planning model had the form of a *constrained optimum* (alternative solutions preferred). We will now show that, in spite of these and other basic differences which appear in more complex models, the two models have a strong formal similarity.

[i] This method bears an obvious resemblance to Johansen's treatment of sectoral production functions [17] and also to the way in which Rimmler, Daniel, and Kornai estimated macrofunctions on the basis of programming models [18].

The close similarity of the two models discussed in this paper stems from two fundamental theorems. The *formal* (mathematical) basis is provided by the Kuhn–Tucker theorem on constrained optima. The *conceptual* (economic) foundation is given by the central proposition of welfare economics which relates Pareto-optimality (efficiency) to competitive equilibrium.

These results are quite well-known—however, we would like to interpret them in a somewhat unusual way. Instead of regarding programming as a special type of competitive analysis, we want to argue that applied general equilibrium modelling is nothing but an alternative method for macroeconomic programming. Even if one faithfully believes in general competitive equilibrium theory the fact that only aggregated producers and consumers can be considered in a computable model implies some theoretical inconsistencies. The only possibility of interpreting such models therefore seems to lie with their programming counterparts.

We shall now demonstrate the formal identity of the two kinds of model in the special case discussed above. In order to do this we will show how the programming model can be transformed into a system of equations formally identical to the general equilibrium model.

For the purposes of our investigation, it is reasonable to assume that the programming problem has a solution and that the conditions of the Kuhn–Tucker theorem on constrained optima are met by our model. Making the normal assumptions, the optimal values of the primal (real) variables in our model will be positive and the constraints (except for some individual bounds) will be fulfilled as equalities. According to the Kuhn–Tucker theorem, differentiating the associated Lagrange function with respect to the primal variables will yield a set of constraints that must be fulfilled by the *dual* variables (shadow prices) associated with the various constraints of the primal problem. In our case the Lagrange function takes the following form:

$$
\begin{aligned}
L = {} & g(C^+) - P_{n+1}\left(\sum_{j=1}^{n} \delta_j K_j + I - X_{n+1}\right) \\
& - \sum_{i=1}^{n} P_i\left(\sum_{j=1}^{n+1} a_{ij} X_j + C_i^- + C_i^+ - X_i\right) \\
& - S\left(\sum_{j=1}^{n} K_j - K\right) - W\left(\sum_{j=1}^{n} N_j - N\right) \\
& - \sum_{j=1}^{n} S_j^+(K_j - K_j^+) - \sum_{j=1}^{n} S_j^-(K_j^- - K_j) \\
& - \sum_{j=1}^{n} W_j^+(N_j - N_j^+) - \sum_{j=1}^{n} W_j^-(N_j^- - N_j) \\
& - \sum_{j=1}^{n} P_j^*(X_j - F_j(N_j, K_j)).
\end{aligned}
$$

Differentiating it then yields the following set of equations:

$$\left(\frac{\partial L}{\partial X_{n+1}}\right) \qquad P_{n+1} = \sum_{i=1}^{n} P_i a_{i,n+1} \tag{1'}$$

$$\left(\frac{\partial L}{\partial X_j}\right) \qquad P_j = \sum_{i=1}^{n} P_i a_{ij} + P_j^* \qquad j = 1, 2 \ldots n \tag{5'}$$

$$\left(\frac{\partial L}{\partial C_i^+}\right) \qquad P_i = \frac{\partial g}{\partial C_i^+} \qquad i = 1, 2 \ldots n \tag{12}$$

$$\left(\frac{\partial L}{\partial N_j}\right) \qquad P_j^* \frac{\partial F_j}{\partial N_j} = W + W_j^+ - W_j^- \qquad j = 1, 2 \ldots n \tag{13}$$

$$\left(\frac{\partial L}{\partial K_j}\right) \qquad P_j^* \frac{\partial F_j}{\partial K_j} = P_{n+1}\delta_j + S + S_j^+ - S_j^- \qquad j = 1, 2 \ldots n \tag{14}$$

where the differentials in parentheses before each equation indicate how the equation was derived.

Since the physical constraints of the two models are formally identical (except for the individual bounds on the sectoral use of primary resources), we only have to show that we can reproduce shadow pricing and behavioral equations similar to Eqs. (1)–(5) in the equilibrium model. Note, first, that Eqs. (1) and (5) of the equilibrium model appear in an identical form in the dual of the optimal planning model, as Eqs. (1′) and (5′). Equations (3) and (4), which represent the necessary conditions for profit maximization in the equilibrium model, have Eqs. (13) and (14) as their planning counterparts. At first glance they seem to be quite different, but closer examination reveals some essential similarities.

Consider Eqs. (3) and (13) first. Their left-hand sides are identical, but on their right-hand sides we find $\omega_j W$ and $W + W_j^+ - W_j^-$, respectively. In the literature concerning the design of (linear) programming models for development planning, the use of individual bounds (like N_j^-, N_j^+) is often criticized because they "pick up shadow prices which have no clear meaning and which, since all dual prices are interdependent, distort the dual solution" [19]. In our case, however, the shadow prices of the individual bounds are easy to interpret in the light of the equilibrium model. Variable W can be interpreted as the general shadow wage rate (i.e., the optimal rate of return on labor). Next, we can define

$$\omega_j' = \frac{W + W_j^+ - W_j^-}{W} = \left(1 + \frac{W_j^+ - W_j^-}{W}\right) \qquad j = 1, 2 \ldots n,$$

where the derived variable ω_j' may be interpreted as an endogenously determined index of the relative shadow wage rate in sector j.

Similarly, we may interpret the dual variable S in Eq. (14) as the optimal (shadow) net rate of return on capital (evaluated at base price). Thus, we can calculate $R = S/P_{n+1}$ so as to obtain the same rate of return at current (shadow) prices, and

$$\beta_j' = \frac{S + S_j^+ - S_j^-}{S} = \left(1 + \frac{S_j^+ - S_j^-}{S}\right) \qquad j = 1, 2 \ldots n$$

can be interpreted as an index of the relative rate-of-return requirement on capital in sector j. In this way we can rewrite the right-hand side of Eq. (14) as

$$(\delta_j + \beta_j' R)P_{n+1} = Q_j',$$

which is equivalent to the right-hand side of Eq. (4).

The "shadow cost" of capital differs from its equilibrium counterpart only in that the relative net rate of return is determined endogenously, not given exogenously. We will return to this difference later.

The only thing we have not yet demonstrated is that the solution of the optimal planning problem can be used to derive a set of special LES "demand" equations; this

may be shown as follows. Observe that the partial derivative of the objective function is $(s_i/C_i^+)g(C^+)$. Thus, Eq. (12) can be rewritten as

$$P_i = \frac{s_i}{C_i^+} g(C^+) \qquad i = 1, 2 \ldots n. \tag{14a}$$

Multiplying Eq. (14a) by the appropriate C_i^+ and adding together the equations for all i yields

$$\sum_{i=1}^{n} P_i C_i^+ = g(C^+). \tag{14b}$$

Next, we can calculate the shadow value of total consumption[j] as

$$\sum_{j=1}^{n} P_j C_j^- + \sum_{j=1}^{n} P_j C_j^+ = E. \tag{14c}$$

From Eqs. (14b) and (14c) we get

$$g(C^+) = \left(E - \sum_{j=1}^{n} P_j C_j^-\right) \qquad j = 1, 2 \ldots n.$$

Finally, substituting the above expression for $g(C^+)$ in (14a) and solving the equation for C_i^+ yields

$$C_i^+ = \frac{s_i}{P_i}\left(E - \sum_{j=1}^{n} P_j C_j^-\right) \qquad i = 1, 2 \ldots n.$$

Thus, the total consumption of commodity i is

$$C_i = C_i^- + \frac{s_i}{P_i}\left(E - \sum_{j=1}^{n} P_j C_j^-\right) \qquad i = 1, 2 \ldots n,$$

i.e., the demand function implied by the specifications of the optimal planning model.[k] However, the parameters are evaluated on the basis of information provided by traditional planning calculations, and they therefore reflect the planners' preferences and commitments.

It is interesting to note here that if we had a linear objective function, for example if instead of maximizing the weighted geometric average we maximized, as is often the case, the level of surplus consumption (y) in a fixed structure ($C_i^+ = yc_i^+$), then the implied demand function would be

$$C_i = C_i^- + \frac{c_i^+}{\sum_{j=1}^{n} P_j c_i^+}\left(E - \sum_{j=1}^{n} P_j C_j^-\right).$$

[j] Incidentally, this indicates how the level of total expenditure is determined endogenously in the general equilibrium model. Since we have only one consumer, the Pareto-optimal solution will simply be the one that maximizes the utility function. The expenditure level is thus determined by the value of this consumption calculated at equilibrium prices.

[k] It is well known in consumer demand theory that the form of utility function that yields LES is $g(C^+)$, as above, or its logarithmic equivalent [20].

This formulation may be interpreted in terms of consumer demand theory as a case where there is no substitutability between different commodities.

We have therefore established the *formal similarity* of the programming model and the equilibrium model. There is, in fact, only one point on which the two models are not formally identical. This is the "mechanism" by which the allocation of primary resources is controlled exogenously. It is even tempting to interpret these different formulations as alternative ways of reflecting the limited intersectoral mobility of primary factors. In an otherwise perfect market economy, this limited mobility would be expressed indirectly, by varying rates of return on the primary factors. In a centrally planned economy, on the other hand, this limited mobility would be accounted for directly, in terms of physical constraints. The planners would separate the sectorally committed (immobile) fraction of the primary factors from the mobile fraction at the outset. We should also stress that the similarity is *formal*, in that the actual meaning of the parameters, variables, and equations of the two models may be completely different, as noted earlier.

We would like to complete our comparison by pointing to the possibility of using *alternative economic policy goals* to measure efficiency gains in general equilibrium resource allocation models just as alternative objective functions are used in programming models. It should be clear from the equilibrium conditions that the model presented is not a completely closed equilibrium system: the distribution and redistribution of income, and their effects on final demand, are not considered in the model. At the same time, total expenditure and consumption are determined endogenously. The programming reformulation sheds some light on the way in which total expenditure may be determined. Since all other possible policy targets, such as net investments, government consumption, levels of primary input use, and the balance of current accounts are exogenous variables, practically all gains (resulting from increased allocational efficiency) will appear as increases in the consumers' utility. In the light of this consideration, it becomes obvious that the general equilibrium model too can be made to reflect different economic policy goals, e.g., increasing government consumption or net investment, decreasing the deficit on current account, or a combination of these. The incorporation of "objective functions" other than consumption would in most cases need changes only in the structure of endogenous and exogenous variables, or perhaps involve the introduction of some new variables and equations. Simple modifications of this type can make the equilibrium model capable of handling alternative policy objectives in the same way as programming models.

4. CONCLUSIONS

We have shown that a certain class of multisectoral general equilibrium models, by proper reinterpretation of elements, can be adopted to support planning in socialist countries. We have also demonstrated how certain nonlinear formulations of substitution possibilities could be utilized in macroprogramming models in order to keep the model relatively small and generate more meaningful dual solutions.

One major advantage of the equilibrium framework is that it makes the dual side of the model less distorted while explicitly taking into account the interaction of real and value variables. Thus, it may help planning modellers to achieve a better linkage between plans for real and value processes. These two main planning functions are usually quite separate from each other in both traditional planning and modelling. Changes in relative prices, costs, tariffs, etc., are not properly reflected in physical allocation models, while the effects of production, import/export, and consumption decisions are not always taken into consideration in price planning models.

The mixed, primal–dual formulation of the resource allocation problem also makes it possible to reinterpret the notion of efficiency (shadow) prices. On the one hand, it allows the model builder to explicitly introduce shadow-price-dependent resource allocation decisions into his model. In our simple model, it was quite easy to see how the efficiency-price-dependent consumption decisions related to the programming program formulation. In more complex cases, it seems to be more useful and straightforward to use such price-dependent (mixed primal–dual) decision rules explicitly (for a more detailed discussion of this issue, especially the treatment of export and import decisions, see Ref. 1).

On the other hand, the equilibrium formulation makes it possible to incorporate price-formation rules that reflect the actual process more accurately than the shadow prices of (linear) programming models. For example, even with constant returns to scale, it is possible to define prices that do contain profits (markup). One can also take into account changes in taxes and tariffs and see how these would affect the allocation decisions.

These comments suggest that the possible use of general equilibrium models is not limited to coordinating a plan. In fact, we believe that these models could also be used for either *ex post* or *ex ante* simulation of various issues of concern to planners. Using statistical estimates of the model parameters, structurally similar models (especially their multiperiod extensions) could be tested in the forecasting phase of planning.[1] A general equilibrium framework could also form a useful basis for analyzing certain kinds of disequilibria. Thus, in short, the study and adaption of general equilibrium modelling techniques may enrich the existing planning modelling methodology of the centrally planned economies in a number of different ways.

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[1] In a central planning context, it seems promising to combine such models with reference path optimization techniques [21]. Research in this direction is currently underway at IIASA.

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THE BASIC LINKED SYSTEM OF THE FOOD AND AGRICULTURE PROGRAM AT IIASA: AN OVERVIEW OF THE STRUCTURE OF THE NATIONAL MODELS

GÜNTHER FISCHER AND KLAUS FROHBERG

Food and Agriculture Program
International Institute for Applied Systems Analysis
Laxenburg, Austria

Abstract—The main aim of the Food and Agriculture Program is to explore alternative solutions to the world's food problem. Towards this end a system of linked models will be used for policy analysis over a medium time horizon. The Basic Linked System of the FAP consists of national models which describe in detail the food and agriculture system of the corresponding country and contain a rather aggregate mapping of the respective nonagricultural sector. The interdependencies of the two sectors are modelled as well. In this paper, the methodological and computational requirements a model should fulfill in order to become linkable with the FAP's model system are discussed and an overview of the structure of the national models of those countries which were built at IIASA by members of the FAP is given. A national model consists of three components: one for supply, one for demand and a third describing the process of policy decision-making. The policy module provides a mapping from the objectives the policymakers pursue while deciding on policies into the space of policy instruments. Endogenizing the process of setting the level of the policy instruments rather than specifying them exogeneously over the whole time span the model runs introduces more realism into the model, because governments react to changes occurring outside and/or inside of their country. The supply module consists of two subcomponents—one for agriculture and one for nonagriculture. The complexity of the decision-making process in agriculture is reduced to a two-stage process. In the first stage, farmers decide on the level of inputs they want to use for production. At the subsequent stage, these inputs are allocated to the various production activities and hence the amount of each commodity produced is decided on. The allocation process is modelled by using a nonlinear optimization program with statistically estimated parameters. Nonagricultural production is described in terms of a Cobb–Douglas function. For modelling demand an extended linear expenditure system is used. For developing countries the total population is divided into two income classes, whereas for developed countries the population is assumed to be homogeneous from the point of view of demand. The models are linked by applying the theory of general equilibrium. The markets are cleared simultaneously at the national and international level. It is assumed that supply cannot adjust during the exchange process and that the policy instruments are predetermined as well.

1. INTRODUCTION

The Food and Agriculture Organization of the United Nations (FAO) has estimated that 16% of the population of the developing countries excluding China, or 462 million people, were undernourished in 1970 [1]. Its forecast for the year 2000 is similarly pessimistic. Depending on prevailing economic growth rates, the FAO estimates that between 7% and 11% of the population of these countries will go hungry at the turn of the century [2].

As these figures show, hunger is a severe problem which cannot easily be eradicated. No completely satisfactory solution to this complex problem has yet been found; indeed, all its causes are not known. Although malnutrition is a national or local problem, solutions to it must also be looked for at the regional and global level. This will become even more important in the future, since the interdependence between national economies is growing.

Exploring alternative solutions to the food problem is the aim of the Food and Agriculture Program (FAP). To be more specific, the objectives of the FAP are to evaluate the nature and dimensions of the world food situation, to identify its underlying factors, and to investigate alternative courses of policy action at the national, regional and global level that may alleviate existing and emerging food problems in the years ahead.

The global system of food and agriculture can be viewed as a set of national agricultural systems which are embedded in national economies and interact with each other. In searching for effective policies for global food security one has to take into account the fact that individual nations affected by these policies might react to them in pursuance of their own objectives. Given the goals of FAP it is therefore necessary to work with a model system for policy simulation in which the main policy decision-making bodies are identifiable (these are usually the national governments[a]) in order to endogenize the process of determining the level of the (main) policy instruments.

Several attempts have been made to model world agriculture. To our knowledge these are the FAO World Price Equilibrium Model [3], the model of the Japanese Ministry of Agriculture and Forestry [4], the Model of International Relations in Agriculture (MOIRA) [5], the University of Illinois Model [6], more recently, the Lundborg Model [7], the Grain–Oilseeds–Livestock Model of the U.S. Department of Agriculture [8] and the World Integrated Model (WIM) [9]. With the exception of MOIRA, in all these models the world is divided into a few regions, so that national governments can no longer be identified. In MOIRA, all agricultural products are aggregated to one commodity by means of protein content. This lack of detail in the existing models made them less suitable for our purposes and, therefore, we built our own system.

It would hardly be feasible to attempt to model all the countries in the world with the manpower and resources available at IIASA. Thus, countries and country groups were selected for inclusion in the simulation system on the basis of population, agricultural land, agricultural production, and exports and imports of agricultural products. The countries and country groups chosen account for 80% of the world total in the categories listed above. They are Argentina, Australia, Austria, Bangladesh, Brazil, Canada, China, the European member countries of the Council for Mutual Economic Assistance (CMEA), the member countries of the European Communities (EC), Egypt, Finland, India, Indonesia, Japan, Kenya, Mexico, New Zealand, Nigeria, Pakistan, Sweden, Thailand, and the USA. It was ensured that economies of varying stages of development are represented and that all regions of the world are included in the system.

The linked system, therefore, consists of national or regional models, hereafter referred to as national models. These models interact through trade, capital flow, aid, and agreements. In this way it is ensured that global and local changes are inseparable and that they mutually influence each other in the system.

As it is beyond the means of the FAP to build a national model for each of these countries and regions, a network of collaborating institutions has been set up which can use all their expertise for developing these models. However, while this work is still

[a]For the European Communities (EC) it is the Council of Ministers.

continuing, a model has been developed by the FAP for almost all of the countries mentioned above. This set of national models, plus those models already finished by the collaborating institutions, is called the Basic Linked System. This intense research was necessary for two reasons. First, a background system had to be provided for any of the national models which were completed before the majority of the other models. Thus a modelling team can test its model in a linked mode, that is, the model can be run in interaction with the other national models. Second, the system provides the opportunity to investigate selected issues in the field of international food policy even before all of the more elaborate models are completed by the collaborating institutions.

