WORKING PAPER

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ABSTRACT

In this paper the usefulness of qualitative calculus for spatial impact analysis will be explored. The major issue of qualitative calculus concerns the sign-solvability problem. Its basic question is whether, and under what conditions, the direction of changes in a set of dependent variables in an interconnected system may be determined solely from the direction of changes in the independent variables and a knowledge of the signs, but not the magnitudes, of the relevant partial derivatives. After an elaboration of this problem in Section 2, an overview will be given in Section 3 of the conditions under which full or partial sign-solvability may occur. A discussion of how this approach might be applied in urban and regional planning is given in Section 4, and the paper concludes with some suggestions for further research.

1. Introduction

The critical assessment of proposed courses of action is an important part of a planning process, and many impact analysis models are available to support these assessments (see, for instance, Batty, 1976; Putman, 1979; and Wilson, 1974). Here the term *impact analysis* means the *a priori* determination of the relevant consequences of system changes that would arise from specific policy measures. *Spatial impact analysis* may be seen as a particular type of impact analysis in that it emphasizes the spatial dimension of these consequences (see also Nijkamp, 1979, 1981, 1982).

Planning-oriented research in general, and spatial impact analysis in particular, must usually be carried out under a number of constraints (see also Voogd, 1982), which include:

* the speed with which results have to be produced;
* the limited amount of suitable quantitative data;
* the limited availability of (skilled) research (support) staff.

As a consequence, very few quantitative mathematical models have actually been used successfully in urban and regional planning. During the last few years, increasing attention has therefore been paid to approaches which:

* are able to deal with *qualitative* information, so that cumbersome data-gathering activities are no longer required in order to reach meaningful conclusions;
* are *flexible* with respect to the inclusion of new information and/or to new circumstances;
* enable decision-makers to assess (the consequences of) the underlying *assumptions* and *value judgements*.

In this paper this line of thought will be continued by investigating the usefulness of ***qualitative calculus*** for spatial impact analysis. The idea of qualitative calculus (or the calculus of qualitative relations) was originally suggested by Samuelson in 1947. The major issue of qualitative calculus, the ***sign-solvability problem***, will be discussed in the next section. The basic question is whether, and under what conditions, the direction of changes in a set of dependent variables in an interconnected system may be determined solely from the direction of changes in the independent variables and a knowledge of the signs, but not the magnitudes, of the relevant partial derivatives. After the elaboration of this problem in Section 2 an overview will be given in Section 3 of the conditions under which full or partial sign-solvability may occur. A discussion of how this approach might be applied in urban and regional planning is given in Section 4, and the paper concludes with some suggestions for further research.

2. The Problem

The theory of qualitative calculus is closely related to the concept of comparative statics usually employed to examine the effect on an equilibrium configuration of a system of a change in one or more of the exogenous variables. The comparative static properties of a system can be derived in a straightforward way by differentiating the equilibrium equations - the qualitative calculus approach is outlined below.

Suppose the system under consideration is described by n endogenous variables, with values x_i $(i=1,2,..,n)$, and m exogenous variables, which are represented as α_k $(k=1,2,..,m)$. The exogenous variables may be described as "system parameters", or briefly "parameters", and the endogenous variables simply as "variables". The variables and parameters are linked by fundamental relationships of the form $f_i(x_1, x_2, \ldots, x_n; \alpha_1, \alpha_2, \ldots, \alpha_m)$.

For given values of $\alpha_k = \alpha_k^o$ $(k=1,2,..,m)$ it is postulated that an equilibrium position can be defined as a set of values $\bar{x}_i$ $(i=1,2,..,n)$ such that

$$f_i(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n; \alpha_1^o, \alpha_2^o, \ldots, \alpha_m^o) = 0 \quad (i=1,2,..,n) \tag{2.1}$$

