NECESSARY AND SUFFICIENT CONDITIONS IN
THE MINIMAL CONTROL FIELD PROBLEM FOR LINEAR SYSTEMS

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I. Introduction

In the article [1], the following problem was posed: given the dynamical system

\[
\dot{x}(t) = f(x,u,t) , \quad x(0) = c , \quad (^*)
\]

where \( x \) is an \( n \)-dimensional state vector, \( u \) is an \( m \)-dimensional control vector, and \( f \) is an \( n \)-dimensional vector function smooth enough to insure a unique solution to (*) for all piecewise continuous \( u \), \( t > 0 \) determine a feedback control \( u \), i.e. \( u = u(x,t) \) such that i) using the feedback law \( u(x,t) \), (*) is asymptotically stable and ii) the argument \( u \) contains the minimal number of components of \( x \) consistent with i).

Obviously, this is a very complex problem and a clear-out solution for general \( f \) seems out of reach at present. Even for linear \( f \) and constant coefficients, i.e. \( f(x,u,t) =Fx + Gu \), the problem is complicated by the fact that the solution is not invariant under coordinate transformations. Nonetheless, for the linear case some headway has been made. In [2] upper bounds for the necessary number of components of \( x \) that need appear (the "dimension" of the control field) are derived making use of special control laws. These bounds are easily

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computable in terms of the characteristic vectors of F. If the condition i) were replaced by i') given a symmetric set of complex numbers A (i.e. if \( \lambda \in \mathbb{A} \), then \( \bar{\lambda} \in \mathbb{A} \)), determine a linear feedback control law K of "minimal dimension," such that \( F + GK \) has A as its characteristic values, then a complete solution is given in [3]. This version of the problem substitutes a rigid placement of the controlled system's characteristic roots for the much weaker requirement of asymptotic stability.

In the current note, we return to the original problem (linear version) and give necessary and sufficient conditions for the eliminations of measurements of certain state variables in a stabilizing linear feedback law. Unfortunately, it does not seem possible to give any single set of operationally useful conditions which are both necessary and sufficient. However, we do present one set of necessary conditions and another set of sufficient conditions which are readily checked using the original problem data (the matrices F and G). Examples are also presented showing that, in some cases, these conditions do enable us to precisely determine the "minimal control field."

The standard method for coping with the foregoing "incomplete" measurement problem is to construct a so-called "observer" [4]. As is well known, the observer compensates for inabilities to measure certain parts of the state and gives the same asymptotic results as for the case of complete state information. However, from a practical engineering
point of view, the introduction of an observer may be objectionable on economic as well as technical grounds. The added hardware and circuitry needed for the observer increases the cost, weight, size, and complexity of the systems under design. Consequently, it seems preferable to first analyze the system from the viewpoint presented above, and afterwards introduce an observer for those components of $x$ which cannot be completely eliminated by the results of this paper and which are not physically measurable.

II. Necessary Conditions

a) Single-Input Systems

To begin with, consider the single-input linear system

$$x = Fx + gu, \quad x(0) = c,$$

(1)

where $x$ is an $n$-dimensional vector, $g$ is an $n$-dimensional constant vector, $u$ is a scalar control function, and $F$ is an $nxn$ constant matrix. We seek conditions such that a linear feedback control law $k$, i.e. $u = k'x$, may stabilize (1) and have some zero components (to avoid trivialities, we assume that $F$ has at least one root with non-negative real part).

To express the basic necessary condition, the following notation will prove useful. Let $A$ be an $nxn$ matrix and $b$ an $nx1$ vector. Then the quantity $[A|b]^{(i)}$ will denote the matrix $A$ with its $i$th column replaced by $b$, i.e.
Furthermore, let $S_j(A)$ denote the sum of the principal minors of order $j$ of the matrix $A$ and let $a_j([A|b]^{(i)})$ denote the sum of the principal minors of order $j$, which contain components of $b$, of the matrix $[A|b]^{(i)}$, $j = 1, 2, ..., n$. For example, $S_1(A) = \text{trace } A$, but $a_1([A|b]^{(i)}) = b_1$. It is well known that the characteristic polynomial of $P$ may be expressed as

$$
\chi_P(z) = z^n + \sum_{j=1}^{n} (-1)^j S_j(P) z^{n-j} = z^n + \sum_{j=1}^{n} a_j z^{n-j}.
$$