This paper describes the structure of those national models of the Basic Linked System which have been built by members of the FAP.[b,c] Models of the Basic Linked System which were built by collaborating institutes are for the following countries (or country groups): the European Communities [11], the European member countries of the CMEA [12][d], Finland [13], India [14 and 15], Thailand [16], and the USA [17].

Before we continue to discuss the Basic Linked System, we will briefly mention the methodological and computational requirements a national model should fulfill in order to become linkable with the FAP's model system.

2. MODEL REQUIREMENTS

Each country model should be built so as to depict perceived realities. In other words, a descriptive model should be set up, indicating the responses of the actors in the system to changes in the economic environment brought about by policy measures and other factors (e.g., weather shocks). This requires empirically based information about the effects of various policies.

The model system will not be used merely for forecasting purposes. It should be designed so as to allow a comparative dynamic analysis of policy alternatives over a medium time horizon (15 to 20 years). Hence, the modelling of the short-term cycles in supply does not receive highest priority.

Since the model system will be used to analyze the impact of policy alternatives on food production and consumption, each model should consist of three components: one for supply, one for demand and a third in which the process of policy decision-making is described.

The models must have an extrapolative robustness, since one might want to test some policy alternatives which lie outside the historically observed ranges but for which one still hopes to obtain realistic results.

It is assumed that supply is given at the time the exchange of commodities takes place; i.e., current demand in all countries must be equal to supply determined in all countries in the previous year, leading to a recursively dynamic system. Although for a few commodities this might not always reflect reality, this assumption is valid for many agricultural products, since their production period is one year. In the nonagricultural sector, the production periods may deviate even more from these annual sequences. However, this assumption has the advantage of reducing the computational burden and

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[b]The data used for building these country models were obtained from various sources, all of which are publicly accessible. The main sources are the Supply Utilization Accounts, Production and Trade Yearbooks, and several issues of *Fertilizer Review*, all of them issued by FAO. In addition, the World Tables of the World Bank, the UN National Account Statistics, and national statistical yearbooks were used.

[c]The Chinese model [10], which also was built at the FAP but whose structure differs from that of all other FAP models, is not included in the discussion.

[d]An aggregate model for these member countries has been built.

of allowing great flexibility in the method chosen to model the supply side. Indeed, the supply modules which have emerged so far cover a wide spectrum of possible techniques. For example, linear programming models, nonlinear programming models with statistically estimated parameters, and conventional, econometrically based supply functions are used.[e]

The linked system is to be used for policy analysis over a medium time horizon. Over such a length of time the input structure is likely to change with altered economic conditions. Hence, emphasis should be placed on modelling the input–output relationships in production.

The models are linked by applying the theory of general equilibrium.[f] The country models must therefore cover the whole economy. In other words, both the agricultural sector and the nonagricultural sector have to be modelled. The policy alternatives to be investigated with the model system affect not only agriculture but also the nonagricultural sector either directly or indirectly through changes occurring in agriculture. Changes in the nonagricultural sector, in turn, have an effect on agriculture. It is therefore necessary to include the nonagricultural sector in the model in such detail as to realistically reflect these interdependencies.

The linkage approach allows the consideration of different income classes. Wherever a significant variation in the preference system of the various income classes is apparent, the population should be classified appropriately to account for these differences.

From the computational point of view it is necessary that all country models adhere to the same commodity classification for the purpose of international trade. It is further assumed that all countries trade at the same time and only once a year, and that trading is achieved instantaneously.

The conditions placed on the demand system of a national model are as follows: Demand must be homogeneous of degree zero and continuous in both prices and income, and a monotonically increasing function of income. There is nonsatiation, i.e., when the price of any commodity drops to zero, weighted total demand exceeds a specified satiation level. One item of demand is considered to be free disposal. This is used as a slack variable if supply exceeds all types of disappearance. But it is assumed that there are no costs for disposing of any quantity of any commodity. The linkage approach requires that the supply system be homogeneous of degree zero in prices.

3. OPERATION OF THE MODEL SYSTEM

Figure 1 describes in a very simplistic way how the model system operates within a year and over time and depicts the interactions between two countries. The information flow between the modules is indicated by arrows.

To begin with, let us assume that for both countries supply has been predetermined based on the events of the previous year, as has the trade deficit. Each government decides on the level of those domestic policy instruments with which it can pursue its goals regarding the outcome of the exchange process.

With the policies and supply given, the exchange process takes place. Equilibria are calculated simultaneously at the national and international level.[g] At the national level, the exchange process solves the demand module describing the preference systems of

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[e]See Parikh and Rabar [18] for an overview of the methods used by the various national modelling teams for modelling supply of agricultural commodities.

[f]A detailed description of the linkage approach can be found in Keyzer [19].

[g]Two different algorithms are used to obtain equilibrium, a complementarity pivoting algorithm at the national and a nonsmooth optimization algorithm at the international level (see Keyzer [19]).

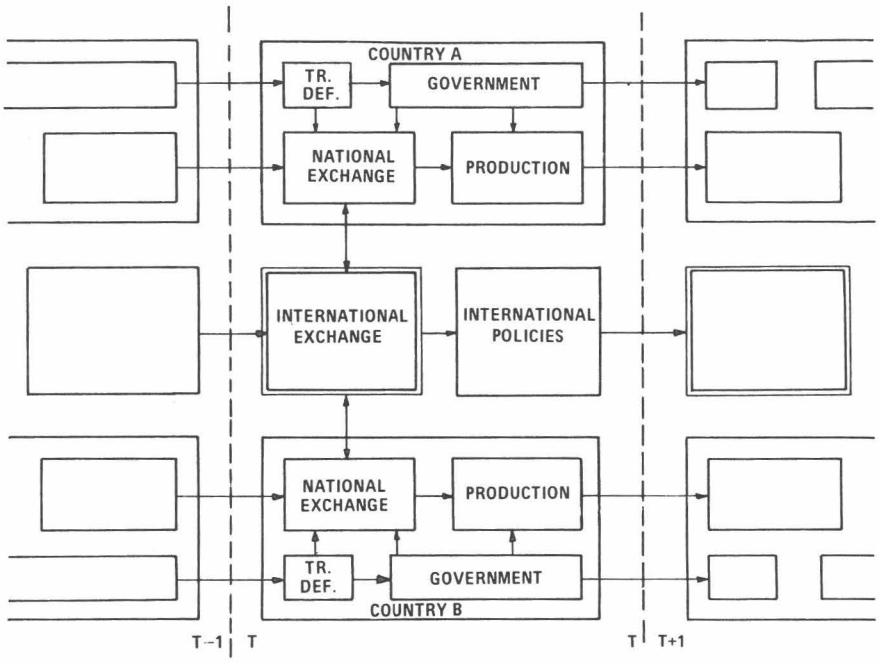

Fig. 1.

the consumers given government policies. Both the preference systems and some government policies (e.g., taxation) can be differentiated according to various income classes. The value of consumption of each consumer is limited by his disposable income. A quantity constraint must hold within the country for each commodity, ensuring that supply plus net import equals total disappearance. Moreover, there is a financial constraint at the national level equating the value of net trade to the (predetermined) trade deficit.[h] Consumption levels, quantities of net trade, national prices and income are the main results of the national exchange process.

At the international level the exchange process finds this vector of world market prices which ensures that for each commodity the values of net trade of all countries sum up to zero.

The information on national prices and income, together with policies implemented by the government to influence the structure of input factors, are further used in the supply module. In this module, net output is determined and in turn used as supply for the next year's exchange process after the carryover in stocks has been added.

4. DESCRIPTION OF THE SUBCOMPONENTS OF A NATIONAL MODEL

4.1. *Policy module*

Many important types of policies can be evaluated by the linked system and may be grouped into three categories. The first group consists of all those policies which have

[h]A negative trade deficit is used for trade surplus.

the purpose of influencing the production schedule. Typical examples of these policies are support prices or the alteration of the input structure, either through modifying input prices or distributing inputs directly. Important in this context is that these policies can be introduced independently of the type of linkage methodology being used.

A second policy group comprises all those national policies which may have an effect on the exchange process and hence depend on the linkage mechanism. The linkage approach used handles the following ones:

- direct and indirect taxes on and subsidies for domestic demand according to income group,
- tariffs on and/or subsidies for imports and exports,
- quotas on imports and exports,
- buffer stock schemes,
- net public demand for food.

International policies depend also on the linkage mechanism and are grouped in the third category. These are

- trade agreements,
- compensatory financing,
- buffer stock agreements,
- agreements on market segmentation.

At present, a detailed version of the policy module is still being worked on. In its final stage, this module will provide a mapping from the space of objectives of the policy decision-making bodies into the space of policy instruments. Some of the instruments used in the exchange part will be set at target levels, e.g., prices, stocks, trade deficit, trade quotas. The reason for this is that if all instruments were to hold as specified, an equilibrium might not exist. Some instruments, therefore, must be allowed to adjust. The order in which policy instruments adjust, i.e., deviate from their target values, is kept flexible in the linkage algorithm.[i] For those instruments which are set at target levels upper and lower bounds can also be introduced within which the adjustment process has to take place.

4.2. *Supply module*[j]

The supply module consists of two components—one for agricultural and one for nonagricultural production. Due to lack of manpower we made the assumption that each of the two sectors can be depicted by similar mathematical structures for all countries.

Agricultural production. The level of annual production in agriculture is typically determined in a sequence of decisions arrived at by a large number of decision makers. Since we cannot model this process in its full complexity, we reduce the decision-making levels to two and limit the number of decision-making units to one—possibly with the risk of some error in aggregating both over time (one year) and space. At the first decision level the quantity of the major inputs to be used in the production activities is decided upon. At the subsequent level these inputs are allocated to the various production activities, and hence the amount of each commodity produced is decided on.

[i] A discussion of this point is given by Keyzer [19].
[j] A more detailed description may be found in Fisher and Frohberg [20].

Assuming that there is only one decision-making unit leads to the aggregation of all production units to a "representative farm."

At the time the farmer makes his production decision, he does not have information on the selling prices of his goods. We postulate that he expects previous year's prices to hold in the current year as well.[k] Expected prices are therefore treated as predetermined variables. At the first decision level input quantities are determined for land, fertilizer, capital, labor and feed concentrates. Other inputs are excluded, for we assumed that their allocation effect was negligible.

An attempt was made to estimate land input into agriculture (measured as total area of crops harvested) with several economic variables as determining factors. However, the t-values showed no significance. The only variable used was therefore time[l]; i.e.,[m]

$$A_t^A = f(t),$$

where A_t^A = total area of crops harvested; t = time (year minus 1960).

Labor input into agriculture is measured by the number of people employed in this sector. A more precise measure for agricultural manpower could not be used due to lack of data. Hence, such important characteristics as skills and total working hours over a year and during peak seasons could not be taken into consideration. As labor input functions we estimated the following relationship[n]:

$$L_t^A = \alpha_{L1} * \left(\frac{Z_{t-1}^A}{Z_{t-1}^{NA}}\right)^{\alpha_{L2}} * L_{t-1}^A,$$

where Z_t^A = income per agricultural laborer in year t $\left(= \dfrac{\mathrm{GDP}_t^A}{L_t^A}\right)$,

Z_t^{NA} = income per nonagricultural laborer in year t $\left(= \dfrac{\mathrm{GDP}_t^{NA}}{L_t^{NA}}\right)$,

L_t^A = agricultural labor force in year t (persons),
L_t^T = total labor force in year t (persons),
L_t^{NA} = nonagricultural labor force in year t (persons),
GDP_t^A = gross domestic product of agriculture in year t (at current prices),
GDP_t^{NA} = gross domestic product of nonagriculture in year t (at current prices).

The ratio of current to previous year's agricultural labor force is explained by the per capita income parity between agriculture and nonagriculture, where we approximate income by gross domestic product.

As with the labor force, we had to assume that capital is a homogeneous input factor,

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[k]The rather short time period covered by our data series did not allow us to elaborate on different price expectation models.

[l]Here as for all the following variables the functional form listed characterizes only the relationships of economical and/or technical processes, but leaves out the random effects which enter the determination of these processes as well. We assume these random effects are additive, identically and mutually independently distributed, and follow a normal distribution with zero mean and finite variance-covariance matrix. Here and henceforth the parameters written in Greek letters are estimated from time series generally covering the period 1961 to 1976.

[m]Moreover, the functional relationships used for all countries are too numerous to be reported here.

[n]The functions vary slightly from country to country. Since we cannot report all of them, we quote the one most often used. This also holds for all the other variables for which we estimated a relationship and the functional relationship is indicated.

since lack of data did not allow us to differentiate between various capital goods. Capital stock is determined in the model in two stages. Gross investment is first decided upon and is then converted into capital stock by using

$$K_t^A = K_{t-1}^A * (1 - d_t^A) + I_t^A,$$

where K_t^A = capital stock of agriculture in year t (at prices of 1970),
d_t^A = depreciation rate for agricultural capital stock in year t,
I_t^A = gross investment in agriculture in year t (at prices of 1970).

Agricultural gross investment is described as a share of total gross investment using the following functional relationship:

$$\frac{I_t^A}{I_t^T} = \alpha_{I1} * \left(\frac{p_{t-1}^A}{p_{t-1}^{NA}}\right)^{\alpha_{I2}} * \left(\frac{\mathrm{GDP}_{t-1}^{A,CO}}{\mathrm{GDP}_{t-1}^{NA,CO}}\right)^{\alpha_{I3}}$$

where I_t^T = gross investment of the whole economy in year t (at prices of 1970),
$\mathrm{GDP}_t^{A,CO}$ = gross domestic product of agriculture in year t (at prices of 1970),
$\mathrm{GDP}_t^{NA,CO}$ = gross domestic product of nonagriculture in year t (at prices of 1970),
p_t^A = price index of agricultural commodities in year t,
p_t^{NA} = price index of the nonagricultural commodity in year t,

and all other variables as defined above. Agricultural gross investment share is determined by the ratio of agricultural to nonagricultural price indices and by the ratio of output of the two sectors. Both explanatory variables are lagged by one year. Investment in agriculture increases relatively to that in nonagriculture if the terms of trade between the two sectors change favorably for agriculture, as is shown by the positive parameter of the price indices. The ratio of output of the two sectors in the previous year is taken as a proxy of the ratio of planned output in the two sectors. According to this specification, agricultural gross investment is higher relative to that of nonagriculture the larger the ratio is.

Total gross investment is estimated as a function of total gross domestic product at current prices, trade deficit, and the change in gross domestic product between last year and the year before; i.e.,[°]

$$I_t^T = f(\mathrm{GDP}_{t-1}, \mathrm{BAL}_{t-1}, \mathrm{DGDP}_{t-1})$$

where GDP_t = total gross domestic product in year t (at current prices),
BAL_t = trade deficit in year t (at current prices),
$\mathrm{DGDP}_t = \mathrm{GDP}_t - \mathrm{GDP}_{t-1}$.

For fertilizer inputs we assumed that nitrogen, potash, and phosphorus are applied in fixed proportions; hence it suffices to consider nitrogen as a variable. However, the unit value of nitrogen consists not only of the nitrogen price but also of the value of potash and of phosphorus applied together with a unit of nitrogen. The function estimated for determining the fertilizer input level is as follows:

$$\mathrm{TF}_t = \alpha_{F1} * (P_{Ft}^r)^{-\alpha_{F2}} * \mathrm{CROP}_{t-1}^{\alpha_{F3}},$$

where TF_t = total fertilizer (nitrogen) bought by agriculture in year t (in mt),
P_{Ft}^r = unit value of fertilizer (nitrogen) in year t relative to that of the nonagricultural price,
CROP_t = volume of crop production in year t at prices of 1970.

[°]Again, the functional relationships used are too numerous to be reported here.

Fertilizer input is a function of the unit cost of fertilizer and previous year's crop production. The latter is considered to be a proxy for planned crop production in the current year.

When calculating the input of feed concentrates we assume that their supply is completely elastic. With this assumption it is possible to determine feed mix per animal unit independently of the level of animal husbandry.

The functional form employed to determine the feed requirement coefficients is derived from a feed cost minimization model which can be written for any animal type i and year t in the following way:

$$\min_{FD_{ikt}}(W_{it}) = \sum_k r_k * FD_{ikt}$$

s.t.

$$(A_{it}) = \rho_i(t) * \prod_j FD_{ijt}^{\epsilon_{ij}}$$

$$\sum_k \epsilon_{ik} = 1.$$

This results in the following equation for feed requirement coefficients:

$$\alpha_{ikt} = \frac{\epsilon_{ik}}{r_{kt}} * \frac{1}{\rho_i(t)} * \prod_j \left(\frac{r_{jt}}{\epsilon_{ij}}\right)^{\epsilon_{ij}}$$

where i = commodity index, $i \in$ animals,
j, k = index of feed concentrates,
α_{ikt} = requirement of feed concentrate k per unit of animal i in year t,
r_{kt} = price of feed concentrate k in year t,
A_{it} = number of animals of type i in year t,
FD_{ikt} = total consumption of feed concentrate k by animal type i in year t,
W_{it} = total feed cost of animal type i in year t,
t = time variable.

The coefficient ρ_i is time-dependent. This is a proxy for measuring the change in the (technical) efficiency in feeding.