The problem is now to determine the changes in the equilibrium values of the variables brought about by changes in one or more of the parameters. If we assume that small changes in α_k occur, then the change in the equilibrium values of the variables can be obtained by differentiating (2.1) totally, which gives:

$$df_i(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n; \alpha_1^o \cdot \alpha_2^o, \ldots, \alpha_m^o) = 0 \quad (i=1,2,..,n) \tag{2.2}$$

$$\sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j} dx_j + \sum_{k=1}^{m} \frac{\partial f_i}{\partial \alpha_k} d\alpha_k = 0$$

It is assumed in equation (2.2) that all of the functions are differentiable. The partial derivatives are evaluated at the equilibrium position $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n; \alpha_1^0, \alpha_2^0, \ldots, \alpha^{o_m})$. In addition, it is postulated that an equilibrium position exists for any configuration of parameter values; i.e., the changes in the variables dx_j are such that (2.1) is in equilibrium at parameter values of both α_k^o and $\alpha_k^o + d\alpha_k$.

Equation (2.2) can be rewritten in matrix notation by defining:

$$\mathbf{A} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \tag{2.3}$$

$$\mathbf{x}' = [dx_1, dx_2, \ldots, dx_n] \tag{2.4}$$

and

$$\mathbf{b}' = \left[\sum_{k=1}^{m} (\partial f_1 / \partial \alpha_k) d\alpha_k \, , \, \sum_{k=1}^{m} (\partial f_2 / \partial \alpha_k) d\alpha_k \, , \, \ldots\ldots \, , \, \sum_{k=1}^{m} (\partial f_n / \partial \alpha_k) d\alpha_k \right] \quad (2.5)$$

which implies that equation (2.2) can be expressed by

$$\mathbf{A}\,\mathbf{x} = -\mathbf{b} \quad (2.6)$$

Equation (2.6) clearly represents a linear system which is well-known in comparative statics. Given matrix **A** and vector **b**, the problem is to solve for vector **x**. If **A** is nonsingular, the solution to (2.6) is given by

$$\mathbf{x} = -\mathbf{A}^{-1}\mathbf{b} \quad (2.7)$$

It is usually assumed in qualitative calculus that the only information available to solve (2.7) is information concerning the signs of the entries in **A** and **b**. These signs are defined as follows:

$$\text{sgn } a_{ij} = \begin{cases} + \text{ if } \partial f_i / \partial x_j > 0 \\ 0 \text{ if } \partial f_i / \partial x_j = 0 \\ - \text{ if } \partial f_i / \partial x_j < 0 \end{cases} \quad (2.8)$$

and

$$\text{sgn } b_i = \begin{cases} + \text{ if } \sum_{k=1}^{m} (\partial f_i / \partial \alpha_k) d\alpha_k > 0 \\ 0 \text{ if } \sum_{k=1}^{m} (\partial f_i / \partial \alpha_k) d\alpha_k = 0 \\ - \text{ if } \sum_{k=1}^{m} (\partial f_i / \partial \alpha_k) d\alpha_k < 0 \end{cases} \quad (2.9)$$

It was Samuelson (1947) who first noted that there are 3^n possible sign patterns for vector **x** in (2.7) and hence qualitative information on **A** and **b** must be sufficient to eliminate $3^n - 1$ of these. Therefore, if the signs of the elements of **A** and **b** are known, it is usually possible to determine one or more of the signs of the entries of vector **x**.

This may be best illustrated by means of a simple example. Consider the following model:

$$A = f(B,p)$$
$$B = g(C,p)$$
$$C = h(A)$$

where A,B and C are endogenous variables and p a system parameter (exogenous variable). Differentiating these equations gives the following result:

$$dA - f_B dB - f_p dp = 0$$
$$dB - g_C dC - g_p dp = 0$$
$$dC - h_A dA = 0$$

where f_B, f_p, g_C, g_p, and h_A are partial derivatives. Displaying this information in the form of equation (2.3) yields the following structure for matrix **A**, where rows and columns are labelled for ease of identification:

