SOFTWARE FOR REGIONAL STUDIES: ANALYSIS OF PARAMETRICAL MULTI-CRITERIA MODELS

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PREFACE

The problems associated with analyzing and managing integrated regional development are multidimensional in character. They stem from (i) the hierarchical relations of the national economic system, (ii) conflicting interests within the region, and (iii) the complex structure of the regional system, whose components have different development dynamics. To solve these problems successfully, it is essential to consider the regional system in a holistic fashion.

Large models of individual components of the regional system are often used for analyzing particular aspects of regional development. Usually, such models are developed independently of each other. If a holistic approach is taken to regional development, however, these independent models must be linked to form a coordinated system; only in this way can consistent results be produced. When attempting this linkage certain mathematical and computer software problems often occur and these problems are the subject of this paper. It is the first of a series of articles focussing on 'software for regional development', whose purpose is to disseminate the results of research on this topic undertaken at IIASA.

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ABSTRACT

This paper describes an approach to analyzing how the balanced states of a multicriteria model depend on the values of exogenous parameters. It provides, consistent with the model criteria, an algorithm that chooses the optimal form of the Pareto set. As an example, the paper explains the use of the approach for a regional water-distribution model.

SUMMARY

The problems of investigating how the equilibrium states of a system depend on external conditions have been investigated for centuries. The mathematical theory of sensitivity was developed and has been used effectively in many applied problems in mechanics, physics, and so on.

Recently new problems--close to the traditional ones--have become of great interest. However, the main feature of these problems is that their equilibrium points are constrained extrema, a fact that makes it impossible to use the classical tools of the theory of sensitivity.

Regional studies have been an important area in which such problems have arisen. In contrast with the approach to global problems, the approach to regional problems considers their interactions with the external environment, which cannot be changed by processes within the regional system. Therefore, one of the most important problems in regional analysis is to study how the optimal states of the regional system depend on its external conditions.

The regional system, as is the case for any system with open inputs and outputs, may have its own internal functional criteria, which, as a rule, are not equivalent to external (or 'national') criteria. Hence, regional problems have a multicriteria character. This is why parametric analysis of multicriteria optimization is an important aspect of regional studies.

The purposes of this paper are:

- to investigate how the equilibrium states of multi-criteria models depend on values of their exogenous parameters;

- to consider the opportunities for finding values of these parameters that are optimal in some appropriate sense (such as supplying the Pareto set with a form that minimizes the 'distance' between the equilibrium point and the 'ideal' point).

This work is based on many sources; however, I would like to emphasize the important role of the ideas of the minimax approach (Fedorov 1979) and the methods of multicriteria optimization (Wierzbicki 1979).

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1. INTRODUCTION

In a qualitative evaluation of the states of a system using several criteria, we usually find at the first stage of the investigation those states that are a compromise between all the evaluations. A point of the Pareto set for a given multi-objective model can be considered as an example of this compromise. The way in which we proceed at the second stage of the investigation depends upon its specific aims. However, a problem that frequently occurs and should be solved at this stage is how these compromise states depend on the values of the model parameters.

This paper describes a method for determining how the equilibrium points move as a result of (not necessarily small) changes to the parameter values. We are then able to find those values for which the equilibrium point has desirable properties.

The major difficulty associated with the problem is that it cannot be solved in a direct way using classical methods of unconstrained optimization or sensitivity analysis. We demonstrate this with an example.

Let us find a compromise use of a resource unit in a system describing two technological processes. The production levels are related to the given volumes of the resource as follows:

$$x_1 \leq u \quad ,$$

$$x_2 \leq 1 - u \quad , \tag{1}$$

where u is the volume of the resource for the first process. For each vector of output, $x = \| x_1 ; x_2 \|$, there are two criteria for evaluating its quality:

$$f_1(x) = 2x_1 - x_2 \quad ,$$

$$f_2(x) = -x_1 + 3x_2 \quad , \tag{2}$$

which may be treated conventionally as profits on two different markets.