For the second decision-making level—the allocation of the inputs—a nonlinear programming model with a nonlinear criterion function and linear inequality constraints is used. This approach seems very suitable for the task of modelling a multiple input–multiple output system of an industry which is characterized by joint production.[p] In modelling for policy analyses over a time span of 15 to 20 years, such an approach has the advantage that both economical and technical relations are included in the mapping. The allocation model can be written for any year t as follows:

$$\max_{F_{it}, K_{it}, L_{it}} (Z_t) = \sum_i nr_{it} * Y_{it}$$

s.t.

$$\sum_{i \in J} F_{it} - F_t^A \leq 0$$

[p]A multiple output technology is joint if the output of any single product depends on the levels of inputs and/or outputs of another product.

$$\sum_i K_{it} - K_t^A \leq 0$$

$$\sum_i L_{it} - L_t^A \leq 0$$

$$\sum_{i \in J} A_{it} - A_t^A \leq 0$$

and with

$Y_{it} = y_{it} * A_{it}$	for each i
$y_{it} = g(F_{it}, t)$ (see footnote q)	for $i \in$ crops
$y_{it} = f(t)$	for $i \in$ animals
$A_{it} = \alpha_{it} * K_{it}^{\beta_i} * L_{it}^{\gamma_i}$	for each i
$\beta_i + \gamma_i < 1$	for each i

where i = commodity index,
t = time index,
J = index set of crops,
Y_{it} = net production of commodity i in year t (gross production minus seed use and waste),
y_{it} = yield per unit (acre, animal) of commodity i in year t,
A_t^A = acreage (harvested) in year t,
F_t^A = fertilizer input in year t,
K_t^A = capital stock in agriculture in year t,
L_t^A = labor force in agriculture in year t,
A_{it} = acreage allocated to crop i in year t, if $i \in$ crops,
= number of animals of type i in year t, if $i \in$ animals,
F_{it} = fertilizer applied to crop i in year t,
K_{it} = capital employed in production of commodity i in year t,
L_{it} = labor employed in production of commodity i in year t,
nr_{it} = expected net revenue per unit of commodity i in year t; defined as expected price minus expected feed cost; if $i \in$ animals
= expected price of commodity i in year t; if $i \in$ crops.

We postulate that the farmer maximizes expected net revenue, which is defined here as the difference between expected gross revenue and expected feed cost. The farmer is assumed to have nonstochastic behavior; in other words, he reaches a decision which does not deviate from the optimal one. We also postulate that there are decreasing returns to scale for a single product. Technical progress in the allocation model is divided into mechanical and biological progress. The latter is measured by a trend variable which affects the intercept and/or the slope of the fertilizer response function. Mechanical technical progress is represented by time-varying parameters. Estimation results indicate that it is labor-saving.

According to the specification of the allocation model, the capital stock employed in the production process of any commodity is determined in each year independently of

[q]The functional relationships are too numerous to be reported on here.

that amount used in previous years. This "putty-putty" approach hypothesizes that the technologies of existing units are interchangeable without cost with the same type of technologies used in newly produced units, i.e., production units must be completely flexible with regard to new technologies. Since the agricultural sector is characterized by a large number of production units, such a high flexibility is likely.

We would like to point out that the specification of the allocation model allows for annual decisions without explicitly considering the dynamics involved in those production processes which cover periods greater than one year (e.g., beef and dairy production). However, implicitly these characteristics are taken care of in the parameters of the corresponding production function and in the lag structure of the respective price expectation model.

By solving the allocation model, we obtain an optimal (with respect to the criterion function) use of the (predetermined) total inputs of land, fertilizer, capital, and labor, and simultaneously net production of each commodity. Net production plus carryover in stocks is treated as supply.

There are various policy instruments which can be considered to affect agricultural production. Among those instruments which give economic incentives for production we would like to mention producer prices and subsidies for investments and fertilizer, as well as income transfer to influence the labor force employed in agriculture. There is also the possibility of affecting the level of inputs directly, e.g., through rationing of fertilizer. In addition, quotas on production may be set.

Nonagricultural production. The nonagricultural sector is aggregated to one commodity. This sector is represented by a Cobb–Douglas production function, i.e.,

$$Y_t^{NA} = \alpha_{nt} * (K_t^{NA})^{\vartheta} * (L_t^{NA})^{1-\vartheta}$$

where Y_t^{NA} = nonagricultural production in year t,
K_t^{NA} = capital stock of the nonagricultural sector in year t,
L_t^{NA} = labor force in the nonagricultural sector in year t,
α_{nt} = a term which includes neutral technical progress.

The calculation of the nonagricultural capital stock is made in the following way:

$$K_t^{NA} = K_{(t-1)}^{NA} * (1 - d_t^{NA}) + I_t^{NA}$$

with

$$I_t^{NA} = I_t^T - I_t^A$$

where d_t^{NA} = depreciation rate for nonagricultural capital stock in year t,
I_t^{NA} = investment in the nonagricultural sector in year t,
I_t^T = total investment in year t.

The labor force of nonagriculture is obtained by determining the difference between the total labor force and that employed in agriculture. Total labor force is a function of total population and the participation rate, each of which varies over time.

$$L_t^{NA} = L_t^T - L_t^A$$
$$L_t^T = f(\mathrm{POP}_t, pr_t),$$

where L_t^{NA} = labor force in the nonagricultural sector in year t,
L_t^A = labor force in the agricultural sector in year t,

L_t^T = total labor force in year t,
POP_t = total population in year t,
pr_t = participation rate in year t.

We assume that the capital stock is always fully utilized and that there is no unemployment.

4.3. *Demand module*

The demand for goods is modelled in the Basic Linked System by using an extended linear expenditure system (ELES). We distinguish between two income classes for developing countries and use only one for developed countries. In the case of two income classes the criterion for differentiating between these classes is their occupation. Those people who work in agriculture and their dependents are grouped into one class and the rest into a second class. This procedure had to be followed since we do not have statistics on the population size of various income classes.

We tried to estimate the coefficients of the extended linear expenditure system but obtained unrealistic results. Therefore, we followed a more pragmatic approach. In an extensive literature search, average expenditure shares for each country—and, where necessary, for each income class—were collected. Together with information on expenditure shares at farmgate level,[r] these expenditure shares at the retail level were taken to determine the value of processing, marketing and distribution per unit of each commodity (hereafter called processing margin). The processing margin determines the amount of nonagricultural product needed to "transfer" a unit of a commodity from the farmgate level to the retail level. Due to lack of information this margin is kept constant over time in the current version.[s]

We also estimated expenditure elasticities for each commodity by fitting nonlinear Engel curves to the time-series data of the corresponding per capita expenditure. The functional forms chosen imply that expenditure elasticities are either constant or decline with increasing expenditure. These expenditure elasticities were then used to obtain the coefficients for marginal budget shares along with committed consumption in the following way.

Given supply to enter exchange, it is assumed that agricultural production is owned by the agricultural income class and that nonagricultural production is owned by the nonagricultural income class. In a first step, we can calculate expected income for each income class by using expected prices. In a second step this expected income is split into expenditures on agricultural goods and nonagricultural goods by means of a two-sector linear expenditure system with habit formation (HLES). Then, expenditures spent in total on agricultural goods are further subdivided into expenditures on each of the nine agricultural commodities using the corresponding expenditure elasticity.

Once the demand at expected prices has been calculated for each of the traded commodities, this information is translated into the parameters of the ELES in the

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[r]Expenditure share at the farmgate of commodity i is defined as consumption of that commodity times its farmgate price divided by total GDP.

[s]An increase in the price of the nonagricultural commodity thereby also increases the price of the agricultural commodity at the retail level.

following way:

$$\text{TEXP}^j = \bar{\varphi} * \sum_{i=1}^{n} \bar{P}_i * YM_i^j$$

$$e_i^j = \eta_i^j * \frac{\text{EXP}_i^j}{\text{TEXP}^j} \qquad i = 1 \ldots n$$

$$XM_i^j = [\text{EXP}_i^j - e_i^j * (\text{TEXP}^j - \text{COMEXP}^j)]/\bar{P}_i \qquad i = 1 \ldots n$$

where η_i^j = expenditure elasticity of commodity i by income class j,
EXP_i^j = expected expenditure on commodity i by income class j, (expected target retail price times expected consumption),
TEXP^j = total expected expenditure by income class j,
COMEXP^j = committed expenditures by income class j at expected prices (obtained from two-sector HLES),
$\bar{P}_i$ = expected target retail prices of commodity i,
XM_i^j = committed consumption of commodity i by income class j,
e_i^j = marginal budget share for commodity i by income class j,
$\bar{\varphi}$ = 1-expected tax rate,
YM_i^j = endowment of commodity i by income class j.

Both feed use and intermediate consumption of each commodity are included in the demand module. Their values are added to the committed demand coefficient of the corresponding product.

5. VALIDATION

For validating the model we adopted a hierarchical procedure following the steps taken in building the model. The first validation was performed during construction of each subcomponent of a national model. At this stage, such conventional criteria as fit of estimated variables, t-values, plausibility of the estimated parameters and tests for autocorrelation were used.

The next validation phase took place when the subcomponents of a national model were linked together. Two tests were performed at this stage. We measured the tracking ability of the model by calculating Theil's inequality coefficient for those endogenous variables which had been exogenous or not generated at all at the former stage (e.g., national equilibrium prices, net trade values). However, this could be done only for the period for which the model had been estimated. In addition, we calculated summary statistics which could be compared with research work done elsewhere. Such statistics are price elasticities of supply and of demand and income elasticities of demand. However, sometimes we encountered problems with regard to comparability of our statistics with those found in the literature, mainly due to differences in the time periods and in the conceptualization of the models used for estimating these elasticities.[1] Nevertheless, these comparisons gave us useful information on the validity of our models with regard to price and income sensitivity. At the last stage, the linking of all national models, we placed heavy emphasis on world trade and world market prices.

[1]For a discussion of this point, see Shumway and Chang [21].

6. CONCLUSION

The work reported on in this paper has now reached a stage where we can begin to analyze the effects of various policy alternatives. To report on the results we have obtained so far would exceed the space limitations of this article; a brief discussion of them could possibly be open to misinterpretation.

We hope that the results of these exercises will give some indication of how to improve the world food situation. The insights thus gained can only be beneficial in practice, however, if they are taken into account by the governments of various nations in their policymaking.

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IDENTIFYING MODELS OF ENVIRONMENTAL SYSTEMS' BEHAVIOUR

M. B. BECK

International Institute for Applied Systems Analysis
Laxenburg, Austria

Abstract—This paper addresses the question of how theories are developed about the behaviour of large, complex systems such as those typically encountered in managing environmental quality. The specific problem considered is that of model structure identification by reference to experimental, *in situ* field data. A conceptual definition of this problem is given in terms of the notion of testing model hypotheses to the point of failure. An approach to solving the problem is proposed in which the use of recursive model parameter estimation algorithms is a central feature. This approach is illustrated by a case study in developing a dynamic model of water quality in the Bedford Ouse River in central-eastern England. The results are organized around the two principles of attempting to falsify confident hypotheses and of speculating about relatively uncertain hypotheses in order to modify inadequate prior hypotheses. The essential difficulty demonstrated by the case study is one of absorbing and interpreting the diagnostic evidence of field data analysis and this is ultimately a difficulty associated with the complex and intrinsically indivisible nature of large-scale systems.

1. INTRODUCTION

According to the spectrum introduced by Karplus [1] environmental systems' analysis lies midway between the two extremes of analyzing socioeconomic systems and electrical network analysis. This gives rise to rather special problems in the analysis of environmental and, more specifically, water quality-ecological systems. On the one hand, *a priori* theory, with its basis in the physical and biological sciences, would seem to be capable of predicting observed behaviour relatively accurately. On the other hand, however, it is especially difficult to conduct planned experiments against which *a priori* theory can be evaluated. In these somewhat ambivalent circumstances there has arisen a growing incompatability between that which can be simulated in principle with a model and that which can be observed in practice. To a great extent this accounts for the gap that has developed between the "larger" simulation models, with which there is little hope of conducting rigorous calibration exercises given currently available field data, and those much "smaller" models that have been so calibrated.

The specific problem to be considered in this paper is that of model structure identification by reference to experimental, *in situ* field data. To see why this is a problem, however, it is first necessary to summarise briefly some limitations in a widely accepted approach to water quality-ecological modelling. According to this approach it is generally assumed that one can (conceptually) subdivide the field system into smaller, individual components, whose (conceptual) behaviour can usually be approximated by laboratory-scale replicas (for example, chemostat and open-channel flow experiments). Submodels for these components are assumed to be "verifiable" against experimental observations of the behaviour of the replica; and the model for the field system can be assembled by linking together the submodels. Thus the content of the model is supported

by arguments that admit extrapolations from laboratory systems and equivalent or similar field systems. At the stage of model calibration the tendency is to assume that *a priori* theory is correct unless *demonstrably* inadequate. It is especially difficult to demonstrate inadequacy, and the need to question the validity of the original extrapolations is thus all too easily likely to remain obscured.

The argument that the extrapolations inherent in the above approach are legitimate would appear to remain in doubt unless one can develop and apply a complementary approach that provides a more direct evaluation of the prior hypotheses about observed system behaviour, without dividing the system into its component parts. Model structure identification is a fundamental part of that complementary approach: it has to do with the questioning so easily set aside because of the imperfections of the available field data; it is a problem for which seemingly few systematic methods of solution have been developed; and, possibly most significant, it requires a subtle but important change of attitude towards modelling. In spite of very many laboratory-scale experiments and a number of major field studies, current knowledge of the structure of the relationships among the mineral, organic, and microbiological components of an aquatic ecosystem is still quite uncertain. Too much confidence has been placed in *a priori* theory. Perhaps, in Popper's terms [2], environmental systems have been modelled as though they were "clocks," being "regular, orderly, and highly predictable," whereas they may well be more like the "irregular, disorderly, and more or less unpredictable clouds." This reflects simply a change of attitude, because, as evident in Somlyódy's papers [3, 4], there is clearly a spectrum of regularity and orderliness associated with the prior knowledge relevant to water quality-ecological modelling (ranging from hydrodynamics to biology). In short, central to the problem of model structure identification is the question: how are theories developed about the behaviour of large, complex systems given the assumption that observations can be obtained (and subsequently interpreted) from experiments broadly similar to the classical form of experimentation in laboratory science?

The work discussed here, then, on the topic of modelling poorly-defined environmental systems ("poorly-defined" being an expression first used by Young [5]), is part of a Task on Environmental Quality Control and Management within the Resources and Environment Area of IIASA. This essentially methodological component of the Task is complemented by a second theme dealing with case studies in lake eutrophication management, that is, for Lake Balaton, Hungary [6, 7, 8] and for a number of Austrian lake systems [9]. A productive interaction between case-study problem-solving and methodological developments is the cornerstone of the Task's research. In the following, although examples drawn from the lake eutrophication studies would be equally appropriate, such as the results reported by Somlyódy [3], we shall illustrate methodological problems associated with modelling the dynamics of water quality in the Bedford Ouse River (U.K.). This river system in turn provides an informal case-study for the development of a third theme of the Task on operational water quality management [10].

Section 2 of the paper discusses both the problem of model structure identification and an idealised approach to its solution based on the use of recursive parameter estimation. Since model structure identification can be viewed as a matter of iteratively falsifying and speculating about hypotheses, Sec. 3 examines the difficulties of interpreting diagnostic evidence on whether a given model structure (set of hypotheses) is demonstrably inadequate.

2. MODEL STRUCTURE IDENTIFICATION

Usually one associates the exercise of model calibration with curve-fitting and parameter (coefficient) estimation. But the word "calibration" is misleading. It suggests

an instrument (here, the model) whose design is complete and whose structure is beyond further argument. All that remains to be done is to make minor adjustments to some of the fittings, i.e., fine-tuning of the parameter values. Calibration of models for water quality-ecological systems, however, is unlikely to be such a simple and straightforward matter. Instead, even before asking the question, "Can I estimate the model parameters accurately?," the analyst must first ask himself whether he knows how the variables of the system are related to each other. In particular, one must ask whether information about these relationships can be identified from the *in situ* field data. Yet most exercises in model calibration have focused solely on the matter of parameter estimation; hence little attention has been paid to the (arguably) more important prior problem of model structure identification.

Let us introduce and qualify a working definition of the problem:

- *Model structure identification* is concerned with establishing unambiguously, by reference to the *in situ* field data, how the measured input disturbances, $\mathbf{u}$, are related to the state variables, $\mathbf{x}$, and how these latter are in turn related both to themselves and to the measured output responses, $\mathbf{y}$, of the system under study.

We may note first that this is significantly different from a definition of what may be called model order estimation, a problem in which, for example, the objective is to estimate the orders of polynomials in an autoregressive/moving-average time-series model (see, for instance, [11, 12, 13]). Second, we may note the importance of the word "unambiguously." A common difficulty in fitting a model to a set of field data is that the error-loss function does not exhibit a well-defined, global minimum. Many combinations of estimates for the model parameter values provide equally good (or bad) descriptions of the observed behaviour; in effect, a uniquely "best" model for the system has not been identified. Such difficulties are often referred to as the problem of identifiability, or the model is said to be overparametrised and to contain surplus content. This is perhaps a matter of no consequence in terms of fitting the model to the data, but it would certainly have significant implications should the model be used for prediction (as has been argued elsewhere [14, 15]). One would expect ambiguous statements about future behaviour, although the effects of uncertainty may preclude any conclusion about significant differences among these statements [15].

The essence of the approach to model structure identification, as discussed briefly here and in much greater depth in [5, 15, 16, 17, 18] is based on a restatement of the original problem definition in terms of a parameter estimation problem. Such an approach, however, depends on the availability of an adequate set of time-series field data, a condition which is by no means always satisfied. Even so, for situations of scarce data the development of a roughly parallel approach is apparent in a recent paper by Fedra [19].

In order to outline the approach, albeit in a conceptual sense, let us imagine that the state variables $\mathbf{x}$ in a model may be represented by the nodes of Fig. 1(b) and that the parameters $\boldsymbol{\alpha}$ are visualised as the "elastic" connections between the state variables. Without going into details, let us also assume that the parameters of the model can be estimated recursively, i.e., such that estimates $\hat{\boldsymbol{\alpha}}(t_k)$ of the parameter values can be obtained for each sampling instant t_k within the sequence of time-series observations (for discussions of recursive estimation, see, for example, [16, 20, 21]).