.	.	dA	dB	dC
f	.	1	$-f_B$	0
g	.	0	1	$-g_C$
h	.	$-h_A$	0	1

and for vector **b**

dp
f_p
g_p
0

The next step is to judge the signs of the various derivatives. Let us take the given signs as they are except that (- f_B) will be positive. Hence, we obtain the following qualitative system:

$$\begin{bmatrix} + & + & 0 \\ 0 & + & - \\ - & 0 & + \end{bmatrix} \cdot \begin{bmatrix} dA \\ dB \\ dC \end{bmatrix} = \begin{bmatrix} + \\ + \\ 0 \end{bmatrix}$$

Since the determinant of the sign matrix is positive, a solution can be found using (2.7). This gives the following result:

$$\begin{bmatrix} dA \\ dB \\ dC \end{bmatrix} = \begin{bmatrix} + & + & + \\ - & - & - \\ - & - & + \end{bmatrix} \cdot \begin{bmatrix} + \\ + \\ 0 \end{bmatrix} = \begin{bmatrix} + \\ - \\ - \end{bmatrix}$$

As we can see, the postulated sign patterns in this example suggest that an increase in variable A and a decrease in variables B and C can be expected. This example is said to be ***fully sign-solvable***. It will be obvious that there are many cases in which only a part of the signs of the solution vector may be determined. Such systems are said to be ***partially sign-solvable***.

So far, qualitative calculus has received particular attention in economics (e.g., see Allingham and Horishuima, 1973; Rader, 1972), ecology (e.g., see Jeffries, 1974; Levins, 1974) and mathematics (e.g., see Klee and Ladner, 1981; Maybee and Quirk, 1969; Michel and Miller, 1977). Apart from the sign-solvability problem, much attention has been paid to the stability properties of the systems under consideration. The main reason for this is that changes in a system might lead to time paths (for the variables) that do not approach any equilibrium position (see, *inter alia*, Quirk, 1981; Quirk and Ruppert, 1965; Jeffries *et al.*, 1977). This stability problem is especially important in long-term predictive studies of large-scale (e.g., global) systems. Since a qualitative spatial impact analysis does not focus on long-term predictions, but more on the expected consequences of a policy measure *given the present structure of the system*, the stability problem will not be considered any further here.

3. Conditions for Sign-Solvability

One of the first attempts to find the necessary and sufficient conditions for full sign-solvability was made by Lancaster (1962), who tried to establish a general sign pattern ("standard form") within which all sign-solvable systems can be accommodated. Instead of using a matrix **A** and vector **b**, he considered an augmented matrix **C** of order n x $(n+1)$, thus converting system (2.6) into

$$\mathbf{C}\,\mathbf{y} = 0 \qquad (3.1)$$

where

$$\mathbf{C} = [\,\mathbf{A} : \mathbf{b}\,] \qquad (3.2)$$

$$\mathbf{y} = \begin{bmatrix} \mathbf{x} \\ d\,\mathbf{b} \end{bmatrix} \tag{3.3}$$

Lancaster (1962) noted that a qualitative solution of system (3.1) does not simply depend on the number of zeros, minus signs and plus signs in **C**, but also on their arrangement in the matrix. He further noted that the following ***manipulations of the pattern of signs*** could be carried out without affecting the signs of the solution vector **y**:

(1) interchanging any two rows of **C**;

(2) interchanging any two columns of **C**;

(3) reversing all the signs in any row of **C**.

The first operation changes only the order in which the equations are written and does not affect the solution vector **y**. The second operation changes the order of the variables and, as a consequence, the order of the entries of the solution vector. The third operation implies multiplying an equation by (-1) and this does not affect the properties of **y** either. Finally, it is also possible to

(4) reverse all the signs in any column of C,

which implies the replacement of a particular y_i by $-y_i$. This will naturally affect the solution vector but can of course be taken into account when interpreting the outcome of a qualitative solution.