Let us specify a multicriteria mathematical model of the system: for a set of pairs of numbers $\| x_1 ; x_2 \|$, subject to:

$$0 \leq x_1 \leq u \quad ,$$

$$0 \leq x_2 \leq 1 - u \quad ,$$

for a given u, maximize the objectives:

$$f_1 = 2x_1 - x_2 \quad ,$$

$$f_2 = -x_1 + 3x_2 \quad .$$

The model is presented in graphic form in Figure 1. Let f_1^* and f_2^* be the optimal (for each criterion) values of the objectives f_1 and f_2, respectively. It is then obvious that

$$f_1^* = 2u, \qquad \text{for } x = \| u ; 0 \| \text{ and}$$

$$f_2^* = 3(1 - u), \qquad \text{for } x = \| 0 ; 1 - u \| \quad .$$

As a compromise between these two solutions, we choose a feasible state for which relative deviations in the values of the criteria (with respect to their ideal values) are equal and as small as possible. In other words, it is necessary to minimize the value of the coefficient of inconsistency μ:

$$\mu(x) = \max \left\{ \frac{f_1^* - f_1(x)}{\mathrm{abs}(f_1^*)} ; \frac{f_2^* - f_2(x)}{\mathrm{abs}(f_2^*)} \right\} ,$$

for the set of all feasible x, or to solve the following linear programming problem.

Minimize, with respect to $\| \mu ; x_1 ; x_2 \|$, the value of the coefficient of inconsistency μ, subject to:

$$0 \le x_1 \le u ,$$

$$0 \le x_2 \le 1 - u ,$$

$$2x_1 - x_2 \ge (1 - \mu) f_1^* ,$$

$$-x_1 + 3x_2 \ge (1 - \mu) f_2^* ,$$

$$0 \le \mu \le 1 . \qquad (3)$$

The solution of problem (3) can easily be found using the following arguments (see Figure 2).

For a given u, the set of points that are solutions to the set of inequalities:

$$\begin{cases} 2x_1 - x_2 \ge (1 - \mu) 2u , \\ -x_1 + 3x_2 \ge (1 - \mu) 3(1 - u) , \\ 0 \le x_1 \le u , \\ 0 \le x_2 \le 1 - u , \end{cases} \qquad (4)$$

is on the intersection of the shaded cone with vertex M and the rectangle 0ABC. If the value of μ is decreased, point M

