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SOME PROBLEMS ON THE STOCHASTIC FLOOD CONTROL

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Some Problems on the Stochastic Flood Control

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1. Let us consider a system of n water reservoirs $R_1,\ldots,R_n$. For a generality let us assume they are connected in such a way that we can transfer an amount of water

$$\int_{t_0}^{T} u_{ij}(t)\, dt$$

from R_i to R_j during the time period $[t_0,T]$ and an amount of water

$$\int_{t_0}^{T} u_{ii}(t)\, dt$$

from R_i to outside where control parameters u_{ij}, $i,j = 1,\ldots,n$ satisfy to constraints

$$a_{ij} \le u_{ij} \le b_{ij} \quad .$$

Therewith a_{ij}, b_{ij} are given non-negative constants--for example, a_{ii} is determined with respect to minimum demands and b_{ii} means the level under which exceeding it might cause a lot of damage. Thus, the total amount of water that passes through the reservoir R_i in a "normal way" during the time period $[t_0,T]$ is

$$\int_{t_0}^{T} \sum_{j=1}^{n} u_{ij}(t)\, dt \quad ,$$

where conventionally $u_{ij}(t) = -u_{ji}(t) < 0$ if actually at the moment t the water comes to the reservoir R_i from the reservoir R_j.

Suppose the target is to accumulate water during some time period $[t_0,T]$. If $\xi_i(t)$ is a corresponding inflow (input) for R_i, then up to the moment T the amount of water $W_i(T)$ in the reservoir R_i is

$$W_i(T) = W_i(t_0) + \int_{t_0}^{T} [\xi_i(t) - \sum_{j=1}^{n} u_{ij}(t)]\, dt \quad .$$

Let the capacity of R_i be W_i^*. Then the negative value $W_i(T) - W_i^*$ means that we do not have enough water at the end of the considered period $[t_0,T]$ and the positive value $W_i(T) - W_i^*$ means that we actually have the "overflow."

The problem is to choose the control functions $u_{ij}(t)$, $t_0 \leq t \leq T$ in such a way as to minimize the mean square value

$$E\,||\,W(T) - W^*||^2$$

of the deviation of the random vector

$$W(T) = \{W_1(T),\ldots,W_n(T)\}$$

from the vector $W^* = \{W_1^*,\ldots,W_n^*\}$.

Of course the optimal control depends very much on the actual input $\xi(t) = \{\xi_1(t),\ldots,\xi_n(t)\}$, $t_0 \geq t \leq T$. We suggest to assume that

$$\xi(t) = x(t) + w(t)$$

where

$$x(t) = \{x_1(t),\ldots,x_n(t)\}$$

is a multi-dimensional, Gamma-distributed stationary random process of the Markov type;[1] and $w(t) = \{w_1(t),\ldots,w_n(t)\}$ is a "big wave" (because of melting snow or rainfall) with components $w_i(t) = w_i(z_i^*,t)$ which are solutions of the corresponding hydrodynamic partial equation[2] of the St. Venant type for discharges $w_i(t,z)$ along "bed-streams" $z_i^0 \leq z \leq z_i^*$ with "initial conditions" of the type

$$w_i(z_i,t) = f_i(t - \tau_i) \quad .$$

Herewith

$$f_i(t) = \begin{cases} P_i(t)\, e^{-c_i t} \quad , & t \geq 0 \\ \qquad 0 \quad , & t < 0 \end{cases}$$

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[1]For example,

$$x_i(t) = \eta_i(t)^2 \quad , \qquad i,\ldots,n$$

where $\eta_1(t),\ldots,\eta_n(t)$ are independent Gaussian stationary Markov processes with corresponding parameters.

[2]See J. Stoker [1].

(where P_i is a polynomial, c_i is a positive constant); the variable τ_i is assumed to be random with the exponential distribution and means the moment at which the wave $w_i(t)$ rises.

In order to deal with the problem on the optimal control in an analytical way, we suggest to assume that

$$w_i(t) = w_i(z_i^0, t) \quad .$$

2. Suppose now the target is "to catch" the wave $w(t)$ in order to protect the overflow, and all reservoirs $R_1, \ldots, R_n$ are some kind of a "trap" for this wave.

Let us assume for simplicity that $R_1, \ldots, R_n$ are not connected so we can pose a problem for $R = R_1$ when we deal with the univariate random process $\xi(t) = \xi_1(t)$ described above.

Suppose that during a time period $t_0 \leq t \leq \tau$, where τ is some unobservable crucial moment,[3] we wish to operate in such a way that the whole inflow $\xi(t)$ goes through[4] so the reservoir has a constant volume (for example, it is empty); but after the crucial moment τ, we wish to stop the outflow (i.e. $u(t) = 0$) so the whole inflow goes in the reservoir.

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[3]For example, $\tau = \tau_1$ is the moment at which the wave $w(t) = w_1(t)$ comes to the system, or τ differs from this moment on a proper time delay.

[4]It means essentially that there are no constraints on outputs $u(t) = u_{11}(t)$.

Because the random moment τ(say with the exponential distribution) is unobservable, we have to use some estimate τ^*. If $\tau^* > \tau$, then some part of the undesirable inflow goes through the system--the corresponding amount of water will be

$$W(\tau^*) = \int_{\tau}^{\tau^*} \xi(t)\, dt \quad ;$$

if $\tau^* < \tau$, then some part of the undesirable inflow goes in the reservoir--the corresponding amount of water will be

$$W(\tau^*) = \int_{\tau^*}^{\tau} \xi(t)\, dt \quad .$$ [5]

The problem is to choose the control parameter τ^* in such a way to minimize--in a proper sense--$W(\tau^*)$, e.g. to minimize the expectation $EW(\tau^*)$ or the probability,

$$P\{W(\tau^*) \geq W^*\} \quad ,$$

where W^* is some crucial level (or to minimize similar probabilities under conditions $\tau^* < \tau$, $\tau^* > \tau$).

[5]This water occupies some part of the reservoir and it leaves space for the "wave" $w(t)$ itself.

References

[1] Stoker, J. Water Waves. New York, Wiley, 1957.