Lancaster showed that matrix **C** is fully sign-solvable if it can be rearranged, using the operations described above, into the following pattern of signs:

$$\text{sgn}\,\mathbf{C} = \begin{bmatrix} - & + & + & + & . & . & . & + \\ 0 & - & + & + & . & . & . & + \\ 0 & 0 & - & + & . & . & . & + \\ . & . & . & - & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & . & - & + \end{bmatrix} \tag{3.4}$$

Any system whose sign matrix can be manipulated into this standard pattern is completely qualitatively determinate. Gorman (1964) showed, however, that pattern (3.4) is a sufficient but not necessary condition for full sign-solvability. Subsequently, in reaction to Gorman's article, Lancaster (1964) improved (3.4) by providing a more general standard pattern, viz.:

$$\text{sgn}\,\mathbf{C} = \begin{bmatrix} \mathbf{C}_1 & \mathbf{C}_2 \\ \mathbf{D}_1 & \mathbf{O}_1 \\ \mathbf{O}_2 & \mathbf{D}_2 \end{bmatrix} \tag{3.5}$$

where:

$\mathbf{C}_1$ is of order 1 x k and contains only negative entries

$\mathbf{C}_2$ is of order 1 x $(n+1-k)$ and contains only positive entries

$\mathbf{O}_1$ is a zero matrix of order $(k-1)$ x $(n+1-k)$

$\mathbf{O}_2$ is a zero matrix of order $(n-k)$ x k

$\mathbf{D}_1$ is of order $(k-1)$ x k and has a sign pattern similar to (3.4) or to $\mathbf{C}$ itself

$\mathbf{D}_2$ is of order $(n-k)$ x $(n+1-k)$ and also has a sign pattern similar to (3.4) or to $\mathbf{C}$ itself.

However, even (3.5) does not represent the necessary conditions for full sign-solvability, as Lancaster himself showed about fifteen years later (Lancaster, 1981).

Sufficient *and* necessary conditions for full sign-solvability have been put forward by Bassett, Maybee, and Quirk (1968), who presented with the help of ***graph theory*** a theorem which asserts that system (2.6) is fully sign-solvable if and only if, using permutations and sign changes, it can be put in the form

$$\mathbf{F}\,\mathbf{z} = \mathbf{h}$$

where $\mathbf{F}$ is of order n x n and

(a) $f_{ii} < 0$, $i=1,2,\ldots,n$

(b) all cycles in $\mathbf{F}$ of length greater than one are nonpositive

(c) $\mathbf{h} \leq 0$

(d) $h_k \neq 0$, which implies that every path $f\,(i \rightarrow k)$ in $\mathbf{F}$ ending in k is nonnegative for every $i=1,2,\ldots,n$; i != k .

This theorem uses the fact that a qualitative matrix can be represented as a signed digraph (see, *inter alia*, Christofides, 1975; Harary *et al.*, 1965; Roberts, 1976). This may be illustrated by means of a simple example. Consider the following system:

$$\mathbf{F} \cdot \mathbf{z} = \mathbf{h} \tag{3.6}$$

$$\begin{bmatrix} - & 0 & 0 & 0 \\ - & - & - & - \\ 0 & + & - & 0 \\ + & + & 0 & - \end{bmatrix} \cdot \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} 0 \\ - \\ 0 \\ 0 \end{bmatrix}$$

The qualitative matrix upon which the graph of this system can be based has the following sign pattern (with structure conform (3.2)):

$$\text{sgn } \mathbf{Q} = \begin{bmatrix} - & 0 & 0 & 0 & 0 \\ - & - & - & - & + \\ 0 & + & - & 0 & 0 \\ + & + & 0 & - & 0 \end{bmatrix}$$

which can be represented by the graph shown in Figure 1, where positive signs are represented by a solid arc and negative signs by a dotted arc.

We can see that condition (a) of the theorem is satisfied by system (3.6) since the diagonal of **F** contains only minus signs. Because matrix **A** must be nonsingular, there exists a nonezero term in the expansion of the determinant of **A** which can be brought to the diagonal and made negative. In order to guarantee the nonsingularity of **F**, all of the terms in the expansion of det **F** must have the same sign. This is due to condition (b).We can see from Figure 1 that matrix **Q** satisfies this condition because neither of the two cycles of length greater than one (i.e., f_{23}, f_{32} and f_{24}, f_{42}) is positive. Condition (c) is evidently also fulfilled. In addition, Figure 1 shows three paths to 5 (f_{25} (+); $f_{32}; f_{25}$ (+ , +); f_{42}, f_{25} (+ , +)), all of which are positive, thus fulfilling condition (d).