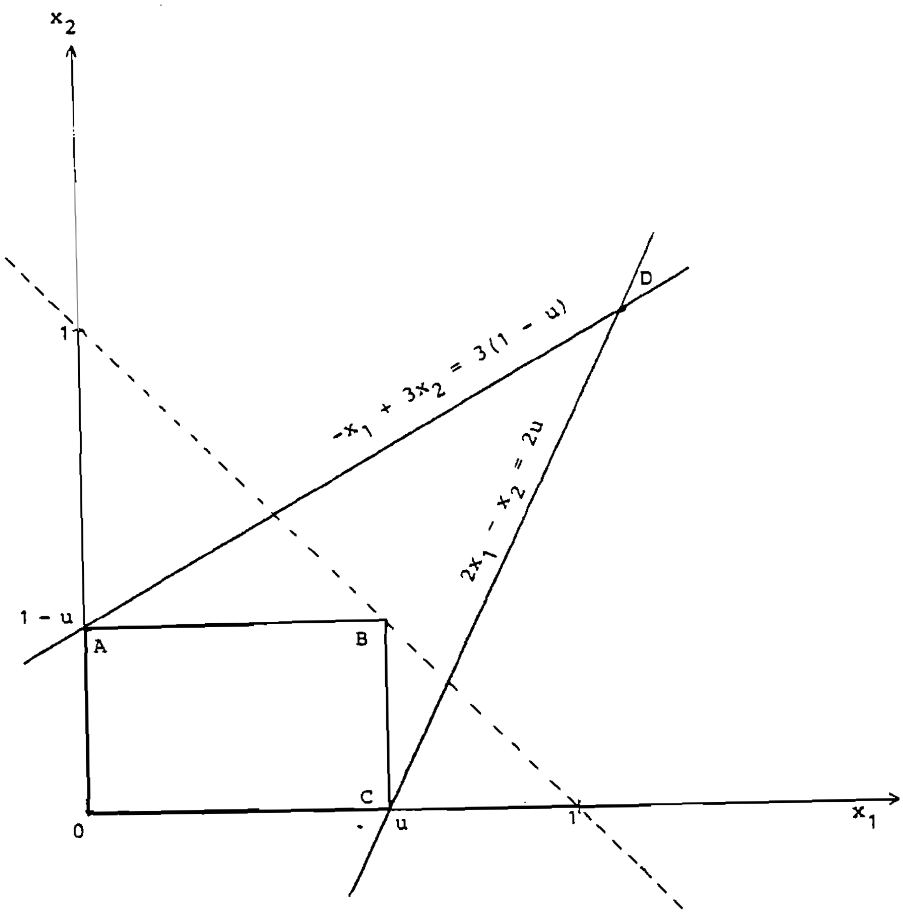

Figure 1. A representation of model (1) - (2).

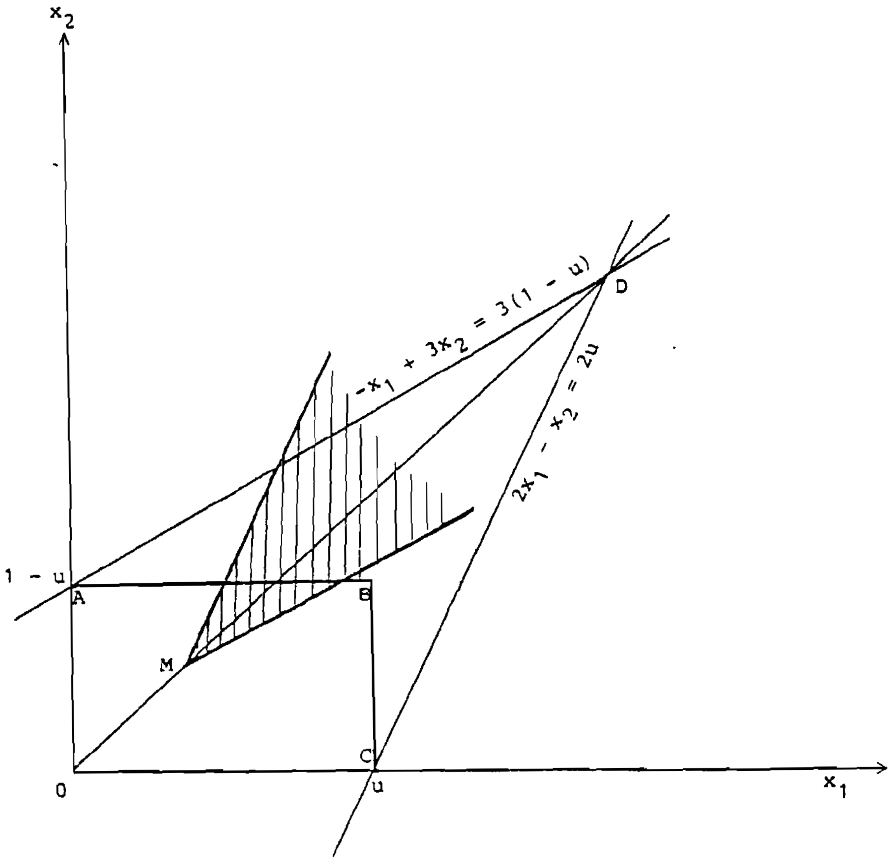

Figure 2. A representation of problem (3).

moves along OD towards D, which is the point of ideal consistency between f_1 and f_2 (with $\mu = 0$).

The minimal value of μ, for which system (4) is still feasible, is defined by the intersection of OD and ABC. Note that the line ABC is the geometrical image of the Pareto set for model (1) - (2).

In solving system (4) with respect to μ, x_1, and x_2 for different values of the parameter u, we find that problem (3) is infeasible for all $u < 0$. For $0 \le u \le 3 - \sqrt{6}$, it has the solution:

$$\begin{cases} x_1 = u \, , \\ x_2 = \dfrac{2u(3 - 2u)}{3(1 + u)} \, , \\ \mu = \dfrac{3 - 2u}{3(1 + u)} \, . \end{cases}$$

For $3 - \sqrt{6} < u \le 1$, it has the solution:

$$\begin{cases} x_1 = \dfrac{3(1 - u^2)}{2(3 - 2u)} \, , \\ x_2 = 1 - u \, , \\ \mu = \dfrac{1 + u}{2(3 - 2u)} \, . \end{cases}$$

For $u > 1$, again there is no solution.