Our necessary condition for minimality of the control law stabilizing $\Sigma$ will be based upon the well known necessary condition for $\chi$ to be a stability polynomial—the that all coefficients be positive. Since we are interested in the smallest number of components of $x$ which can generate a stabilizing law, let us first consider the necessary condition for the dimensions of the minimal field to be one. This condition is given by
Theorem 1. Let $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ be the non-positive coefficients of $x_F(z)$. Then a necessary condition for $z$ to be stabilizable by a linear law $k' = (0 \ 0 \cdots 0_k \ 0 \cdots 0)$ is

$$
\sigma_i([P|g]^{(j)}) \neq 0 \quad i = i_1, i_2, \ldots, i_k.
$$

Proof. Using a control law of the above form, the $i^{th}$ coefficient in $x_F+gk'(z)$ is

$$
a'_i = a_i + (-1)^i k_j \sigma_i([P|g]^{(j)}) \quad (1)
$$

Hence, if $a_i \leq 0$, $\sigma_i([P|g]^{(j)})$ must be non-zero in order for any choice of $k_j$ to influence the magnitude of $a'_i$.

Remarks. (i) The condition of Theorem 1 only enables us to eliminate certain components of $x$ as potential "single-measurement" stabilizers. However, in view of $(1)$, the string of inequalities implied by Theorem 1 is actually somewhat stronger since we must have

$$
(-1)^i [S_i(F) + k_j \sigma_i([P|g]^{(j)})] > 0 \quad i = i_1, i_2, \ldots, i_k. \quad (2)
$$

As the example below illustrates, these stronger inequalities impose additional restrictions on $k_j$ that may result in the further elimination of candidate components of $x$ that satisfy the necessary condition of Theorem 1. For example, if no number $k_j$ satisfies $(2)$, then component $j$ cannot stabilize $z$ even if $\sigma_i([P|g]^{(j)}) \neq 0$ for $i = i_1, i_2, \ldots, i_k$. 
ii) To test whether a "two-measurement" control field is possible, that is, a law of the form $k' = (0 \cdots 0 k_j 0 \cdots 0 k_i 0 \cdots 0)$, it is easy to see that the $i^{th}$ coefficient of $X_{F \epsilon K}(z)$ is

$$a'_i = a_i + (-1)^i k_j \sigma_i([F|g]^{(j)})$$

$$+ (-1)^i k_i \sigma_i([F|g]^{(k)}).$$

Thus, the appropriate necessary condition is

$$|\sigma_i([F|g]^{(j)})| + |\sigma_i([F|g]^{(k)})| > 0, \quad i = i_1, i_2, \ldots, i_k.$$ 

Now instead of the single constraining inequality (2), we have

$$(-1)^i [S_1(F) + k_j \sigma_i([F|g]^{(j)}) + k_i \sigma_i([F|g]^{(k)})] > 0,$$

$$i = i_1, i_2, \ldots, i_k. \quad (3)$$

Again, (3) may impose enough additional constraints that some possibly feasible components of $x$ are eliminated.

iii) Unfortunately, the foregoing conditions become rather unwieldy for high-dimensional systems when we want to test feasibility of a control field of dimension greater than one or two, since the number of combinations of components grows at an alarming rate, i.e. factorially. However, since all the operations and inequalities are linear, it should be possible to computationally check all possibilities for systems of moderate size—e.g. if $n = 20$ and we want to check the possibility of stabilizing by 10 components of $x$, we have $\binom{20}{10} = 184,756$ combinations to check (the worst case) and if it takes $10^{-3}$ seconds to check one combination, then the total
time required is about 3 minutes.

Example. To illustrate application of Theorem 1 and its consequences, consider the system \( Z \) described by the matrices

\[
F = \begin{bmatrix}
-10 & 1 & 0 \\
1 & -2 & 0 \\
0 & 0 & 2
\end{bmatrix}, \quad g = \begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}.
\]

By inspection, it is clear that this system can be stabilized by measurements of \( x_3 \) alone but it is instructive to use Theorem 1 to obtain this result. We have

\[
x_p(z) = z^3 + 10z^2 - 5z - 38.
\]

Since \( a_2 < 0, a_3 < 0 \), we have (in the notation of Theorem 1), \( k = 2, i_1 = 2, i_2 = 3 \). Our conjecture is that a control of the form \( k' = (0 \ 0 \ k_3) \) will stabilize. Calculating the relevant quantities, we have

\[
S_2(F) = -5,
\]

\[
S_3(F) = 38,
\]

\[
\sigma_2[F|g](3) = \det \begin{bmatrix}
-10 & 1 \\
0 & 1
\end{bmatrix} + \det \begin{bmatrix}
-2 & 1 \\
0 & 1
\end{bmatrix} = -12 \neq 0,
\]

\[
\sigma_3[F|g](3) = \det \begin{bmatrix}
-10 & 1 & 1 \\
1 & -2 & 1 \\
0 & 0 & 1
\end{bmatrix} = 19 \neq 0.
\]