It should be noted that full sign-solvability is not always possible given only the sign patterns of **A** and **b** or **C**, respectively. Often only some of the signs of the solution vector may be determined. A sufficient, but not always necessary, condition for partial sign-solvability is given by Maybee (1981) as follows:

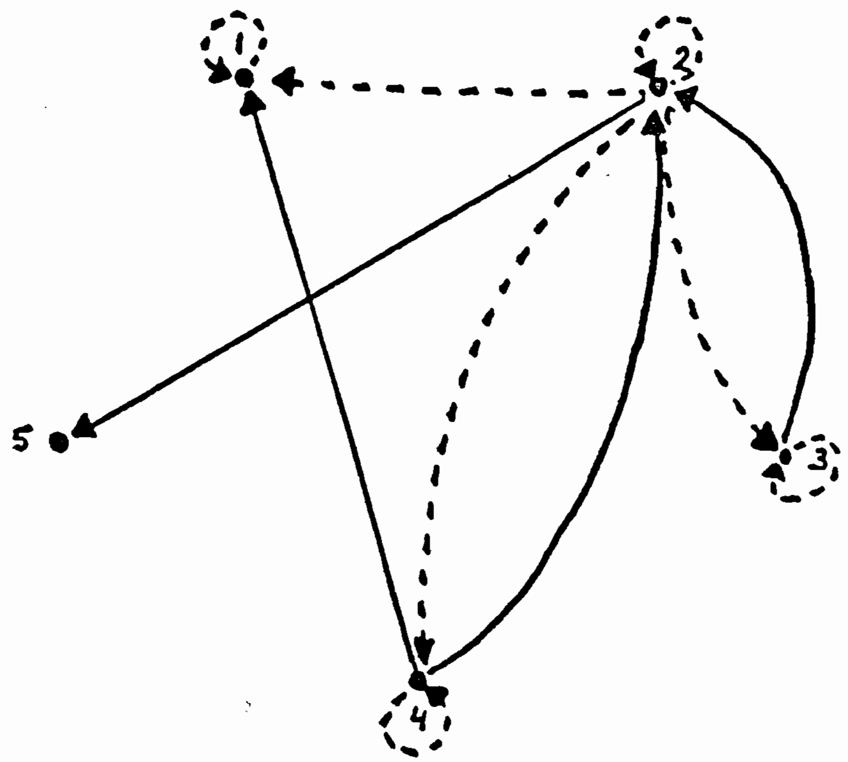

Figure 1. *A graph of qualitative matrix* **Q**

The system $\mathbf{Ay} = \mathbf{b}$ is partially sign-solvable if by admissible qualitative operations (as discussed before) $\mathbf{Ay} = \mathbf{b}$ can be transformed into the system $\mathbf{Ex} = \mathbf{c}$, where $e_{ii} \leq 0$, $c_i \leq 0$ for $i=1,2,..,n$ and $x_i \geq 0$ for any signed variable, and where $\mathbf{Ex} = \mathbf{c}$ can be partitioned into

$$\begin{bmatrix} \mathbf{E}_{11} & \mathbf{E}_{12} \\ \mathbf{E}_{21} & \mathbf{E}_{22} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \end{bmatrix}$$

where

(a) diagonal elements in $\mathbf{E}_{11}$ are negative and all cycles in $\mathbf{E}_{11}$ are nonpositive;

(b) if x_k is a signed variable then k is an index appearing in $\mathbf{E}_{11}$;

(c) either $\mathbf{E}_{12}$ or $\mathbf{E}_{21}$ is a block of zeros;

(d) all positive cycles in $\mathbf{E}$ are contained in $\mathbf{E}_{22}$ and if $\mathbf{E}_{22}$ contains any nonzero cycles, then every index in $\mathbf{E}_{22}$ appears in a positive cycle;

(e) $c_i = 0$ for every index i appearing in $\mathbf{E}_{22}$;

(f) $c_i < 0$ and x_k is a signed variable implies that every chain $e(k \rightarrow i)$ is nonnegative.