If we consider the optimal value of μ as a function of u (for $0 \le u \le 1$) (see Figure 3), it is evident that it has a minimal value at $u^* = 3 - \sqrt{6} \approx 0.5505$. The inconsistency here equals $(4 - \sqrt{6})/(4\sqrt{6} - 6) \approx 0.4825$, so that both criteria of the model can be up to 51.75% consistent with the vector of the production level:

$$\| x_1(u^*) \; ; \; x_2(u^*) \| = \| 3 - \sqrt{6} \; ; \; \sqrt{6} - 2 \| \approx \| 0.5505 \; ; \; 0.4495 \| \, .$$

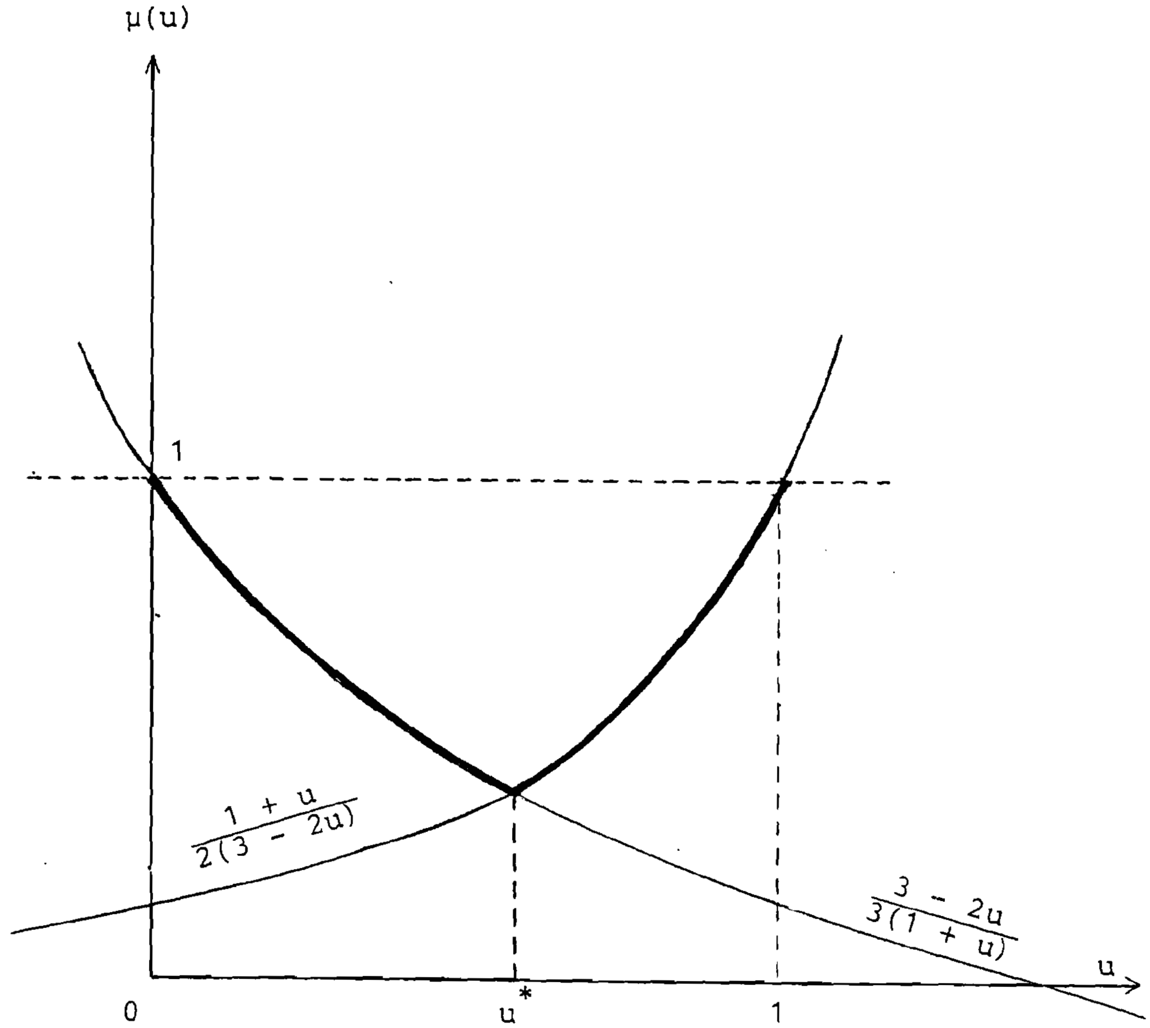

Figure 3. The dependence of optimal μ on u for problem (3).