Thus, the necessary conditions of Theorem 1 are satisfied.
The strengthened condition (2) yields
\[ -38 -19 k_3 > 0 , \]
\[ -5 -12 k_3 > 0 , \]
\[ 10 - k_3 > 0 , \]
which together imply \( k_3 < -5/12. \)

It is easily calculated that the necessary conditions of
Theorem 1 are also satisfied for measurements of \( x_1 \) alone or
\( x_2 \) alone. However, the inequalities (2) yield

\[ x_1 \text{ alone: } 10 -k_1 > 0, -5 -k_1 > 0, 6k_1 -38 > 0 \text{ which are } \]
\[ \text{ inconsistent,} \]
\[ x_2 \text{ alone: } 10 -k_2 > 0, -9k_2 -5 > 0, 22k_2 -38 > 0, \text{ which } \]
\[ \text{ are also inconsistent. Thus, the only possibility to stabi-} \]
\[ \text{ lize } \Sigma \text{ by measuring a single coordinate is to measure } x_3. \]

To check that indeed \( \Sigma \) can be stabilized by measurements
of \( x_3 \), we use the necessary and sufficient conditions
\[ a_1 a_2 > a_3 , \quad a_1 > 0 , \quad a_2 > 0 , \quad a_3 > 0 . \]
We need only check the first of these inequalities as the
others have already been seen to be fulfilled when \( k_3 < -5/12. \)
The relevant inequality is
\[ (10 - k_3) (-5 - 12k_3) > 2 - 19k_3 \]
or
\[ k_3^2 - 6k_3 - \frac{13}{3} > 0 . \]
This inequality is satisfied for
\[ 4 - \frac{1}{2} \sqrt{244/3} < k_3 < 4 + \frac{1}{2} \sqrt{244/3} . \]

Since \( \sqrt{244/3} \approx 9.018 \), we see that any value of \( k_3 \) in the range
\[ -1/2 - \varepsilon < k_3 < -5/12 , \]
will stabilize \( \Sigma \), where \( \varepsilon = \sqrt{244/3} - 9 \approx 0.018 \).

b) Multiple-Input Systems

Now we assume that \( \Sigma \) has \( 1 < m \leq n \) inputs, i.e. \( G \) is an \( nxm \) matrix and \( K \) is an \( m \times n \) matrix. Again let \( a_{i_1}, \ldots, a_{i_k} \) be
the non-positive coefficients of \( x_F(z) \). Then the counterpart of Theorem 1 is

Theorem 1'. A necessary condition for \( \Sigma \) to be stabilizable by a linear law
\[ K = \begin{bmatrix} 0 & \cdots & 0 & K_{1i} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots & \ddots & \ddots & \vdots \\ \end{bmatrix} \]

is
\[ \sum_{r=1}^{m} |a_i[[F|G^{(r)}(j)](j)| > 0 , \quad i = i_1, i_2, \ldots, i_k , \]
where \( G^{(r)} \) denotes the \( r \)th column of \( G \).

Proof. It is easily verified that the \( i \)th coefficient in
\( x_{F*GK}(z) \) is
Consequently, no measurements on \( x_j \) can influence \( a_i \) if all the terms \( g_i(\mathbf{F})g_j(\mathbf{F}) \) vanish, \( i = 1, 2, \ldots, m \).

In a completely analogous manner, we can obtain similar results for the case of more than one state measurement.

III. Sufficient Conditions

To derive sufficient conditions for the elimination of state variables from a stabilizing feedback law, we utilize the integral equation representation for the solution of \( L \):

\[
x(t) = e^{\mathbf{F}t}c + \int_0^t e^{\mathbf{F}(t-s)}\mathbf{G} \mathbf{x}(s) \, ds.
\]

We now assume that \( F \) is a normal matrix, i.e. \( F \) is diagonalizable as

\[
F = T \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) V',
\]

where \( \text{Re}\lambda_1 > \text{Re}\lambda_2 > \ldots > \text{Re}\lambda_p > 0 > \text{Re}\lambda_{p+1} > \ldots > \text{Re}\lambda_n \).