The proof of this theorem can be found in Maybee (1981).

4. Spatial Qualitative Calculus

As already mentioned in the introductory section, qualitative calculus is of great potential interest in spatial planning because of its limited demand for data. The relationship of qualitative calculus to comparative statics also has many parallels in urban and regional modeling, in which an equilibrium situation is very often postulated (see, for example, Andersson and Persson, 1979; Haveman and Hollendeck, 1980; Putman, 1979; and Wilson, 1974).

Urban and regional equilibrium models can be divided into two broad categories: *spatial interaction models* and *spatial allocation models*. Spatial interaction models can be (roughly) characterized by the following general equilibrium structure:

$$f_k(x_{iz}, x_{iz'}, d_{zz'}, t_{zz'}) = 0 \quad (z(z') = 1,2,..,Z; i = 1,2,..,I; k = 1,2,..,(Z.Z)) \qquad (4.1)$$

whereas spatial allocation models have the following structure:

$$f_l(x_{iz}, x_{iz'}, d_{zz'}, q_{az}, Q_a) = 0 \quad (z(z') = 1,2,..,Z; i = 1,2,..,I; l = 1,2,..,(A.Z)) \qquad (4.2)$$

The variables in (4.1) and (4.2) can be defined as follows:

x_{iz} is the value of variable i for zone z $(z(z') = 1,2,..,Z)$

$d_{zz'}$ is the value of the friction (usually distance) between zones z and z'

$t_{zz'}$ is the number of interactions (e.g., trips) between zones z and z'

q_{az} is the amount of activity a $(a = 1,2,..,A)$ to be allocated in zone z

Q_a is the total amount of activity a.

The number of variables in (4.1) and (4.2) is much greater than that usually found in the systems to which qualitative calculus is generally applied. As a result, it is impossible to establish the qualitative determinacy of a spatial system manually. Computer algorithms are necessary, but there are - unfortunately - hardly any appropriate algorithms available. Some steps in this direction have been taken by Lancaster (1965,1966), but his algorithms are not particularly suitable for large systems due to the fact that their complexity increases with the number of variables involved.

The decomposition principles developed in graph theory and matrix algebra (e.g., see Himmelblau, 1973) may be very useful in this case. Decomposition involves a rearrangement of rows and columns in a large system of equations such that it is possible to solve a group of small systems of equations instead of the original large system. For example, under some special conditions, the matrix **A** associated with a particular system can be rearranged and then partitioned such that all of the submatrices other than those on the main diagonal are filled with zeros. Then **A** is said to be *block diagonal*, and the system it represents is said to be completely decomposable into its constituent subsystems. If permuting the rows and columns of **A** leads to a set of equations that can be displayed as

$$\begin{bmatrix} \mathbf{E}_{11} & \mathbf{E}_{12} \\ 0 & \mathbf{E}_{22} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \end{bmatrix} \tag{4.3}$$

where $\mathbf{E}_{11}$ is a r x r submatrix, $\mathbf{E}_{22}$ is a $(n-r)$ x $(n-r)$ submatrix, $1 \le r \le n$, of $\mathbf{E}$ and $\mathbf{x}_1, \mathbf{x}_2, \mathbf{c}_1$ and $\mathbf{c}_2$ are suitably permuted and partitioned versions of $\mathbf{x}$ and $\mathbf{c}$. The second of the equations derived from the above matrix equation can now be solved for $\mathbf{x}_2$, and substituting this value in the first equation yields $\mathbf{x}_1$. Thus, the solution of the original matrix equation is reduced to the solution of two lower-order matrix equations. A square matrix $\mathbf{E}$ is said to be reducible if there exists a *permutation matrix* $\mathbf{P}$ such that

$$\mathbf{PEP}^{\mathrm{T}} = \begin{bmatrix} \mathbf{E}_{11} & \mathbf{E}_{12} \\ 0 & \mathbf{E}_{22} \end{bmatrix} \tag{4.4}$$

where the $\mathbf{E}_{ij}$'s are submatrices of **E** as defined above. Otherwise, **E** is said to be irreducible (Vemuri, 1978).