If now the assumption has been made that all the parameters have values that are constant with time, yet a recursive algorithm yields an estimate of one or more of the parameters that is significantly time-varying, one may question the correctness of the chosen model structure. We can argue this point as follows. The general tendency of an estimation procedure is to provide estimates $\hat{\mathbf{x}}$ of the state vector, or some functions thereof, i.e., $\hat{\mathbf{y}}$, that track the observations $\mathbf{y}$. Hence, if any *persistent structural*

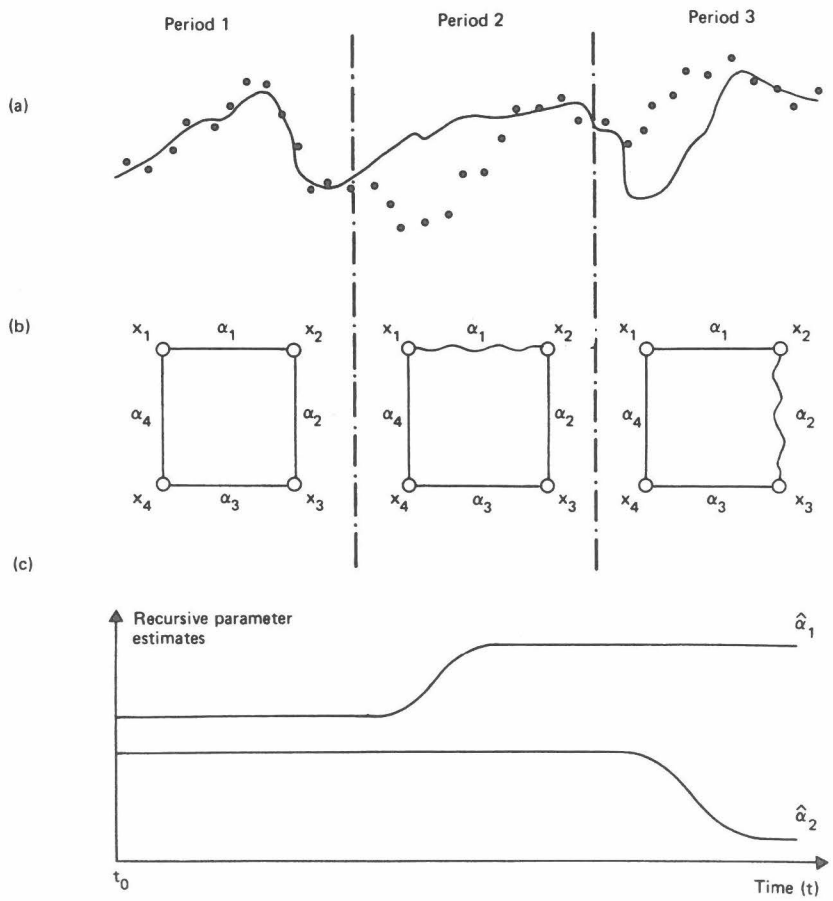

Fig. 1. An illustrative example showing the concept of using a recursive parameter estimator in the context of model structure identification: (a) hypothetical model response and observations (dots); (b) conceptual picture of model structure; (c) recursive parameter estimates.

discrepancy is detected between the model and "reality" (in other words, the errors $\boldsymbol{\epsilon} = (\mathbf{y} - \hat{\mathbf{y}})$ exhibit a significantly nonrandom pattern), this will be revealed in terms of *significant adaptation of the estimated parameter values.* There may well be good reasons for why the parameter estimates vary with time, and, indeed, that is precisely what one is looking for.

Starting with Period 1 of Fig. 1(a), however, let us continue to sketch the outline of the approach. The model responses ($\hat{\mathbf{y}}$) and output observations ($\mathbf{y}$) are essentially in agreement over this period and there is no significant adaptation of the parameter estimates [according to Fig. 1(c)]. At the beginning of Period 2, however, there is a persistent discrepancy between $\hat{\mathbf{y}}$ and $\mathbf{y}$. It might be supposed, for example, that the underlying cause of the discrepancy is an inadequacy in the behavior simulated for x_1 and x_2, that α_1 is sensitive to this discrepancy [Fig. 1(b)], and that (persistent) adaptation of the estimate $\hat{\alpha}_1$ [Fig. 1(c)] partly compensates for the error between $\hat{\mathbf{y}}$ and $\mathbf{y}$. Again in

the third period there is disagreement between the observations and model responses, which leads to adaptation of the estimate $\hat{\alpha}_2$.

The example of Fig. 1 is clearly an ideal view of how a recursive estimation algorithm should be employed for model structure identification. In fact it is an idealised framework developed largely, but not entirely, from a particular case-study in modelling the dynamics of water quality in the River Cam, U.K. [17, 22]. Generalisation from a single example is undoubtedly not without dangers and certainly the results to follow challenge the usefulness of this ideal view. Nevertheless, cast in this particular fashion such an approach has intuitively appealing interpretations. First, and by analogy with the analysis of physical structures, the aim is to expose inadequacy in terms of the "plastic deformation" [Fig. 1(c)] of the model structure. Second, and of deeper significance, testing the model structure to the point of failure, that is, the failure of one or more hypotheses, can be said to be consistent with Popper's view of the scientific method [23]. And Popper's view of the scientific method is in turn exercising a growing influence over the discussion of modelling the behaviour of environmental and similar systems [5, 19, 24, 25, 26].

Especially pertinent here is Holling's remark that "... the model is [to be] subjected to a range of tests and comparisons designed to reveal where it fails" [24]. This, with emphasis on the words "range" and "designed to reveal" sets a suitable guiding principle for solving the problem of model structure identification. But to have revealed that the model structure is inadequate is merely a part of the solution, and actually a *relatively* easy part. If we extend the example of Fig. 1 one further step, let us suppose that the first (model) hypothesis has been identified as failing, according to Fig. 2(a). Now assume that a second hypothesis can be generated in some way—which is a complementary part of the solution—and that it has the structure of Fig. 2(b) with an additional state variable (x_5) and two new parameters (α_5, α_6). It may well be that calibration of the second model

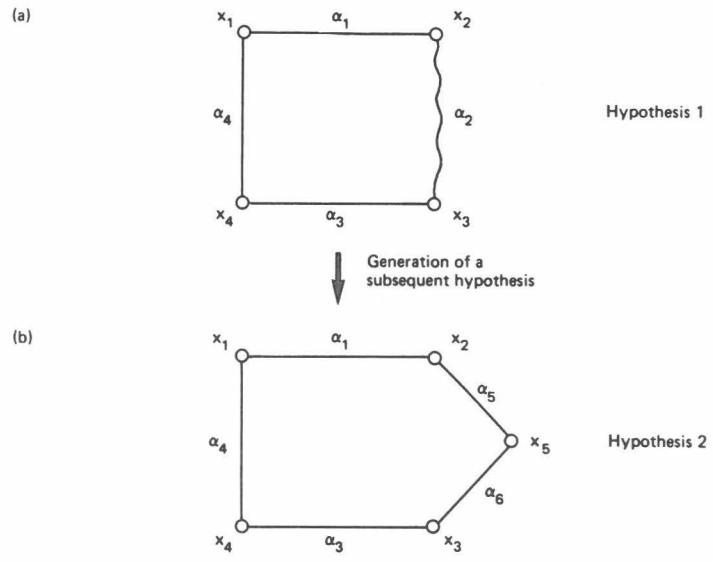

Fig. 2. The process of model structure identification: revision of the model structure and reestimation of the associated parameters (b) on the basis of diagnosing how the prior model structure fails (a).

against the field data yields essentially invariant parameter estimates and hence the analyst can accept the adequacy of this model structure as a conditionally good working hypothesis.

The basic aim of model structure identification is thus to seek plausible hypotheses for apparently "unexplained" relationships in a set of field data. The approach outlined above exploits the idea of curve-fitting as a "means-to-an-end" and not as an "end" in itself. Falsifying the model structure, or components thereof, rests partly upon judgements about absurd parameter values, or about implausible variations in the parameter values. Unless these variations and values can be defended by logical argument, then it must be conceded that the structure of the model does not match the structure underlying the observed patterns of behaviour.

It would be wrong, however, to assume, because of the exclusive discussion of an approach based on recursive parameter estimation, that this approach is a panacea. The benefits to be derived from a range of procedures have already been emphasized and are apparent in the Cam case study [22]. This is only one approach applicable to a certain sector of the overall problem defined for a restricted set of conditions; yet it is an approach that has yielded considerable insight into the nature of the problem.

In the following we shall focus on two types of critical difficulties in applying the above approach to model structure identification, that is, the difficulty of revealing that a hypothesis is absurd, which is really the most demonstrable form of inadequacy; and the difficulty of synthesizing the diagnostic evidence in order to speculate about how to modify an inadequate prior hypothesis. Our purpose is to expose weaknesses and limitations both in the technical effectiveness of recursive parameter estimation as a method of solution and, more fundamentally, in the appropriateness of the approach. As with model structure identification, so too with the approach itself, establishing what is wrong or inadequate is the key to improvement and progress.

3. DESIGN FOR FAILURE AND SPECULATION

If solving the problem of model structure identification depends strongly upon revealing absurd hypotheses, an easily recognizable difficulty is that *in situ* field data subject to high levels of uncertainty are hardly likely to yield such revelations. There are, however, more subtle aspects of the nature of field data from environmental systems that place equally, if not more, awkward constraints on the likelihood of success in model structure identification. The patterns of time-series observations typically available for analysis reflect experiments—if indeed they can be so called—that are successively less good approximations of the classical, planned experiments of laboratory science [15]. In all but a few cases the observed perturbations in system behaviour do not conform with the desirable attributes of data usually expected for the identification of models for, for example, aircraft and industrial process control [12, 27]. And since it is in areas such as these latter that many of the methods of analysis have originated [28], recursive estimation included, one finds that there is an impressive array of techniques that perform well on well-posed problems, yet a dearth of techniques that can perform adequately on the ill-posed problems of environmental systems analysis.

It is tempting to blame a lack of success on poor data and inappropriate analytical methods. But this would be misleading and, in any case, current constraints are not destined to persist into the future. Consider, for example, the ever-growing potential for generating data from environmental monitoring networks and consider also the principal asset of a recursive estimation algorithm, that is, to generate model parameter estimates at *each* instant of time t_k in a time-series. There is every possibility that future critical constraints will be dominated by the inability to absorb and interpret the diagnostic

evidence of data analysis. In fact, these constraints are ultimately a function of the complexity and indivisibility of large-scale systems. It is to the difficulties of conducting an analysis in the face of such problems that we now turn.

From the generalisation of the River Cam case study, to which passing reference has been made earlier, it is possible to propose a tentatively broader organising principle for the procedure of model structure identification. Hence, let us simply suggest that the analyst is concerned with conducting experiments (in a loose sense) on and with the model structure, where these experiments can have the following two distinctly different orientations (or objectives):

(i) in the process of *falsifying* a given model structure;

(ii) in the process of (creative) *speculation* about alternative hypotheses.

These two processes are probably best viewed as mutually exclusive, for reasons we shall discuss later, and, quite appropriately, they reflect the two-step nature of solving the problem.

The case of the Bedford Ouse River in central-eastern England is a natural extension of the Cam study. From 1972 to 1975, the Department of the Environment in the United Kingdom and the Anglian Water Authority jointly funded a major study of the Bedford Ouse River system in order to evaluate the effects of developing a new city (Milton Keynes) in the upper part of the catchment [29]. It is in the light of tackling this substantially more complex problem of field data analysis that we shall be able both to judge the usefulness of the above organising principle and to illustrate the difficulties of interpreting the diagnostic evidence of analysis.

3.1. *Failure of the model structure*

Let us look first at the notion of testing the model structure to the point of failure, that is, the process of falsifying a given set of hypotheses. For the Bedford Ouse example the model structure to be evaluated contains various confident assumptions about the transport and dispersive properties of the river, reaeration, the decay of waste organic matter, and the growth, death, and photosynthetic properties of a population of phytoplankton. That these should be "confident" assumptions, which has a quantitative counterpart in the specification of the *a priori* error statistics associated with the model, is an important point. Given the conceptual outline for model structure identification (see also Figs. 1 and 2) this is a very deliberate tactic of stressing a relatively rigid structure so that the probability of detecting a significant failure is maximised. In this step of the analysis it would not appear to be particularly useful to express little confidence, *a priori*, in the model and then to try and identify unambiguously where failure occurs. In such a case the postulated model structure is, as it were, too flexible. Adaptation of the parameter estimates may, or may not, be significant, because one has little confidence in the model, and clear-cut answers cannot be obtained because, in effect, clear-cut questions are not being asked. Flexibility would be more of an advantage at the stage of creative speculation and this is why separation of the two steps is desirable.

Altogether six parameters are to be estimated in identical model structures for the behaviour of interactions between dissolved oxygen (DO), biochemical oxygen demand (BOD, a measure of degradable organic matter), chlorophyll-*a* (as a measure of phytoplankton populations), and suspended solids concentrations in each of the three reaches of the river system (a total, therefore, of 12 state variables and 18 parameters). Figure 3 shows the recursive estimates of these six parameters for the third (downstream) reach of river. Comparing Fig. 3 with the enviable idealised simplicity of Fig. 1, one would have great difficulty in answering the question "at what point does the model structure fail?" without even asking the question why it might have failed. The results

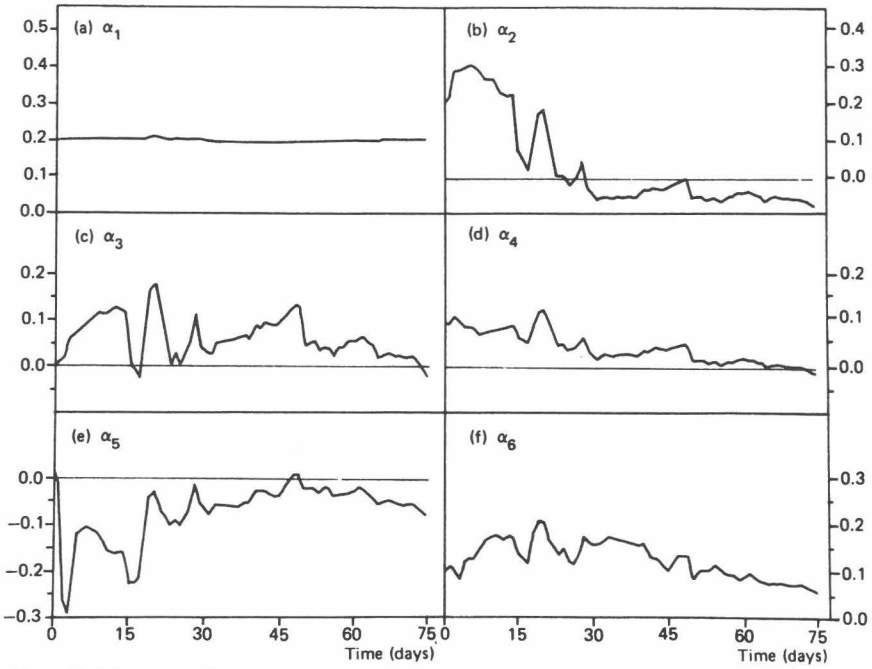

Fig. 3. Model structure identification (the process of falsifying confident hypotheses) in the Bedford Ouse case study (third reach): (a) reaeration rate constant (day^{-1}); (b) maximum specific growth-rate constant for phytoplankton (day^{-1}); (c) BOD decay rate coefficient (day^{-1}); (d) rate constant for addition of BOD to reach from suspended solid matter (day^{-1} [g^{-3} BOD] [g^{-3} SS]$^{-1}$); (e) death-rate constant for phytoplankton (day^{-1}); (f) rate constant for "loss" of suspended solids from the reach (day^{-1}).

are a peculiar mixture of both insufficient and redundant hypotheses in the model structure—of, at the same time, under- and overparameterisation. The considerable nonstationarity of the parameter estimates clearly indicates that the model structure is inadequate. Yet the similar patterns of variability among the different parameters is a symptom of surplus content in the model, i.e., one inadequacy compensates for another. In other words, certain critical features of the structure of the relationships underlying the field data are not included in the model, while no single parameter estimate unambiguously compensates for the obvious inadequacy.

There are apparently some absurd hypotheses. For instance, the recursive estimates of both the maximum specific growth-rate (nonlinear Monod kinetics) and first-order, death-rate constants for the phytoplankton population [Figs. 3(b) and 3(e), respectively] become negatively valued. One could argue, as a result, that the former is barely significantly different from zero and that the latter—a linear, negative, death-rate—is perhaps evidence of a preferred linear growth-rate function for the phytoplankton (at least for all but the initial period of the data). But the analyst would be hard pressed to attach great confidence to such conclusions. On balance it might be more appropriate to conclude that the algal population is in a state of equilibrium with neither of the rates of growth and death being independently identifiable from the data.

The principal issue raised by the results of Fig. 3 is one of misplaced confidence in *a priori* theory. It has a specific aspect associated with these results and a more general aspect relating to the introductory comments of the paper. Thus, for example, the

remarkable stationarity of the recursive estimate for the reaeration rate constant [Fig. 3(a)] is a function of having assumed relatively more *a priori* confidence in this particular parameter. In other words, the analyst has assumed that if the model is to fail it is unlikely to be a function of an inadequate description of the reaeration process, a point to which we return later. This might be a reasonable assumption since, together with the assumption concerning BOD decay, about which similar questions will be raised shortly, it is a basic component of the *classical* studies (conducted in 1925) of Streeter and Phelps [30] on river pollution and self-purification. That these assumptions have been used for a long time creates a resistance to challenging their validity. Yet there are good reasons, as demonstrated elsewhere [15], for arguing that the classical assumptions of Streeter and Phelps, and the equally classical assumptions of dispersion in flowing media, represent patterns of behaviour that are not identifiable from this particular set of *in situ* field data. In this case the problem of identifiability arises because other dominant modes of behaviour—here, especially in the first and second reaches of the river, the growth of a phytoplankton population—almost entirely obscure these less significant modes of behaviour. In a sense, therefore, the assumptions of Streeter and Phelps are, for this example, not testable propositions, and their inclusion in any subsequent model structure is tantamount to an act of faith.

It seems important in a more general sense, therefore, to question the motives for maintaining hypotheses that are not, strictly speaking, falsifiable. The reluctance to set aside convention is strong indeed, and Fig. 3(c) illustrates well the conflict that can occur—Young [31] has put forward a cogent and challenging argument on the same point. Given prior experience that the hypothesis of BOD decay is probably not identifiable, a BOD decay rate constant is still retained in the model structure, but with an *a priori* estimate of zero (day^{-1}). It would be difficult to argue from Fig. 3(c) that the subsequent pattern of the recursive estimates prompts the assumption of a significantly nonzero value for this parameter. The problem can thus be summarised as follows. The results of Fig. 3 are founded upon the premise that

(a) "We have confidence in the hypotheses of Streeter and Phelps, but consider current hypotheses about mechanisms of phytoplankton growth as highly speculative."

Such a premise could be reoriented to either of the following:

(b) "We are confident about our hypotheses for phytoplankton growth, but consider the assumptions of Streeter and Phelps to be highly speculative;"

(c) "All hypotheses are equally speculative."