Standard results in the theory of integral equations allow us to express \( x(t) \) in the form

\[
x(t) = e^{\mathbf{F}t}c + \int_0^t R(t,s) e^{\mathbf{F}s}c \, ds, \tag{4}
\]

where the resolvent kernel \( R(t,s) \) satisfies the integral equation

\[
R(t,s) = e^{\mathbf{F}(t-s)}\mathbf{G} + \int_0^t e^{\mathbf{F}(t-s')\mathbf{G}} R(s',s) \, ds'.
\]
The representation (4) plus the normality assumption on $F$ enable us to write the $i^{th}$ component of $x$ as

$$\begin{align*}
x_i(t) &= \sum_{j=1}^{n} e^{\lambda_j t} (c_j V_j) t_{ij} \\
&+ \int_0^t \sum_{j=1}^{n} r_{ij}(t,s) \left[ \sum_{k=1}^{n} e^{\lambda_k s} (c_k V_k) t_{kj} \right] ds,
\end{align*}$$

where $T = [t_{ij}]$, $V_j$ is the $j^{th}$ row of $V$, and $(,)$ denotes the usual vector inner product. We are now in a position to assert

**Theorem 2.** Assume that $t_{ij} = 0$, $j = 1,2,\ldots,p$ and that $E$ is controllable. Then component $x_i$ may be omitted from a stabilizing linear feedback law.

**Proof.** If $t_{ij} = 0$, $j = 1,\ldots,p$, no increasing exponentials appear in the first term of (5). Also, the condition implies that no increasing exponentials will appear in any of the integrals which involve the $i^{th}$ column of $R$. Thus, in selecting a stabilizing $K$, we may choose it so that the $i^{th}$ column of $R$ is zero. However, the integral equation for $R$ shows that this implies that the $i^{th}$ column of $K$ may be chosen equal to zero, i.e. component $x_i$ does not appear in the law $K$.

**Example.** To illustrate Theorem 2, let $E$ be formed from the matrices

$$F = \begin{bmatrix} -2 & -1 & 1 \\ -3 & -2 & 3 \\ -3 & -1 & 2 \end{bmatrix}, \quad g = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$
Then $X_p(z) = (z - 1)(z + 1)(z + 2)$ and

$$F = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} = T_{A}''.

For this example, $p = 1$ and the form of $T$ indicates that it is possible to stabilize $F$ by a control law of the form $k = (0 \ k_2 \ k_3)$, i.e. with the component $x_1$ not appearing. To check this, we compute the characteristic polynomial of $(F + gk')$ obtaining

$$X_p + gk'(z) = z^3 + (2 - k_3)z^2 + (-4 - 3k_2 - 1)z + (5k_3 - 3k_2 - 2).$$

Application of the Hurwitz conditions shows that for stability it is necessary and sufficient that $k_2$ and $k_3$ satisfy

$$k_3 < 2, \quad 4k_3 + 3k_2 < -1, \quad 5k_3 - 3k_2 > 2, \quad 4k_3^2 - 3k_2k_3 - 12k_3 + 3k_2 > 0.$$

It is not hard to see that there are many solutions to this system, e.g. $k_2 = -2, k_3 = (6 - \sqrt{132})/4$ is one. It is also interesting to observe that the above inequalities have no solution if either $k_2 = 0$ or $k_3 = 0$, i.e. both $x_2$ and $x_3$ must appear in the control law to stabilize $F$. 
IV. Discussion

The preceding results offer the possibility to precisely determine the "dimensions" of the minimal control field for many linear systems. As has been emphasized, the conditions presented are not simultaneously necessary and sufficient. It should be noted, however, that the necessary conditions of Theorem 1 may be combined with the usual Hurwitz conditions to form a single set of necessary and sufficient conditions. But, checking the sufficiency requires checking the consistency of a set of polynomial inequalities which, for large systems, is an operationally intractable situation. The most practical way to combine Theorems 1 and 2 is to use Theorem 2 to determine those components of \( \mathbf{x} \) which may certainly be eliminated from the control law, then employ Theorem 1 to check the remaining components for possible elimination. This procedure also has the desirable side benefit of reducing the number of combinations needed for applications of Theorem 1 from \( \sum_{k=1}^{n-m} \binom{n}{k} \) to \( \sum_{k=1}^{n-m} \binom{n-m}{k} \), where \( m \) is the number of components of \( \mathbf{x} \) which are eliminated by Theorem 2.
References