Finally, it is clear that classical methods of smooth optimization cannot be used to evaluate u^* because of the non-differentiability of $\mu(u)$ at u^*.

In geometrical terms, minimization of the inconsistency by choosing the values for the exogenous parameters of the model may be treated as an optimization of the form of the Pareto set in order to minimize a 'distance' between the set and the ideal point of consistency.

The metric may be chosen, for example, by letting the distance between the set of feasible points of the model and the ideal point D be (see Figure 4).

$$S = \inf_{\| x_1 ; x_2 \|} \sqrt{(x_1^D - x_1)^2 + (x_2^D - x_2)^2}$$

$$= \sqrt{(x_1^D - x_1^B)^2 + (x_2^D - x_2^B)^2} ,$$

where

$$\| x_1^B \; ; \; x_2^B \| = \| u \; ; \; 1 - u \| \quad \text{and}$$

$$\| x_1^D \; ; \; x_2^D \| = \| \frac{3 + 3u}{5} \; ; \; \frac{6 - 4u}{5} \| .$$

Substituting these at S, we find:

$$S(u) = \sqrt{(\frac{3 + 3u}{5} - u)^2 + (\frac{6 - 4u}{5} - (1 - u))^2} .$$

It follows that the minimum distance is reached at $u = 1$. The optimal form of the Pareto set (in the sense of the chosen metric) is the segment $\{0 \le x_1 \le 1 \; ; \; x_2 = 0\}$.

It is obvious that the method demonstrated for model (1) - (2) is not applicable for problems of real value. The dependence $\mu(u)$ cannot be found in an explicit form for most practical problems. The use of numerical techniques such as the Taylor approximation is strongly limited by the undesirable properties of $\mu(u)$, which are indefinite for any u and non-differentiable for those u where $\mu(u)$ is defined.