Perhaps one should cling to the first premise and not reject convention until it is demonstrably inadequate. The obvious dilemma is that just such a clutching at convention, especially in the context of water quality-ecological modelling, may preclude the possibility of revealing inadequacy. And the shift in emphasis as to where greater confidence is placed, from premise (a) through (b) to premise (c), is a specific interpretation of the change in attitude towards modelling discussed in the introduction to the paper.

3.2. *Creative speculation*

The process of *speculation* can be illustrated with results drawn likewise from another part of the Bedford Ouse analysis. It is again assumed (implicitly) that premise (a) above is reasonable so that speculation can be conducted in terms of a vector of *lumped* parameters representing all the other mechanisms of behaviour (in this case, sources and sinks of DO, BOD, and chlorophyll-*a*) that are considered to be speculative assumptions. The objective then is to generate plausible hypotheses about why the

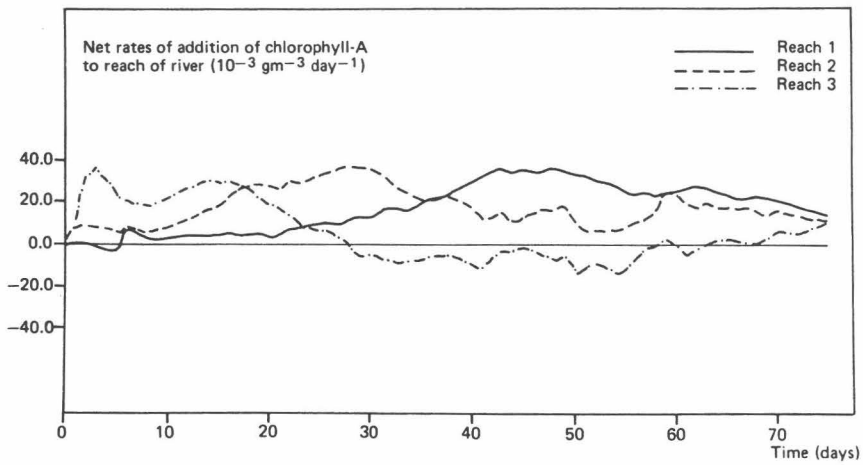

Fig. 4. Model structure identification (the process of speculation) in the Bedford Ouse case study: recursive estimates for the net rates of addition of chlorophyll-*a* to each reach of the system.

estimates for these lumped parameters exhibit variations with time (or space), if that is so; to formalise these hypotheses; and to proceed to a subsequent step in the process of falsifying the revised model structure. For the three reaches of the Bedford Ouse system, part of the diagnostic evidence from analysis of this speculation is gathered together in Figs. 4 and 5. One could tentatively conclude from these recursive estimates that

(i) The rate of addition of chlorophyll-*a* to the system reaches a maximum first (in time) in the third (downstream) reach, then in the second, and lastly in the first (upstream) reach (Fig. 4);

(ii) The rate of addition of dissolved oxygen to the first reach is roughly proportional to the observed concentration of chlorophyll-*a* at the downstream boundary of that reach [Fig. 5(a)]; the rate of addition of dissolved oxygen to the second reach is roughly proportional to the observed concentration of chlorophyll-*a*, except over the middle period of the record [Fig. 5(b)]; the rate of addition of dissolved oxygen to the third reach is not obviously proportional to the observed chlorophyll-*a* concentration for most of the time [Fig. 5(c)].

It would certainly be a bold and imaginative hypothesis that could be synthesised from such evidence and hence lead to the restructuring of the model for the purposes of again attempting to falsify the revised hypotheses. And this is actually a relatively simple example, when compared with the complexity of models frequently discussed in the literature. We have presented the evidence of Figs. 4 and 5 primarily so that one can ask the rhetorical question: how would the analyst absorb and interpret this relative wealth of diagnostics? As earlier, to have drawn the possible conclusion that the model fits the data subject to arbitrary variations in one or more of the parameters (as typified by the recursive estimates of Figs. 3, 4, and 5) is of no consequence. Rather, it is the process of speculating about why such variations occur that should be highly valued.

3.3. *Reconstructing the experiments of laboratory science*

In introducing the problem of model structure identification, it was assumed that observations could be obtained (and subsequently interpreted) from experiments broadly similar to the classical form of experimentation in laboratory science. We shall further

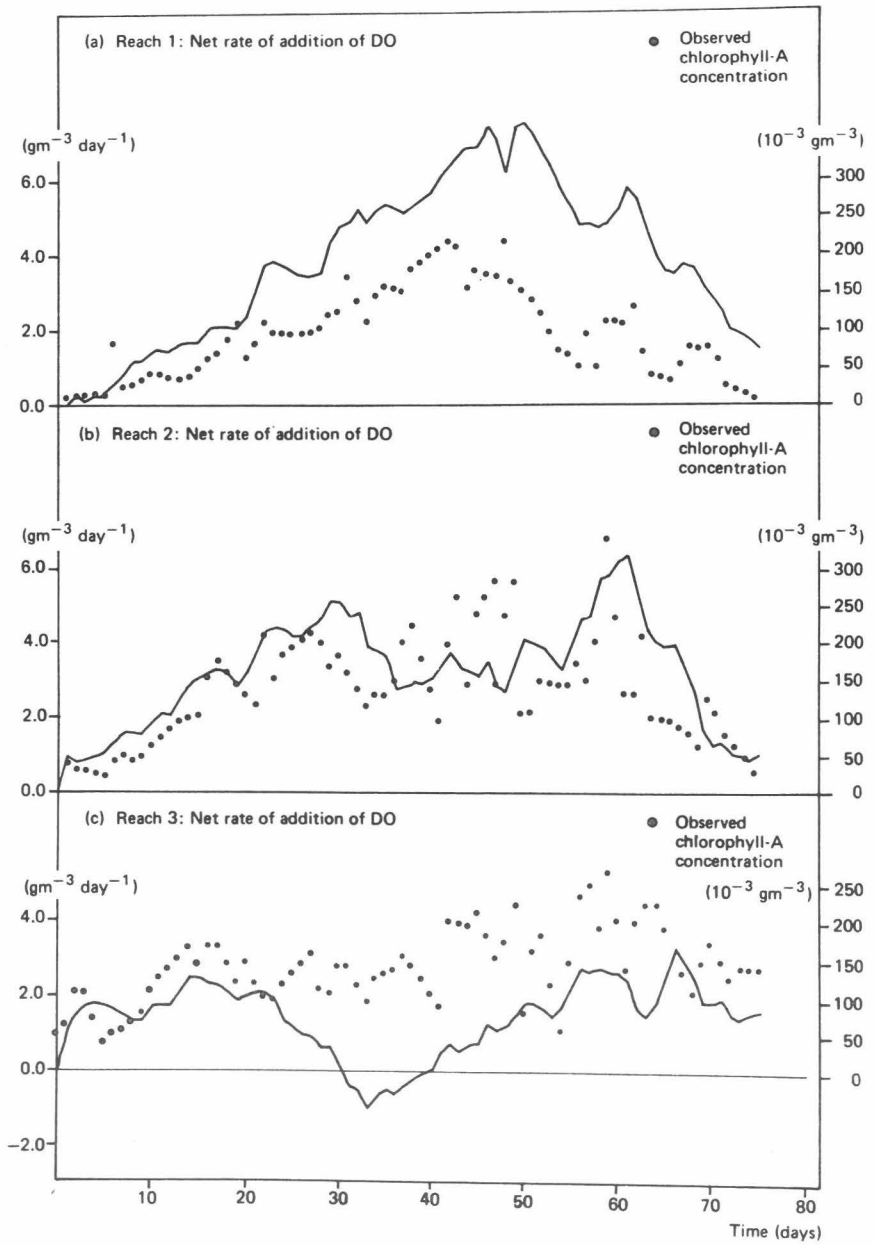

Fig. 5. Model structure identification (the process of speculation) in the Bedford Ouse case study; comparison of recursive estimates for the net rates of addition of DO to each reach of the system with the observed chlorophyll-*a* concentrations at the downstream boundary of each respective reach.

assume that a laboratory experiment is usually designed to test the relationship between, say, two variables (cause and effect) while all other variables associated with the system are maintained at steady, constant values. Clearly the field data available from environmental systems reflect quite imperfect experiments. Let us suppose, nevertheless, that model structure identification is a procedure for reconstructing *in situ* "experiments" from observed data by (mathematical) analytical methods. In other words, it seems reasonable to attempt to design the analysis of model structure identification such that it compensates for the unsteady and extraneous disturbances originating from the "environmental conditions" of the laboratory-type "experiment." An apt example is premise (a) associated with the Bedford Ouse analysis in Sec. 3.1, where the "experiment" would be concerned with identifying the mechanisms of phytoplankton growth and the Streeter–Phelps assumptions would be absorbed into the analytical compensation for the "experimental environment." Another apt example is given in Somlyódy's paper [3], where the "experiment" is to identify the relationship between wind stress at a point on the surface of Lake Balaton and the distribution of suspended solids measured in the vertical water column below that point. All other phenomena affecting the vertical distribution of solids, that is, other than sedimentation and the wind-induced resuspension of particles from the bed of the lake, are assumed to be included in the "environmental conditions." This latter would include, for example, solids transported horizontally into the vertical water column and that fraction of the observed suspended solids concentration due to living organic matter, such as a phytoplankton population. In fact, the model for this "experiment," as defined, is sufficiently well posed that the analysis might more fruitfully be "inverted" in order to identify better the relationships assumed in the given definition of the "environment."

In either of the two examples quoted, the skill of the analyst would lie in arranging the analysis such that extraneous interference with the analysis could be filtered out. At first sight, this is perhaps a rather attractive view of the true purpose of system identification and time-series analysis. But it presupposes, of course, that that part of the model required to compensate for the experimental "environment" is known *a priori* with sufficient confidence to permit the full power of the analysis to be directed towards identification of the relationships defined as the "experiment." Such assumptions themselves have to be evaluated. The distinction between what is "known well" and what is "speculation" thus becomes vanishingly small. It is unlikely, as with premise (c) in Sec. 3.1, that all prior hypotheses are equally speculative; rather, a spectrum of degrees of confidence is probable. The freedom to manipulate where greater prior confidence should be placed, however, can thus be seen to be both an advantage and a disadvantage. In its worst form it allows the possibility of prejudicing the diagnosis of failure, as apparent with the results of Fig. 3. It is difficult to claim, however tempting it may be, that there is just one "experiment" and its complementary "environment." Instead, it is only possible to state that a number of more or less significant "experiments" are proceeding in parallel. This does not mean that partially isolated experiments cannot be conducted on large-scale field systems—the study of wind-induced resuspension of lake sediments in Somlyódy's paper [3] typifies what is possible in this respect. But it does mean that if the analyst aspires to the development of a model for the field system as a whole, then his analysis of the data will have to contend with the intrinsically indivisible character of the system's behaviour.

4. CONCLUSIONS

Many contemporary exercises in water quality-ecological modelling have been conducted without serious consideration of the significance of calibration. It is not an empty

appendix to the mainstream developments in water quality modelling. It may only be considered so if one chooses to attach great confidence to *a priori* theory, thereby renouncing, in effect, much of the questioning that should accompany calibration.

The "questioning" process of model calibration, to which considerable importance is attached, is what has been called here the problem of model structure identification. The procedure proposed for solving (in part) this problem has two primary features: (a) the use of recursive parameter estimation algorithms for the analysis of time-series field data; and (b) the alternative objectives of examining the model structure from the point of view of either falsifying confident hypotheses or creatively speculating about uncertain hypotheses.

The paper has illustrated this approach to model structure identification with a case-study of the Bedford Ouse River system. The relative complexity of this study defines it as what might be called a second-generation study in model structure identification; indeed it raises more questions than it answers. In particular, the Bedford Ouse example challenges the usefulness of the procedure outlined for model structure identification and draws attention to the crucial difficulty of focusing and interpreting the diagnostic evidence of analysis. This example also illustrates the problem of distinguishing between which are confident and which are speculative prior hypotheses, a distinction that is important for implementing the proposed approach. Finally, consideration of an analogy with the planned experimentation of laboratory science, although superficially attractive as an interpretation of model structure identification, leads to the conclusion that the analyst has to contend with the multiplicity of "experiments" inherent in a set of field data from an environmental system. Clearly, complexity, an intrinsically indivisible nature, and not merely uncertainty, are inescapable problems in modelling such large-scale systems.

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MODELLING A COMPLEX ENVIRONMENTAL SYSTEM: THE LAKE BALATON STUDY

L. SOMLYÓDY*

International Institute for Applied Systems Analysis
Laxenburg, Austria

Abstract—A systems approach is introduced into eutrophication modelling and is illustrated by the example of Lake Balaton, Hungary, one of the world's largest shallow lakes. One of the major features of the problem is its complexity. Many interrelated processes should be considered in the lake and in the corresponding watershed, both on the level of scientific understanding and policy making. The other essential feature is the presence of various kinds of uncertainties. The approach developed is off-line in character, which avoids the direct coupling of the detailed descriptions of all the subprocesses. The procedure starts with the decomposition of the entire model into smaller, tractable units, forming a hierarchical system. This step is followed by aggregation, the aim of which is to preserve and integrate only essentials in the higher stratum. The various levels involve the modelling of biological phenomena, the sediment–water interaction, hydrodynamic and transport processes and the nutrient loads, with the corresponding calibration and validation steps. The approach accounts also for the influence of natural and man-made influences, as well as for the propagation of uncertainties. The procedure can lead to a realistic but yet simple model on the higher level of the hierarchy, where an optimization problem should be solved (e.g., how the maximum water quality improvement can be achieved under given budget constraints). The various steps of the study are illustrated by examples.

1. INTRODUCTION

The eutrophication of lakes, a typical, unfavorable manifestation, symptomatic of the past few decades, is a consequence of the increase in the amount of nutrients (such as phosphorus and nitrogen compounds) reaching water bodies. This increase is closely related to the generally rapid development of industry, agriculture, and tourism within the watershed or in short, to a change in the infrastructure of the region. Eutrophication, an in-lake phenomenon, the origin of which lies outside the lake, causes unpleasant consequences, e.g., a rise in algal biomass, water discoloration, taste and odor problems, and bacterial contamination. These can greatly limit the use of the lake's water for recreation, water supply, etc., and lead to a drastic change in the ecosystem.

The phenomenon of eutrophication and mathematical modelling of it have been quite well explored for deep lakes, but only to a lesser extent for shallow lakes. Here, due to the low depth and the generally strong wind action, stratification rarely occurs. The dynamics are more complex and are faster; consequently, shallow lakes are much more affected by changes in environmental factors and show less consistent patterns from year to year. The recognition of this gap in knowledge led IIASA's Resources and Environment Area to initiate research on the eutrophication of shallow lakes. Lake Balaton in Hungary, the largest lake in Central Europe, which has exhibited the signs of

*On leave from the Research Centre for Water Resources Development (VITUKI), Budapest, Hungary.

man-made eutrophication, was chosen to be one of the two case studies. The case study is being carried out in cooperation with the Hungarian Academy of Sciences and with the participation of several Hungarian institutions.

The general features of the problem from a methodological standpoint are as follows: (i) The system composed of the water body and the watershed is large in a physical sense. There are strong interactions among biological, chemical, and hydrophysical in-lake processes; furthermore, between the watershed and water body. (ii) There are several stochastic influences (e.g., meteorology, hydrology). (iii) The data available are often inadequate and scarce. Uncertainty consequently plays an important role. (iv) Based on these features, understanding and managing the system cannot be accomplished solely on an empirical basis. Modelling is inevitably needed and should be in harmony with in situ and laboratory measurements, and data collection, respectively. (v) The vicinity of the lake is the major tourist resort in Hungary, so there is a strong economic interest in a practical solution to the problem. In other words, the management of the system is of primary interest.

From these characteristics it follows that the lake and its region form a complex environmental system; the problems related to it require a systems analytical approach. The aim is to handle the problem in both a research and management context, that is, to better our understanding on the scientific level and then to utilize this knowledge for working out optimal control strategies for improving the water quality of the lake. The elaboration of such a model or set of models is not an easy task and meets serious methodological difficulties.

The objective of this paper is to illustrate with the example of Lake Balaton, how such a complex system can be understood and managed in its entirety, i.e., together with the related methodological questions. The paper is organized as follows: First, the main characteristics of the system and the modelling approach are outlined (Secs. 2 and 3). In Sec. 4, the individual steps of the analysis are illustrated through examples. These will be related to the sediment–water interaction, spatial mass exchange, nutrient loading problem, and the lake eutrophication model. In most of the examples, the influence of uncertainties and stochastic effects will be accounted for. At the end of this section, it is shown how the lake water quality model can be incorporated into the management framework.

It has to be mentioned that the study has not yet been completed. Therefore, in some cases, reference will be made to results which are preliminary only.

2. MAJOR CHARACTERISTICS OF THE SYSTEM

The lake and its watershed are illustrated in Fig. 1. The length of the lake is 78 km, the average width around 8 km (surface area nearly 600 km^2) and the average depth 3.1 m. The major inflow of the lake is the Zala River at the southwestern end of the lake which drains half of the total catchment area ($\sim$5800 km^2). There is a single outflow at the other end of the lake, Siófok, through a control gate. The mean residence time of water is about two years.

The fluctuation in the water's temperature is high. There is a relatively long ice-covered period (around two months), while the temperature in summer may exceed 25°C. Concerning the chemical composition of the water, the high calcium carbonate content and pH value (8.3 to 8.7) should be mentioned. Wind action is important, resulting in favorable oxygen conditions and a permanent back and forth motion (seiche) along the lake and a complicated circulation pattern. Wind strongly influences sedimentation and release of various materials from the sediment layer (its organic material content is relatively low).