Given matrix **E**, systematic procedures (e.g., from graph theory) can be used to determine the permutation matrix **P** (see Bellman and Cooke, 1970). The following algorithm is a very attractive example of this.

First, an *adjacency matrix* **A** must be derived from a graph G (**E**) based on a given sign matrix **E**. The elements of **A** are defined by

$$a_{ij} = \begin{cases} 1 & \text{if sgn } e_{ij} \neq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.5}$$

The second step is to determine a *reachability matrix* **R**, which can be derived from the adjacency matrix **A** as follows:

$$\mathbf{R} = f\,(\mathbf{I} + \sum_{i=1}^{n} \mathbf{A}^{i}) \tag{4.6}$$

where **I** is an identity matrix and n is the dimension of **A**. The function $f\,(x)$, is defined as:

$$f\,(x) = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.7}$$

In addition, the so-called *strong components* of $G(\mathbf{E})$ must be determined. This can be done by constructing an auxiliary matrix **Q**:

$$\mathbf{Q} = \mathbf{R} \times \mathbf{R}^{\mathrm{T}} \tag{4.8}$$

where **x** denotes a product of elements. The nonzero entries q_{ij} of **Q** indicate that vertices i and j in graph $G(\mathbf{E})$ are mutually reachable. A set of vertices that are mutually reachable is called a strong component. Note that a strong component may consist of only one vertex.

By using these strong components as vertices, a condensed graph $G^{*}(\mathbf{E})$ of $G(\mathbf{E})$ can be constructed, with a condensed adjacency matrix $\mathbf{A}^{*}$ of $G^{*}(\mathbf{E})$. Assume that the vertices in the original graph $G(\mathbf{E})$ are denoted by $n_1, n_2, \cdots,$

etc.

The next step is to locate a column of zeros in $\mathbf{A}^*$ and to relabel the original vertices n_i included in this strong component as $v_1, v_2, \ldots, v_k$, where k represents the number of vertices included in the particular strong component. Then delete this row and column in $\mathbf{A}^*$ and repeat this step. Relabel the next group of vertices as $v_{k+1}, v_{k+2}, \ldots$, etc.. This step therefore reorders the old vertices n_i as new vertices v_i. This information can be used to construct a binary matrix **P**, whose entries p_{ij} are defined as:

$$p_{ij} = \begin{cases} 1 & \text{if } n_i = v_j \\ 0 & \text{otherwise} \end{cases} \qquad (4.9)$$

Matrix **P** is the desired permutation matrix.

In a discussion included in the proceedings of a symposium on computer-assisted analysis, Maybee put forward an algorithm that may possibly be used to identify large sign-solvable systems (see Greenberg and Maybee, 1981, pp.321). The algorithm first identifies a nonzero term in the expansion of the determinant of the **A** matrix, and moves it, using an algorithm of Duff and Reid (1978) to the principal diagonal. Next, the strong components of the resulting matrix are determined in order to check that all cycles are negative. The so-called "depth-first search" algorithm of Tarjan (1972) is recommended for this task.

In general, it can be concluded that the algorithmic properties of qualitative calculus represent a still largely unexplored area, and the sign-solvability of large systems is especially problematic. If this approach is adopted for spatial impact analysis to any extent, then particular attention must be paid to the development of efficient algorithms.