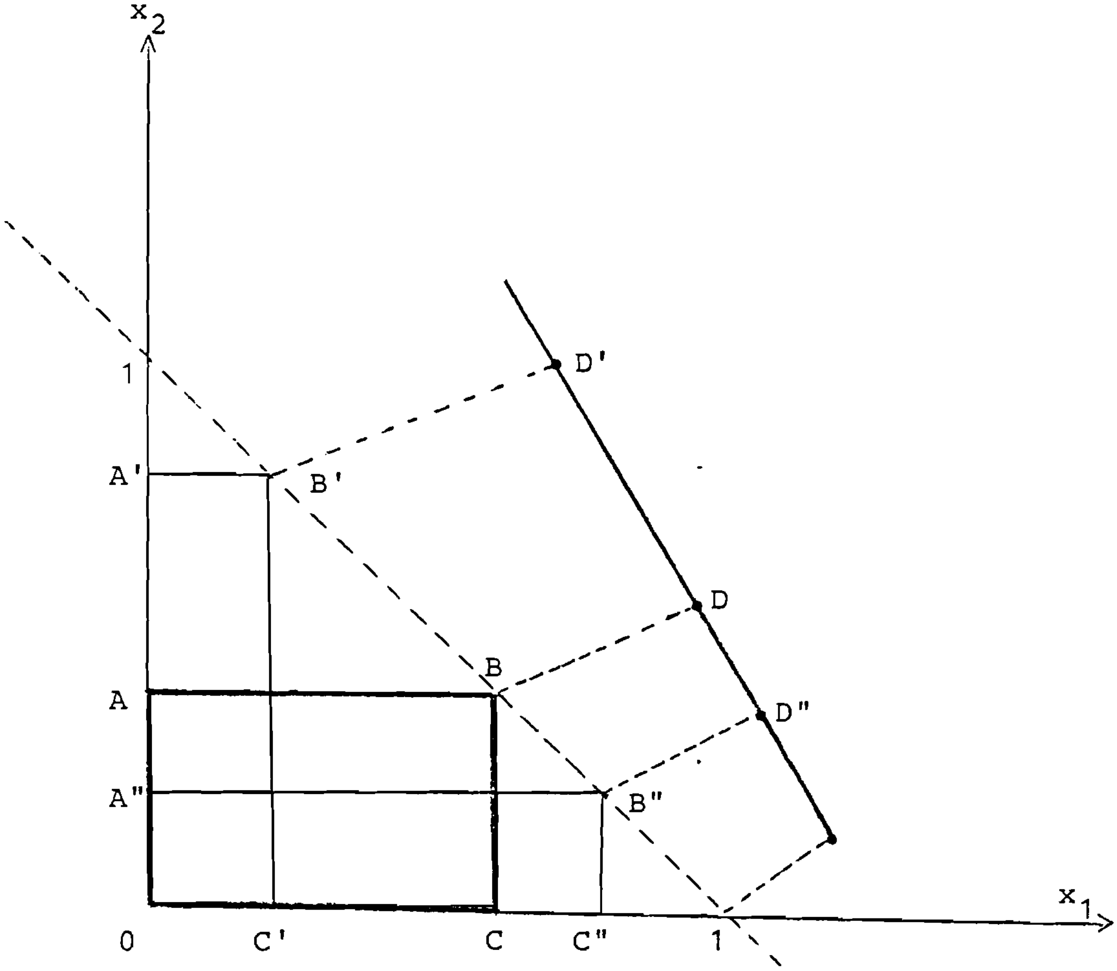

Figure 4. The dependence of the distance between the Pareto set and the ideal point on u.

This paper describes a numerical algorithm that permits us to solve the problem of parametrical analysis for multicriteria models. The method is based on changing the initial problem to an equivalent one (in the sense of the solution), which has properties that allow us to use any of the classical schemes of sensitivity analysis.

2. PARAMETRICAL ANALYSIS OF NONLINEAR MODELS

2.1. Statement of the Problem

Let us consider a multicriteria model, the state of which is described by means of a vector of variables $x \in E^n$ and a vector of exogenous parameters $u \in \Omega \subset E^L$.

In terms of these vectors the model description is reduced to a definition of its constraints, delimiting feasible states of the model

$$y_s(x, u) \geq 0 \quad , \quad s = [1,m] \qquad \text{and} \tag{5}$$

a set of criteria evaluating the quality of these states:

$$\text{maximize with respect to } x , \quad f_k(x,u) \quad , \quad k = [1,N] \ .$$

We assume that all the functions $y_s(x,u)$ and $f_k(x,u)$ are sufficiently smooth within their domain of definition.

Let f_k^* be the optimal value of the objective for the following mathematical programming problem:

$$\text{maximize} \quad f_k(x,u) \quad \text{with respect to } x \ ,$$

$$\text{subject to} \quad y_s(x,u) \geq 0 \quad , \quad s = [1,m] \ . \tag{6}$$

We are now able to define the coefficient of inconsistency μ for a feasible state x of model (5) as

$$\mu(x,u) = \max_k \left\{ \frac{f_k^*(u) - f_k(x,u)}{\mathrm{abs}(f_k^*(u))} \right\} \ .$$

The point x^* will be used as the equilibrium point, such that

$$\mu(x^*,u) = \min_x \mu(x,u) \ .$$

Note that the use of absolute values allows us to make no distinction between cases $f_k^*(u) > 0$ and $f_k^*(u) < 0$. However, the case when $f_k^* = 0$ must be excluded.

According to the definition given above, the procedure for finding the equilibrium state for model (5) can be viewed in terms of the following mathematical programming problem:

$$\text{minimize the value of } \mu \text{ with respect to } \| \mu ; x \| ,$$
$$\text{subject to } y_s(x,u) \geq 0, \; s = [1,m], \text{ and } R_k(\mu,x,u) \geq 0 ,$$
$$\text{where } R_k(\mu,x,u) = f_k(x,u) + \mu \text{abs}(f_k^*(u)) - f_k^*(u) ,$$
$$k = [1,N] \; . \tag{7}$$

This problem is an unusual one because the problem statement includes values of $f_k^*(u)$ defined by the solutions of problems (6).