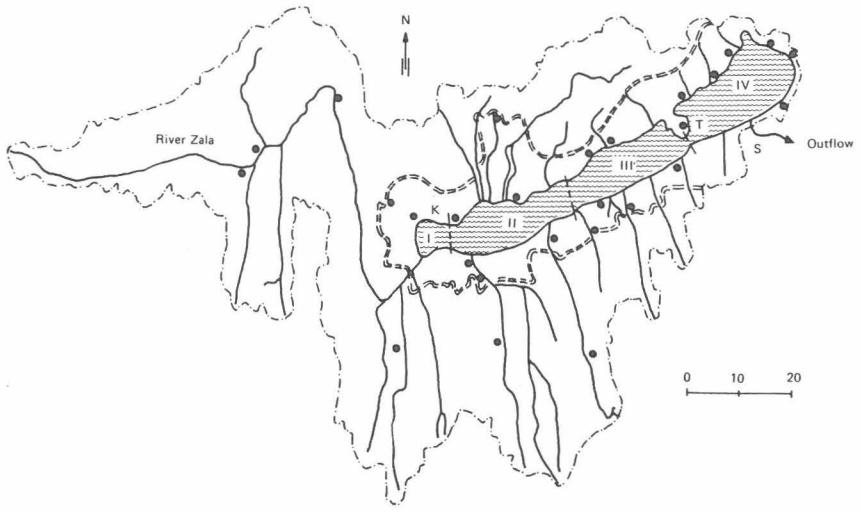

Fig. 1. Major characteristics of the system: (K) Keszthely, (T) Tihany, (S) Siófok, (I . . . IV) typical basins of the lake, (–·–) boundary of the catchment area, (===) boundary of the recreational area, (·) sewage discharges in the region.

In recent years, remarkable changes have been observed in the water quality due to the rapid increase in tourism, sewage discharges, fertilizer use, and other factors. The algal biomass (algae is the most important primary producer in this case) increased by a factor of 10 when compared with the past 15–20 years. The trend in primary production is similar and at the most polluted western basin, peaks of up to 13.6 g C/m^2 were observed, a hypertrophic value [12]. In short, the average lake conditions moved from mesotrophic to eutrophic, thus endangering the use of the lake for recreational purposes, the prime water use in this case.

Phosphorus plays a dominant role in the eutrophication of the lake. Thus, both from the point of view of understanding and managing the system, tracing the phosphorus compounds in the lake and on the watershed is of primary interest. The total phosphorus load of the lake is around 1000 kg/d [17], half of which is estimated to be available for algal uptake. The load has many components: 33% is derived from sewage, 27% from diffuse sources, 22% is related to runoff processes in the direct vicinity of the lake, while the contribution of atmospheric pollution is 18%. The ratio of sewage discharges in the available load is higher; only the sewage released in the recreational area (Fig. 1) accounts for 36% of the available load. This direct load varies quite a lot, following the fluctuations in population due to tourism, and has a 2–4 times higher value in summer than during the off-season. The load distribution along the lake is approximately uniform; however, the volume related value is twelve times higher at Keszthely Bay (Fig. 1) than at the other end of the lake, due to differences in the volume of the four main basins. This fact is also reflected in the pronounced longitudinal gradient of various water quality parameters, e.g., for chlorophyll-*a* the ratio of the maximum and minimum values ranges between 4 and 20 [30]. The gradient observed at the same time indicates that the strong wind action and the mixing associated with it are still not sufficient for leveling out the spatial nonuniformities.

From an analysis of the data it is clear that there is not only a critical state of the water quality at Keszthely Bay, but also a spreading deterioration process which extends towards other areas of the lake where the water quality is still good. Thus, action is urgently required from the view of the entire lake.

Based on hydrologic and water quality considerations, the lake was divided into four basins, as indicated in Fig. 1. The application of the principle of segmentation proved to be a useful tool for modelling, data collection, and data handling.

Concerning data, extensive records are available on hydrology and meteorology. Regular water quality monitoring started ten years ago, in two network systems consisting of 9 and 16 spatial sampling points, respectively (10–20 measurements per year), but irregular data are also available back to the early sixties. Several other in situ and laboratory measurements were also taken (primary production, extinction, sediment–water interaction, velocity, etc.). A survey was done on the nutrient load between 1975–1979, which involved 20 tributaries and 27 sewage discharge points [17] (indicated in Fig. 1). On the major tributary, daily observations were made during this period [16].

On the basis of data collected by Hungarian institutions recently—often in connection with modelling—the "Lake Balaton Data Bank" was created at IIASA, serving as a starting point for the modelling work.

For further details on the case study, the reader is referred to [30, 31].

3. THE MODELLING APPROACH

3.1. *Decomposition and aggregation*

The previous sections demonstrated the complexity of the water quality problem. Here, methodological questions will be discussed. For the purpose of illustration the in-lake processes will be considered. From this example, conclusions will be drawn, leading to the modelling approach to be adopted.

The water quality of a lake is the result of several physical, chemical, and biological processes. The development of a model is generally based on the appropriate combination of knowledge gained from theory and measurements, respectively (the so-called theoretical and measurement knowledge [8]). Depending on the solidity of the theory of various processes, quite different approaches may be employed. For example, in most of the hydrodynamic applications (hydrophysics can be called a "hard" science [3] within the spectrum of water quality) the model structure is basically determined by the partial differential equations (PDEs) of continuity and momentum. The PDEs are solved by some numerical techniques through discretization of the domain in question where the discretization is determined primarily by mathematical aspects (convergency, stability, accuracy, etc.). The grids or segments are relatively small and thus result in a continuouslike approach (the PDEs are replaced by a large number of ordinary differential equations, ODEs).

Within the domain of water quality, the theoretical background of both biology and chemistry are less satisfactorily established than for hydrophysics and model development should be supported more extensively by the measurement knowledge. The development is generally based on testing hypotheses and uncertainty plays an important role [4, 9, 14, 31]. Under such conditions, steps such as model structure identification, parameter estimation and the analysis of error propagation [2] are inevitably required—techniques which are available in practice for ODEs if the number of equations is relatively small. This latter requirement leads to models which distinguish only some limited number of segments (or series of mixed reactors) established mainly by intuition but without proper justification. The structure of such models is determined by the ODE

character as contrasted to hydrodynamic and transport models where the main features are associated with the original PDEs.

In the majority of the water quality models, the description of physical, chemical, and biological processes is necessary; the dilemma then is [27]: how should the modelling procedure of biology and chemistry (ODE structure with interactive use of data) be combined with the "precise" treatment of hydrodynamics, the PDE structure of which excludes the application of most of the techniques required for the scientific cognition of phenomena of the other group and consequently the exploration of the entire process? The answer seems to be relatively easy, namely, one should start elaborating detailed models for both groups (hydrodynamics and transport; and biology and chemistry, respectively) and keeping in mind the main features of the other group (time and length scales [11], etc.), simplify them. The effort aims at finding the simplest description of processes according to their relative importance in the entire water quality problem (e.g., the contribution of mass exchange associated with water motion to changes caused by biochemical phenomena). This step can be done through sensitivity analysis and is called aggregation. It can be performed in two stages: aggregation of the submodels, first, followed by the aggregation of the coupled models, second (see below). An essential point is that at the end of this procedure, one should arrive at a version of the model which is also consistent from a methodological viewpoint (e.g., describe the mass exchange properly and preserve the applicability of parameter estimation techniques).

From this conclusion, an *off-line* modelling approach follows, which avoids the direct coupling of all the detailed submodels. The procedure starts with a reasonable *decomposition* of the complex structure into smaller, more tractable units which are accessible for separate and detailed studies. This is followed by *aggregation*, the aim of which is to preserve and integrate only essentials, ruling out the unnecessary details.

In the course of submodel development, experiments of "isolated" character are certainly required (see, e.g., Sec. 4.1), but then the coupled, aggregated model is validated against its detailed version and data (a tedious task which is rarely documented in the literature).

Each modelling step is associated with model and data uncertainties; consequently, the aggregation is required in this respect as well (see Sec. 4): another guideline in our approach.

The tactics of decomposition are especially important if the objective of the study involves such different levels as the scientific understanding of a system and the decision making related to the same problem: a situation that we face here. It is noted that in ecological modelling, there is a certain gap between "larger" models (incorporating some ten state variables and nearly one hundred parameters) and "smaller" models (see Beck [4] in this issue). In the first case, there is nearly no hope for a precise calibration, but "smaller" models can also be just as unrealistic for complex problems because of their simplicity. Accordingly, the solution cannot be sought in this contrast, i.e., "larger" or "smaller" model. It is felt that the alternatives offered here provide a reasonable approach for such cases.

3.2. *The modelling framework*

The application of this approach for the Lake Balaton problem is explained with the help of Fig. 2, which shows the framework of the research [31]. The first decomposition that directly comes to mind is the distinction between lake and watershed, since as mentioned before, the water quality problem lies in the lake, but the causes, and practically all control possibilities are to be found in the watershed. Next, the various units should be separated and the essential results put together on a successively higher

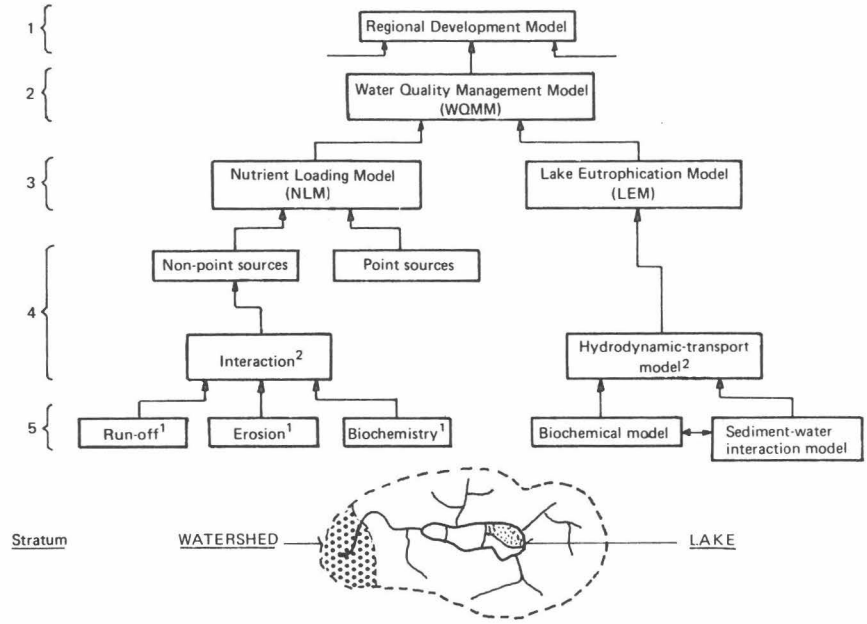

Fig. 2. The method of decomposition: hierarchy of models. (1) Submodels for uniform segments (dotted areas) and (2) coupling of the submodels.

level of integration. The procedure involving five strata will be discussed in greater detail for the Lake Eutrophication Model (LEM), with reference to models now being elaborated. The parallels in the Nutrient Loading Model (NLM) can be found through Fig. 2.

Stratum 5. First, those segments of the lake should be isolated which can be considered approximately uniform from the viewpoint of water quality (complete mixing inside each unit) and from the factors influencing them. The objective of the models on this stratum is to describe the algal dynamics and nutrient cycling for all the segments, involving both the water body and the sediment, since the latter is a sink and source of various materials and their interaction plays an important role in shallow lakes (also from the point of view of management, after a reduction in load, the new equilibrium of the lake will be determined by the nutrient release of the sediment). These kinds of models based on the mass conservation principle and formulated through a set of nonlinear ODEs are well known in the literature [22]. In the frame of the present study, three submodels, BEM, BALSECT, and SIMBAL were developed (see [13, 20, 32], with respect to their comparison [31]) which differ basically in the number of state variables (between 4 and 7) and essential parameters (10–17), as well as in the mathematical formulation of various processes and in the parameter estimation technique adopted. It is noted here that some of the parameters can be derived from further isolation up to a lower level with appropriately designed experiments. As examples, the estimation of algal growth parameters from vertical primary production measurements [33] and the study of wind induced sediment–water interaction (see Sec. 4.1) may be mentioned.

To end the discussion on stratum 5, it is stressed that several steps of the analysis are based on intuition (starting from the segmentation to the formulation of various biochemical processes), thus the inclusion of data with their uncertainties is inevitably required for justification of the assumptions made.

Stratum 4. On the next level the segment-oriented biochemical and sediment models are coupled by involving mass in- and outflows at the boundaries of the units. For this purpose, a hydrodynamic-transport model can be used. In light of the experiences gained from the study of the Great Lakes [7], it was decided not to use a coupled multidimensional hydrodynamic-transport model incorporating the submodels of the lower stratum: the gain in information is not proportional to the increase in complexity; furthermore it causes methodological difficulties as explained in Sec. 3.1. Here again, an off-line technique is applied. The basic assumption is that it is sufficient to subdivide the lake in a longitudinal direction only. This is supported by the riverine shape of the lake and the presumably extensive transversal mixing, since the prevailing wind direction is nearly perpendicular to the lake's axis (the description of the shoreline effects is not the objective here). Consequently, the parallel development of an unsteady three-dimensional (3D) and one-dimensional (1D) hydrodynamic model was decided on [23, 28]. The first can be used to derive through several simulations, the longitudinal dispersion coefficient as an "empirical" function of the major wind parameters, while the much simpler 1D model could describe convection (see Sec. 4.2). Subsequently, the submodels of stratum 5 will be incorporated in a straightforward way into a set of longitudinal dispersion equations on stratum 4. It is noted that the 1D hydrodynamic model also allows an uncertainty analysis on the wind data, a methodologically remarkable aspect (Sec. 4.2).

At this level, the 1D model was aggregated from the 3D version. A further aggregation (see Sec. 3.1) can be arrived at through the use of the coupled dispersion–biochemical model being elaborated. Provisionally, in all three biochemical models (see Sec. 2) four segments (or mixed reactors, see Fig. 1) are assumed; their coupling is based on hydrologic throughflow and a wind influenced mass exchange process described globally. Since the model structure based on ODEs has many advantages, one of the objectives of the study on the 1D coupled model is whether the four basins concept can be maintained or not.

Stratum 3. The involvement of mass exchange among segments as described before will result in the Lake Eutrophication Model (LEM) (Fig. 2) which has several forcing functions, such as solar radiation, water temperature, wind, etc. (natural or uncontrollable factors) and the nutrient load. Since the latter is the only factor to be controlled, it plays a distinguished role; however, less modelling work was done on it in the frame of the case study. This can be explained by the relatively high contribution of the sewage load (modelling is basically not needed here because of the nature of the problem) and the limited amount of watershed data available for non-point source modelling. A thorough data collection and the derivation of a nutrient balance for the whole lake were preferred (for details see [16]), the results of which were already summarized in Sec. 2. This study also involved an uncertainty analysis in relation to the unobserved contribution of floods to the load (Sec. 4.3). The research allowed the derivation of the temporal and spatial pattern of the load components in a descriptive fashion, both for LEM and the Water Quality Management Model (WQMM) on stratum 2. In fact, at the first stage of lake model development the modelling of any of the driving functions is not necessarily required; both for calibration and validation, historical data can be employed. For planning purposes the situation is different, therefore the stochastic

effects of both the load and meteorology were involved through Monte-Carlo simulation (Sec. 4.4).

Stratum 2. The objective of WQMM is to generate alternative management options and strategies (the effect of these being expressed through NLM which should be used here in a planning mode) and to select from among these alternatives, on the basis of one or more objectives. Both water quality and costs can be used as objective functions or constraints, and quite often their weighting is required. Frequently the load can replace the lake's water quality in the optimization in which case LEM is used merely to check the reaction of the lake and WQMM may have a simpler structure. Admittedly, however, the inclusion of water quality is more obvious because of the nature of the problem. This formulation, however, leads to a dilemma: how should a complex dynamic model be incorporated into the optimization framework [29]?

At this step aggregation is also needed. This starts with the selection of certain water quality indicators characterizing the large scale and long-term behavior of the system serving as a basis for decision making. Different parameters (yearly peak, different averages, duration of critical concentrations, frequency distributions, etc.) of typical water quality components (primary production, algal biomass, chlorophyll-*a*, etc.) can be employed as indicators. Subsequently, the dynamic model LEM can be used in terms of indicators established, I, under reduced loading conditions or in another way under several loading scenarios, L. Since the definition of indicators introduces temporal averaging, it is expected that the lake's response will be less complex compared to the dynamic simulation and a simple, direct $I(L)$ type relationship can be found for the new equilibrium. If such a solution has already been attained, LEM could be replaced by $I(L)$ in WQMM; an essential aggregation (see Sec. 4.5).

Among the management alternatives, only the two most important options are mentioned here: (i) tertiary treatment (point source load reduction), (ii) establishing reservoirs (consisting of two segments serving for the removal of both particulate and dissolved nutrient forms [30], respectively, at the mouth of rivers which are the recipients of point and nonpoint source pollutants. The optimization should then be based on the trade-off between the two basic alternatives, with respect to their locations (e.g., regional versus local treatment) and the spatial variation of the lake's water quality.

Stratum 1. For the sake of completeness it has to be mentioned that WQMM could be thought of as being a part of a regional development policy model forming the top of the pyramid, a field which is beyond the scope of this study.

4. ILLUSTRATION OF THE DIFFERENT STEPS OF THE APPROACH

4.1. *Wind induced sediment–water interaction* (*stratum* 5)

For studying the sediment–water interaction in lakes, several approaches are possible (see, for example [25]). In this study, yet another method was chosen [26], in recognition that when eutrophication is considered, more than just the physical processes should be examined. Daily measurements were taken for 6 months, at the midpoint (depth $H = 4.3$ m) of the Szemes basin (Basin II, Fig. 1). The measurements involved Secchi depth, temperature, suspended solids (SS), chlorophyll-*a*, and phosphorus fractions at different vertical locations. Wind velocity and direction were recorded continuously, from which hourly averages were calculated. The objective of the first part of the analysis was to describe the dynamics of the suspended solids as a function of wind.

This then allowed for a characterization of the temporal changes in the light conditions, the deposition, and resuspension of other particulate material and to some extent, also the releases of dissolved components. Here only the behavior of SS will be reported.