Another issue raising by considering qualitative calculus from the perspective of spatial impact analysis concerns the qualitative determinacy of a spatial system such as (4.1) or (4.2). A large number of zero entries may be expected in the **A** matrix, since spatial activities are often concentrated in a limited set of zones. Consequently, it is not certain that **A** will be nonsingular, which is, however, necessary to solve the complete system according to (2.7). If nonsingularity is not guaranteed on the basis of the sign-pattern of **A** alone, one can nevertheless proceed by decomposing **A** as follows (cf. Maybee, 1981):

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \tag{4.10}$$

where $\mathbf{A}_{11}$ and $\mathbf{A}_{22}$ are square matrices, $\mathbf{A}_{12}$ is a block of zeros and $\mathbf{A}_{21}$ has a nonzero determinant. Hence, the system will be partially sign-solvable.

Given the large number of variables involved in (4.1) and (4.2) it seems realistic to expect that full sign-solvability will seldom occur unless additional information about the system is available (see also Lancaster, 1962). The *incorporation of additional (quantitative or qualitative) information* can be done in two ways:

(1) by specifying and calibrating the equations which are not sign-solvable;

(2) by using the fact that linear combinations of equations are allowed in matrix theory.

The first method is basically a rather conventional modeling approach. Although effective in solving the qualitative determinacy problem, it does not seem to be very efficient in this case. If one part of the system has to be modeled quantitatively, it is only a small step to treat the whole system in the same way. Note that this approach is only useful if complete equations are calibrated: quantification of single entries of **A** is pointless. Even with many quantitative elements in **A** the system may still remain completely unsolvable !

The second approach is more interesting because in this case only additional *ordinal* information must be provided. For example, suppose the following qualitative matrix is given:

$$\text{sgn } \mathbf{D} = \begin{bmatrix} - & + & + \\ + & 0 & + \\ - & - & + \end{bmatrix}$$

Row 2 may, for instance, be subtracted from row 3, which results in the following pattern:

$$\text{sgn } \mathbf{E} = \begin{bmatrix} - & + & + \\ + & 0 & + \\ \text{sgn}(d_{31}-d_{21}) & - & \text{sgn}(d_{33}-d_{23}) \end{bmatrix}$$

If it is known that $d_{31}>d_{21}$ and $d_{33}<d_{23}$ then the third row of sgn **E** will be completely different from the third row of sgn **D**, i.e.,

$$\text{sgn } \mathbf{E} = \begin{bmatrix} - & + & + \\ + & 0 & + \\ + & - & - \end{bmatrix}$$

By using the principle of linear combination of rows in the correct way, many qualitatively unsolvable systems may become qualitatively solvable with additional information. The necessary ordinal data may be obtained quite easily, especially if it concerns a comparison of zonal variables in spatial systems.

5. Some Concluding Remarks

The preceding section suggests that qualitative calculus may be a promising method of dealing with non-numerical information. It is very attractive in urban and regional analysis, because the output of a qualitative analysis will often be just as useful as the results of a more sophisticated quantitative analysis. Qualitative calculus yields information about the *direction* in which a variable may change. Quantitative models will, of course, provide quantitative output, not only about the direction but also about the *size* of the changes. But every competent modeler will know that this quantitative information is intrinsically unreliable due to modeling errors, specification errors, data-measurement errors, calibration errors, and so forth. Hence, the only useful output of a quantitative model will often be the "direction" of the expected changes. If this information can be obtained more easily by a less demanding approach, like qualitative calculus, it is evidently worth investigating.

This paper suggests one important topic that should receive urgent attention in the near future: the development of fast algorithms for the sign-solvability of large (spatial) systems. Further work also needs to be done on the incorporation of additional qualitative and quantitative information into the problem, especially in connection with constraints (e.g., physical barriers). One possibility might be an extension to so-called "matricial forms" (cf. Greenberg, 1981). Finally, it might be interesting to explore in more detail the relationships between qualitative calculus, structural models (see, for example, Linstone *et al.*, 1979 and Kane, 1979), and various kinds of path analyses (see, for example, Blalock, 1972; Joreskog, 1977; Folmer, 1980), since these three approaches all employ graph theoretical ideas. An interesting open question is to what degree qualitative calculus might be extended into these areas of research. It is clear that qualitative calculus is opening several very intriguing new avenues of spatial methodological research which invite further investigation.

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