The analysis started from a simplified transport equation for describing the temporal and vertical changes of the average SS concentration for the basin, neglecting inflow and outflow. It was recognized, however, that the problem has an undefined boundary condition at the bottom, $z = H$, [26]

$$wc - E\frac{\partial c}{\partial z} = \phi_d - \phi_e, \tag{1}$$

where c is concentration, w is settling velocity, E is vertical eddy viscosity, and ϕ_d and ϕ_e are the fluxes of deposition and resuspension, respectively. In fact, one of the objectives of the measurements was to formulate the boundary condition. From the observations made, it appeared that the temporal changes governed the system (see Fig. 3a for the depth integrated values and wind speed, W). The $c(z)$ vertical profiles were quite uniform, except close to the bottom where the expected, but sudden increase could be observed. Accordingly, it was decided not to determine the unknown boundary condition from the PDE formulation (a rather tedious procedure), but to integrate the

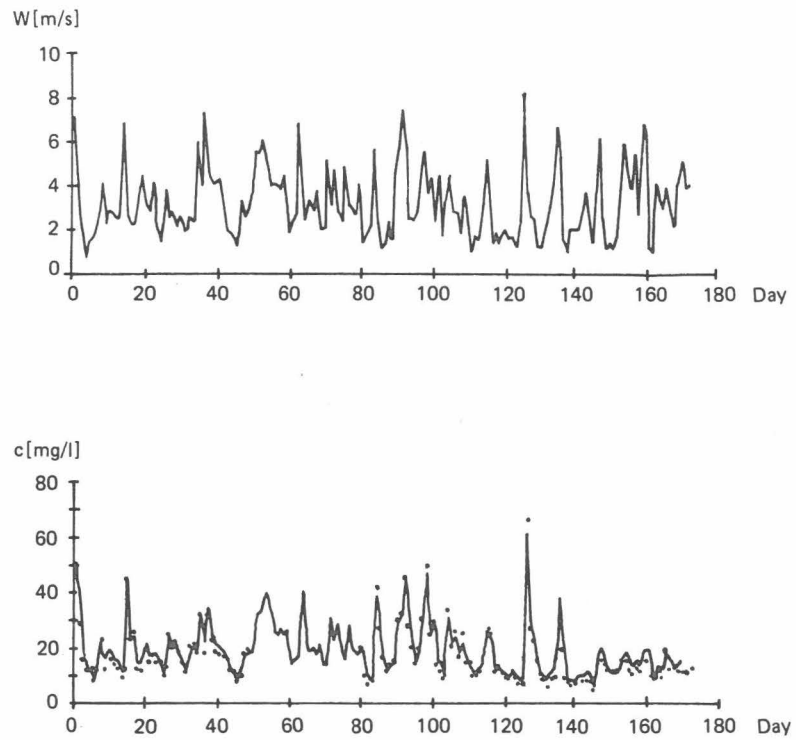

Fig. 3a. Identification and parameter estimation of a model for wind induced sediment–water interaction for Lake Balaton: recursive estimate of the suspended solids concentration; (W) daily average wind speed, (c) suspended solids concentration, (·) observations.

turbulent diffusion equation along the depth, and use the ODE derived, which thus directly involves the boundary condition itself.

In order to carry out this step, hypotheses were needed for the fluxes ϕ_d and ϕ_e. The deposition was characterized by the coefficient P expressing which portion of the particles reaching the interface would remain there:

$$\phi_d = Pw\tilde{c} \tag{2}$$

(here, the tilde indicates depth averaged value), while ϕ_e by an empirical relationship [19]

$$\phi_e = k\rho_w \frac{\rho_s}{\rho_s - \rho_w} w_e, \tag{3}$$

where ρ_s and ρ_w are sediment and water densities, and w_e is entrainment velocity. To find w_e, the concept of energy transformation between potential and turbulent kinetic energies often employed for stratified lakes was adopted [6]. Accordingly, under simplified conditions:

$$w_e \sim \frac{1}{H} W^n, \tag{4}$$

where the power depends on the Richardson number. Using these hypotheses, the depth integrated transport equation takes the following form [26]:

$$\frac{\mathrm{d}\tilde{c}}{\mathrm{d}t} = -K_1\tilde{c} + K_2 W^n, \tag{5}$$

where K_1 and K_2 comprise the unknown coefficients, derived from the hypotheses (Eqs. 2 to 4). Consequently, the structure of the model should be identified and the parameter values, K_1, K_2, and n, estimated from measurements. The feasibility of Eq. 5 can be appreciated from Fig. 3a, which clearly shows the influence of the wind velocity on the concentration. However, a simple regression between the W and SS is not precise enough; the involvement of SS in a time series fashion improves it, thus suggesting the influence of settling and deposition.

First a nonrecursive deterministic estimation technique was adopted to derive the unknown coefficients which resulted in realistic values but without proving the correctness of the hypotheses (*a posteriori* model structure identification, see [4]).

For this purpose, as a second step, the Extended Kalman Filter (EKF) method was applied [2, 5]. For the power n a value near to 1 was derived which corresponded to the small Richardson number [27]. Subsequently, n was fixed to 1 since in this case the physical interpretation of the results is more obvious. The recursive estimation started from the estimates of the deterministic technique. The results are illustrated in Fig. 3a. As is apparent, the agreement between observations and model calculation is reasonably good, and the parameters become approximately constant after the first 40–50 days (Fig. 3b), proving that the model structure is adequate [2, 4] and the data do not contain more information than described by the model. Some slight parameter changes can be observed at the end of the period; this may be caused, e.g., by the exclusion of inflow–outflow processes (or by other phenomena such as algal blooms). This suggests that the isolation of subprocesses is generally not complete. From the analysis, a realistic order of magnitude follows for all the essential physical quantities; in this connection see [27].

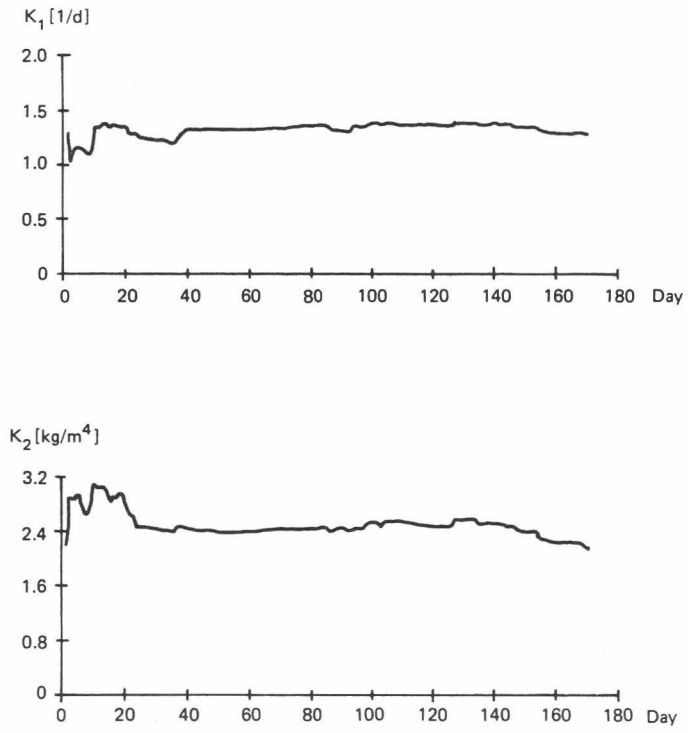

Fig. 3b. Recursive parameter estimates for the sediment–water interaction model.

As can be observed in Fig. 3, for one month in the middle of the total period, no measurements were available, so the model was used for prediction. The appropriateness of the model is also illustrated by the fact that after getting new data, the parameter values did not change. This second period served for validation, following the identification and calibration procedure.

The study, which underlines the definite need to combine both theoretical and measurement knowledge, resulted in two basic achievements: (i) the estimation of the unknown boundary condition of a transport problem (which was then also solved by using an implicit finite difference method but the "submodel" was not maintained for the complex study as the vertical changes are not essential in this case from the point of view of eutrophication), and (ii) the description of the processes of deposition and resuspension through an ODE which can be easily incorporated into the biochemical submodel with a similar ODE structure (e.g., for characterizing transparency conditions or nutrient release from the sediment).

In addition to the wind induced interaction discussed here, the sediment biochemistry is also of major importance, a field where further research is required.

4.2. *Application of a* 1D *hydrodynamic model* (*stratum* 4)

The results gained from the 3D and 1D models (for details see [23] and [28]) showed that the two models could be equally calibrated against dynamic water level data and

suggested that the cross-sectionally averaged discharge, $Q(t)$, can be properly derived from the much simpler 1D model. For establishing this latter, the complete one-dimensional equation of momentum and continuity was solved by using a conventional implicit finite difference scheme [28]. For the matrix inversion, an effective decomposition technique was developed, resulting in efficient computations [28]. Dynamic input is the longitudinal component of the wind force, while the output is the water level and streamflow rate at each cross section ($\Delta x = 2000$ m). The two parameters of the model (drag coefficient and bottom friction) were calibrated on the basis of the work of Muszkalay [21]. From the data of nearly ten years of observations, he derived empirical relationships between some typical wind parameters of a storm and the corresponding maximum reduction in water level and velocity at the Tihany peninsula (Fig. 1), respectively. For validation, more than ten historical events were selected. The results of one example are shown in Fig. 4. This event can be characterized by a long-lasting longitudinal wind followed by smaller shocks of different directions (Fig. 4a). It is apparent that the agreement between measured and simulated water levels is satisfactory (Fig. 4b). The discharge shows a striking oscillation between -2000 and $3000\ m^3/s$ (Fig. 4c) associated with the seiche phenomenon which is higher by two orders of magnitude than the hydrologic throughflow and may cause considerable fluctuation in the volume of the basins (Fig. 1). The seiche contributes to mixing only to a limited extent (note that here nonconservative materials are considered). A more essential mixing is associated with different kinds of backflow and circulation. These are not captured by the 1D model and their influence should be derived from the 3D model. In fact, a longitudinal dispersion coefficient, D_L, can be approximately calculated from the velocity field by a method similar to that employed for rivers [10]. Based on the initial experiences [24] it seems to be sufficient to perform the computations for some typical stormy events and afterward to correlate D_L to wind parameters. The relationship $D_L[W(t)]$ gained (where W is the wind speed), will allow replacement of the coupled 3D hydrodynamic-transport model by a 1D version (see Sec. 3.2).

The seiche type phenomenon also has another effect—at a given location the rapid back and forth motion will cause an oscillation of various constituents within a day, which strongly depends also on the longitudinal gradient. This may result in quite a critical error in the concentration determined through instantaneous sampling, a recognition which will allow definition through the model of an uncertainty range of historical measurements.

To find a more satisfactory agreement between model simulation and observation than in the previous example is often impossible. The reason is quite simple: a small error in the wind direction may cause a drastic change in the wind force, if the direction is far from the longitudinal one, i.e., transversal wind conditions. In fact, there are many kinds of uncertainties in the wind direction, such as random fluctuation (turbulence), the influence of hills on the northern side of the lake, which cause nonuniformities in the wind field, measurement errors, etc., with the effects illustrated in Fig. 5. A deterministic simulation did not prove acceptable. Bearing in mind the possible role of uncertainties, a random component was subsequently added to the wind direction (Gaussian distribution, zero mean, 17° standard deviation: a modest value) and a Monte Carlo simulation was performed. Figure 5, which shows the major results for 100 runs, does not require detailed discussion: it stresses the extreme sensitivity to input data uncertainty (compared to this the parameter sensitivity is negligible) and illustrates how difficult it is to validate a deterministic model (to a lesser extent this is also true for a multidimensional model). This behavior also suggests that some time averaging should be introduced for the transport model part. According to our analysis, the mean and variance of the flow rate time series on a daily basis shows limited sensitivity only. Since generally one is not interested in

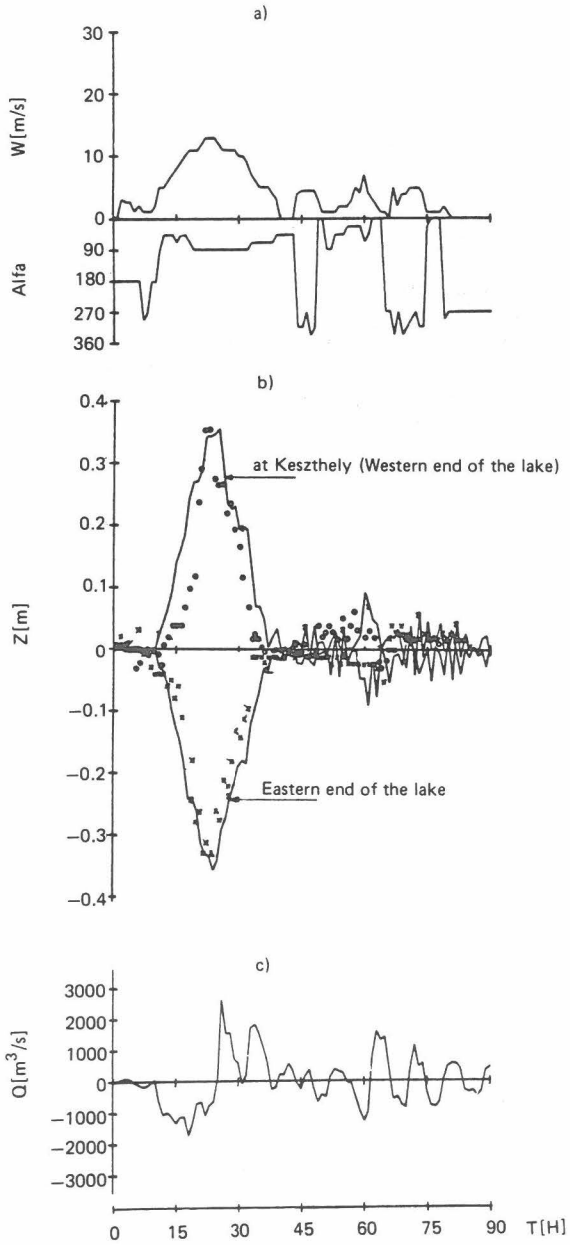

Fig. 4. Simulation of a historical event: longitudinal wind conditions. (a) Wind record [21], W speed, Alfa angle (North $\equiv 0°$); (b) comparison of simulated and observed water levels ($T = 0$ corresponds to 16/11/1966, 8 a.m.). Dots indicate measurements [21]; (c) computed streamflow rate at Tihany.

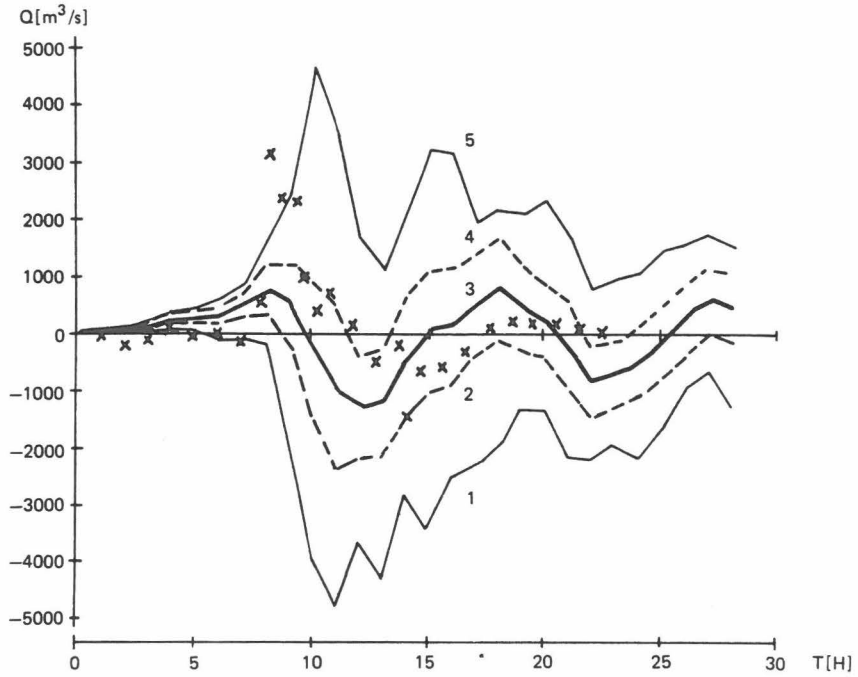

Fig. 5. The influence of wind data uncertainty on the discharge at the Tihany peninsula ($T = 0$ corresponds to 8/7/1963, 8 a.m.); (×) discharge derived from measurements [21]; (3) mean value; (4 and 2) ± standard deviations; (5 and 1) extreme values.

short-term concentration changes (the character of the sampling problem mentioned before is different) this allows use of the mean value for convection in the dispersion model, while the effect behind the variance can be incorporated in the dispersion coefficient.

The example suggests that although a deterministic hydrodynamic model can hardly be verified in a strict sense, the verification can be done for a transport model in a water quality study by filtering out the influence of uncertainties.

4.3. *The nutrient load under uncertainty and stochasticity* (*stratum* 3)

In accordance with the nutrient loading estimate done [17], more than 40% of the total phosphorus (TP) load reaches the lake through tributaries. As is well known, floods play a decisive role in the yearly total transport, their contribution ranging generally between 70% and 90%. This fact is in most of the cases not reflected in the monitoring strategy; generally one or two observations made monthly at the mouth of the river are available. Thus, the influence of floods remains unobserved. The infrequent data collection is characteristic for 19 of the 20 tributaries of Lake Balaton, while for the major pollution source, the Zala River which accounts for more than half of the tributary load, daily measurements were performed (1975–1979 [16]).

As the "accurate" load for different averaging periods (such as a month or year) can be derived from this data set, it allows one to study the error caused by scarce

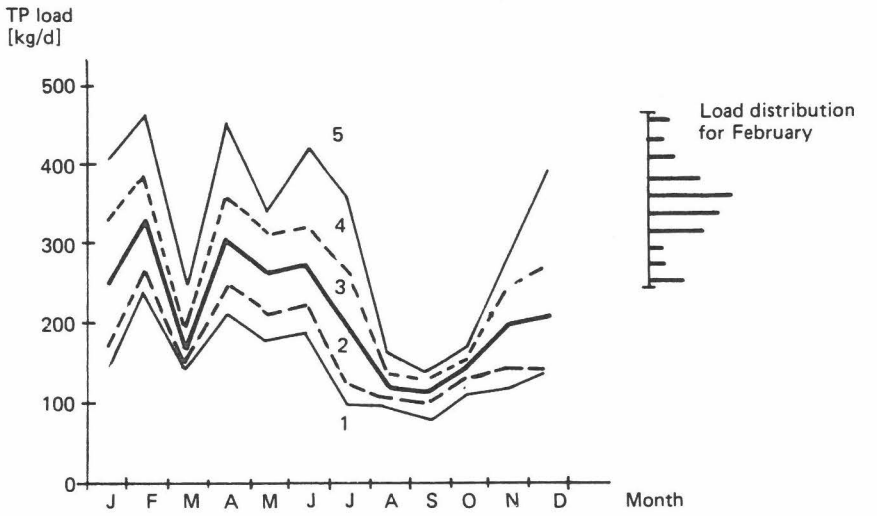

Fig. 6. Monthly average TP load: uncertainty caused by scarce observations (Zala River, 1976–1979); (3) mean value; (4 and 2) ± standard deviations; (5 and 1) extreme values.

observations. The procedure is a straightforward Monte Carlo type technique which starts with a random selection from the detailed data set following the sampling strategy of the other tributaries and calculates the load of the period in question. After making a sufficient number of random selections the statistical parameters of the load can be determined. The results for the long-term monthly average load (on the basis of a four-year-long observation period and 200 data selections for each month) are illustrated in Fig. 6. The choice of a month was made for two reasons: (i) being in possession of two monthly observations for a period of several years only the monthly average load can be estimated at best; (ii) from a sensitivity study on LEM [29], it turned out that it is sufficient to use this load type as a forcing function. As can be seen from Fig. 6, which shows the mean and extreme values, as well as the domain of ± standard deviation, the error is quite high and its fluctuation follows the change in the mean value. On the basis of this study and a similar analysis for the yearly averages, the annual load of other tributaries was corrected (by assuming that most of the floods are unobserved for these rivers) in an extrapolative character and a random component was added to the monthly average load component [29].

To incorporate the stochastic influence of the hydrologic regime a regression analysis was done on the data set of the Zala River. It was found that the monthly average TP load correlated satisfactorily with the corresponding streamflow rate. Accepting the statistics of the monthly average streamflow from long-term observations [1] the load can be calculated in a stochastic fashion. Figure 7 shows the average annual characteristics of the load pattern for 1976–1979 (from observations) and the 90% confidence levels derived for the long-term load. To illustrate the influence of the hydrologic regime, an event of low probability in July 1975 is likewise indicated.

As a final output of the research outlined in this section a load scenario generator was developed for the whole lake, which accounted for both uncertainty and stochasticity, discussed above. For further details see [29].

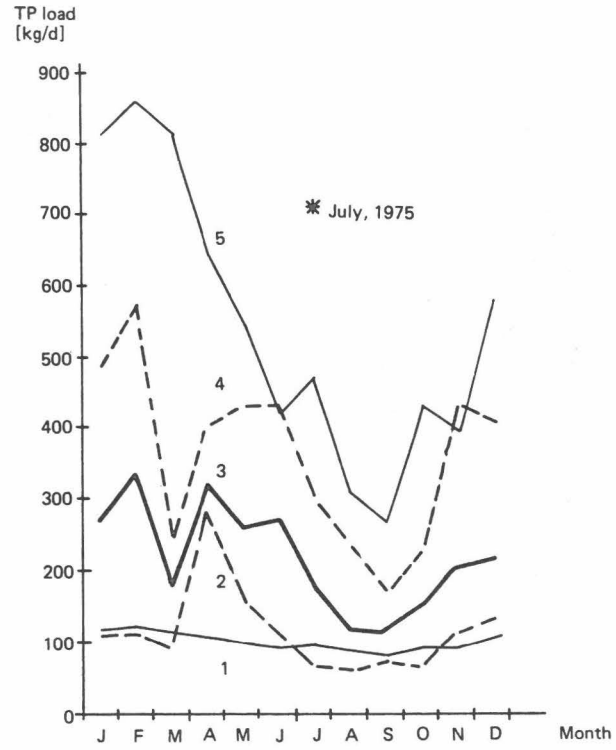

Fig. 7. Influence of the hydrologic regime on the monthly average load, Zala River; (3) average load; (4 and 2) maximum and minimum observed load; (5 and 1) 90% confidence interval.

It is noted here that using historical data, a similar analysis was made on climatic (uncontrollable) factors, which allowed the water temperature and daily radiation to be generated in harmony with each other, in a random fashion [29]. Thus, future scenarios can be generated for all the essential forcing functions of the lake model—an essential tool for planning purposes.

4.4. *The lake eutrophication model (stratum 3)*

The preliminary results gained with the simplest model, SIMBAL [32], developed for Lake Balaton are given below. The model is a phosphorus cycle model, that is, all the state variables (the essentials are two algal groups, detritus, and dissolved inorganic phosphorus) are expressed in terms of phosphorus, for the four basins indicated in Fig. 1 (see Sec. 3.2). A Monte Carlo simulation is incorporated into the model to find areas in parameter space where the model produces results fully within specified boundaries drawn around the data to account for data uncertainty (see, e.g., Sec. 3.2) and thus easily applicable to test various hypotheses ([32] and also [9, 14]).

Among the calibration runs, results for the phytoplankton phosphorus, PPP, for the

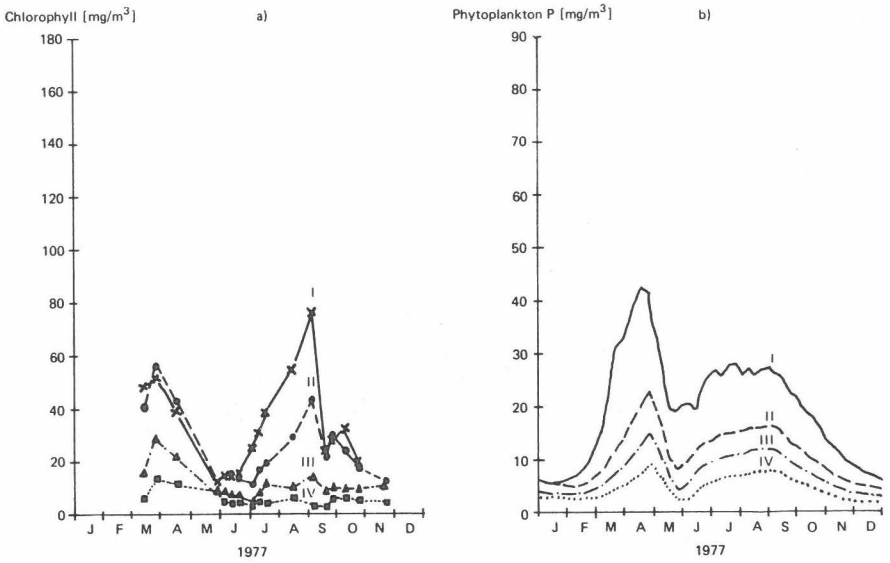

Fig. 8. Results from SIMBAL. Comparison of field data for four basins (I–IV) (left) and average model for runs satisfying the behaviour definition (right). Adopted from van Straten [29, 30].

four basins are given in Fig. 8 (1977 forcing data were used) together with the corresponding observation variable, chlorophyll-*a* (basin average values). It is pointed out that chlorophyll-*a* and PPP cannot be directly compared to each other; however, since a more or less linear measurement equation is expected among them, PPP should follow the pattern of chlorophyll-*a*: a trend which can be generally observed. Further discussion on the calibration and model improvement required can be found in [32].

For management purposes, historical data cannot be used. Either some critical, unfavorable environmental conditions should be introduced or the model should be considered stochastic through input data which are basically random variables when future planning is in question. Here the latter approach was adopted and the generators outlined in the previous section coupled to the lake model. Two essential results for Basin 1 are presented in Figs. 9 and 10.

In the first case, uncertainties caused by natural factors were considered and the 1977 load was maintained. The summary of 100 Monte Carlo runs (mean, ± standard deviation, and the extremes of PPP) suggests the relatively large sensitivity of the lake's water quality to meteorological factors and explains the essential year to year changes observed in the behavior of the lake even when the load remained unaffected. The second case (Fig. 10) involved the random generation of both natural and controllable factors. While for the previous example the specific 1977 load was adopted, here the mean load of the input generator was derived from data for the period 1975–1979 (Sec. 4.3). This is the reason why the average trajectory shown in Fig. 10 differs from that shown in Fig. 9. The inclusion of load randomness had an obvious influence: the range of uncertainty of the water quality simulation results became much wider.

Compared to the role of parameter uncertainty (Fig. 8), the meteorological factors represent the same order of magnitude, while the contribution of load to the model

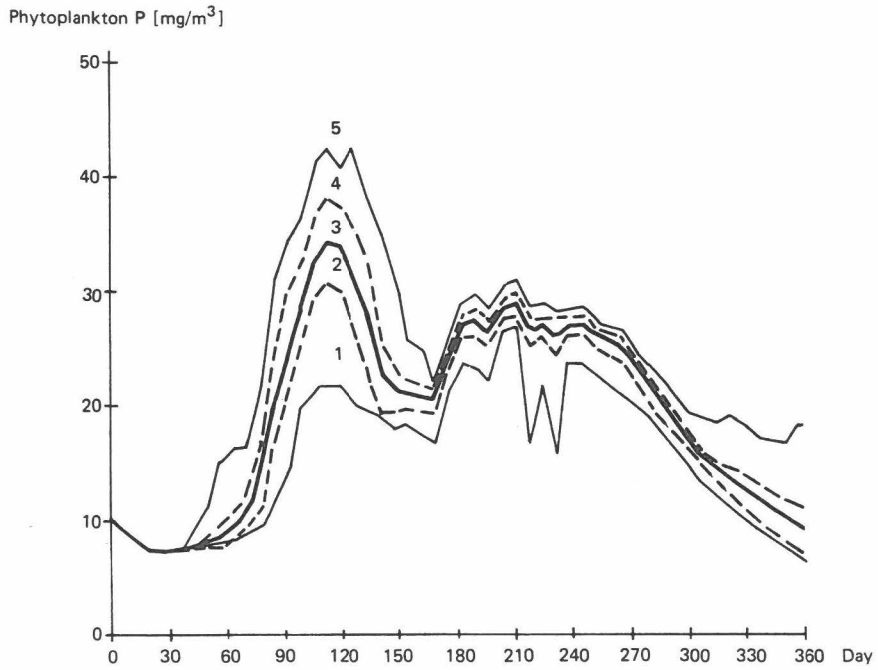

Fig. 9. The influence of meteorologic factors on the water quality (Basin 1); (3) mean value; (4 and 2) ± standard deviations; (5 and 1) extreme values.

uncertainty is twice as high. It is stressed that in the frame of the present example, no load reduction was employed. Thus, Fig. 10 shows in which domain the water quality may range under the present conditions, since the changes in trend are already relatively small from year to year. Control alternatives will certainly modify not only the mean load but also the related uncertainties [e.g., the smoothing effect of a retention pond or the uncertainty of the effectiveness η, of a given management option (Fig. 11)], an issue which should be taken into account on the level of decision making.

4.5. *The incorporation of the lake model into* WQMM (*stratum* 2)

The question that we are planning to answer here was raised in Sec. 3. In the course of the analysis outlined subsequently, the different parameters of the PPP(t) distribution were selected as water quality indicators characterizing the algal behavior (see Sec. 3.2), and deterministic simulations were performed with the dynamic lake model, SIMBAL, under reduced loading conditions. It turned out from the study that the lake's response expressed in terms of the yearly average or peak of PPP is quite linear on the load in a wide range [29] (the same experiences were also gained with the two other models). It is noted that a similar linearity is expressed for total phosphorus, TP, by several empirical models [34]. However, for shallow lakes, TP may not properly characterize the process of eutrophication since this component is strongly influenced by wind induced interaction at the bottom (Sec. 4.1).

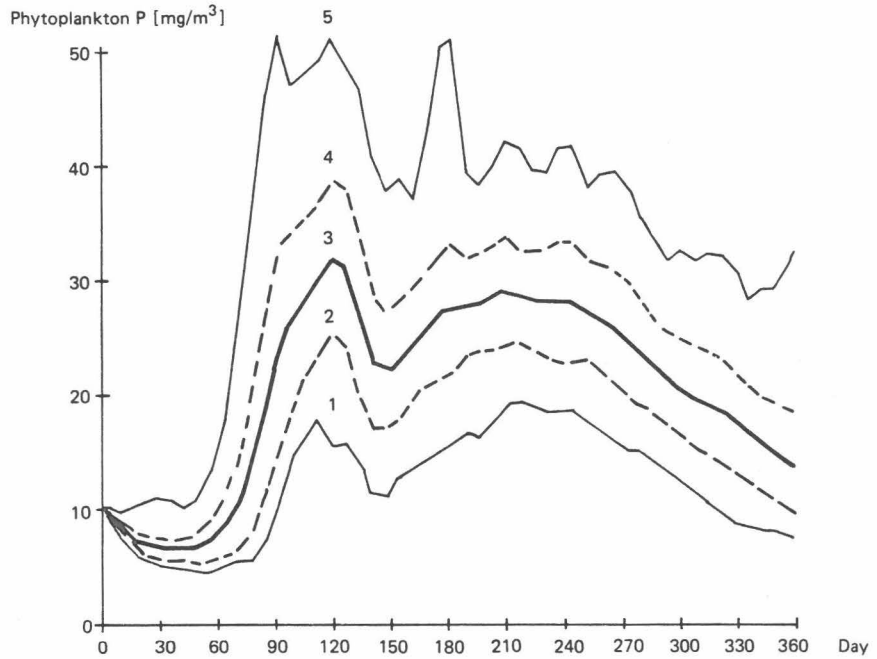

Fig. 10. The combined influence of uncertainty and stochasticity in the meteorology and loading, respectively, on the water quality (Basin 1). (3) mean value; (4 and 2) ± standard deviations; (5 and 1) extreme values.

The recognition of the linearity leads to an important aggregation: the dynamic lake model can be replaced at the level of WQMM (see Fig. 2) by a simple linear equation (see Fig. 11):

$$\mathbf{c} = \mathbf{c}_0 + \mathbf{A}(\mathbf{L} - \mathbf{L}_0), \tag{6}$$

or

$$\Delta\mathbf{c} = \mathbf{A}\,\Delta\mathbf{L}. \tag{7}$$

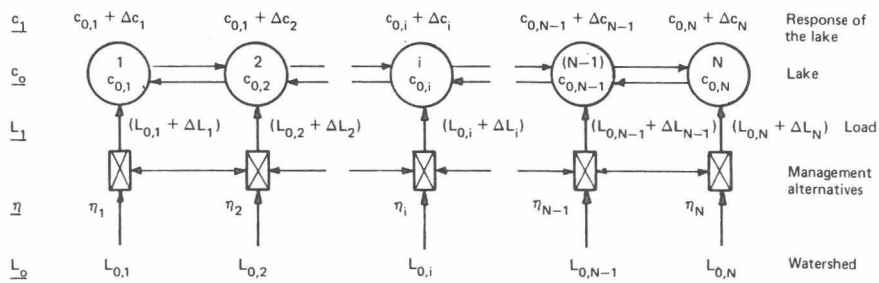

Fig. 11. Incorporation of the lake model in WQMM. (*i*) uniform segments in the lake, (⇆) interaction among basins through mass exchange and hydrologic throughflow, (←→) interaction among load components through control activities.

Here $\mathbf{L}_0$ and $\mathbf{c}_0$ are the initial load and concentration "vectors," respectively, defined by the number of uniform segments assumed in LEM (at present $N = 4$, Fig. 1); $\Delta\mathbf{L}$ represents the reduction of load achieved by various control alternatives, while $\Delta\mathbf{c}$ is the corresponding long-term response of the lake (expressed in terms of water quality indicators) characterizing the new equilibrium of the system. The elements of the A matrix maintain the essence of LEM: a transition from a "large" model to a "smaller" on a higher level of the hierarchy of Fig. 2. The first remarkable feature of Eqs. 6 and 7 is that they clearly preserve the influence of subprocesses on the lower strata and show the subsequent steps of aggregation. The elements of the main diagonal are determined primarily by biochemical processes and sediment–water interaction (stratum 5), while the other elements mainly express the influence of hydrodynamics and associated mass exchange (stratum 4), showing that a management action at the region of the ith segment will affect the water quality of other segments as well (Fig. 11). It is briefly noted that through LEM (stratum 3), the uncertainties discussed in the previous section can also be included in Eq. 6.

The second remarkable feature of Eq. 6, and this is of primary importance on stratum 2, is that its linear structure allows its direct involvement in an optimization framework: a solution that we were looking for (see Sec. 3.2).

At the end of this section, it is noted that optimization of this kind is not an easy task. First of all each $L_{0,i}$ element is composed of various types of nutrients (sewage, diffuse sources, etc.) which in a general case has different spatial locations within the corresponding region. Thus, $L_{0,i}$ itself is a vector (in other words, several homogeneous segments of the watershed may belong to a given ith uniform segment of the water body). Second, there are several options for loading reduction and the same ΔL_i could be arrived at by different control strategies. In addition, just as with the linkage of the c_i elements through in-lake mass exchange processes (Fig. 11), the $L_{1,i}$ components are coupled through realization of the desired management strategy implemented in the watershed (e.g., regional treatment and the related sewage system). All these are features of the problem that should be accounted for at the level of WQMM. In fact, such an approach is also being elaborated within the case study [15, 18].

A further difficulty is associated with the problem of how to handle the various kinds of uncertainties discussed before, in the course of optimization: an issue which is not yet explored satisfactorily for water quality problems.

5. CONCLUDING REMARKS

In this paper, a system characterized by two major factors was considered: (i) complexity due to the multifarious interactions of in-lake and watershed processes, respectively, and different levels of interest such as scientific understanding and decision making, and (ii) the uncertainty of various sources. The objective is also twofold, not only to study the specific issues of this system, but also to answer general methodological questions for systems with similar features.

The modelling strategy adopted is an off-line procedure where the complex structure is decomposed into units, tackled separately in detail, but preserving and coupling only the essentials on the higher level of the modelling hierarchy. The method is characterized by several steps of aggregation (based on sensitivity analyses in order to find the relative importance of various subprocesses), and validation (detailed models against field data and the aggregated models against their detailed versions and data on the higher level of the modelling hierarchy).

A harmony is also needed in the methodologies employed; none of them should exclude the application of techniques required after the coupling (e.g., parameter

estimation) and they should allow the proper combination of theory and observations in the model. Uncertainty should be involved at each step and also aggregated simultaneously with the off-line construction of the complex model.

Admittedly the coupling, aggregation, and validation of the linked model are the crucial points in the development: it is not certain whether the simplifications planned can be performed and whether a model of realistic structure and complexity can be arrived at on the higher level.

Although the study is not yet finished and future research is certainly required (the role of sediment, completion of coupling the segment models and that of the water quality management part) the individual examples introduced show that the modelling principles outlined here can be successfully adopted for the different strata and suggest that this is also valid for the entire model incorporating the level of decision making. It is believed, too, that other systems characterized by complexity and uncertainty can be treated in a similar way; thus models of reasonable size and structure can be gained.

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