Now let us consider a problem of parametrical analysis for model (5). We will examine the dependence of the optimal value of μ on the vector of exogenous parameters u.

Suppose that we find a vector u^*, such that

$$u^* = \underset{u \in \Omega}{\text{argmin}} \; \mu^*(u) \quad ,$$

where $\Omega \subset E^L$ is a set of considered u. In formal terms, u^* is the solution of the problem:

$$\begin{aligned} &\text{minimize the value of } \mu \text{ with respect to } \| \mu ; x ; u \| , \\ &\text{subject to} \quad y_s(x,u) \geq 0 \; , \quad s = [1,m] \; ; \\ &\qquad\qquad\quad R_k(\mu,x,u) \geq 0 \; , \; k = [1,N] \; ; \\ &\qquad\qquad\quad \text{and } u \in \Omega \; . \end{aligned} \tag{8}$$

Although the statements of (7) and (8) appear to be similar, the latter problem is considerably more complex. First, problem (7) is linear, if all functions f_k and y_s are also linear, but problem (8) is always nonlinear. Second, statement (8) contains $f_k^*(u)$, which are not defined for all u and are not differentiable, even if all f_k and y_s are defined and differentiable. Finally, the dimension of problem (8) is greater because all components of u are unknown.

The described approach essentially consists in solving problem (8) in an indirect way. The following two-level iterative scheme is suggested as a means of evaluating u^*.

At each step of this process vector u is fixed; this permits us to return to the simpler problem (7). Then using a special analytical procedure for solving (7) for the given u, a better approximation of u^* is found. If necessary, the process is repeated several times. This approach is described in detail below. It should be noted that it is impossible to link components of x and u in a common vector, since this would create a different problem, which will have another interpretation.

It has already been shown that the methods based on the Taylor approximation cannot be used directly to solve problems such as (8), because of the indefiniteness and nondifferentiability of $\mu(u)$. To overcome these difficulties, an approximation of $\mu(u)$, $\hat{\mu}(u)$, is used. It has the following properties:

-- it is uniquely defined for all $u \in E^L$;

-- it is differentiable for all $u \in E^L$;

-- it is close to $\mu(u)$ in the sense of a metric wherever $\mu(u)$ is uniquely defined.

The main problem is to find this new function $\hat{\mu}(u)$, which should be convenient for practical use. In the proposed approach, the approximate solution of problem (7), found by the Smooth Penalty Function Method --SPFM (Fiacco and McCormick 1968), is used for $\hat{\mu}(u)$.

2.2. The Smooth Penalty Function Method

The solution of problem (6) by means of SPFM consists in unconstrained minimization of the auxiliary function

$$E_k = -f_k(x,u) + \sum_{s=1}^{m} P(T,y_s(x,u)) \quad , \tag{9}$$

where function $P(T,\alpha)$, usually referred to s the penalty function, is defined for all $T > 0$ and all α and satisfies the relation

$$\lim_{T\to+0} P(T,\alpha) = \begin{cases} 0\,, & \alpha > 0 \\ +\infty, & \alpha < 0 \end{cases} .$$

Following from the known properties of SPFM, point $\bar{x}_k(T,u)$, at which function (9) has its minimum i.e.

$$\bar{x}_k(T,u) = \underset{x}{\text{argmin}}\; E_k(T,x,u) \quad ,$$

exists for all u.

Subject to additional weak constraints on $P(T,\alpha)$ (Fiacco and McCormick 1968), pointwise convergence will take place:

$$-E_k(T,\bar{x}_k(T,u),u) \xrightarrow[T\to+0]{} f_k^*(u) \quad ,$$

where $f_k^*(u)$ exists.

If, moreover, the $P(T,\alpha)$ is such that $\frac{\partial P}{\partial \alpha} = F(\frac{\alpha}{T})$, i.e., the first partial derivative of P with respect to α depends only on the ratio α/T, then uniform convergence will also take place:

$$-E_k(T,\bar{x}_k(T,u),u) \xrightarrow[T\to+0]{} f_k^*(u) \quad ,$$

for all u, where $f_k^*(u)$ is uniquely defined. If, as a result of theorem (4) (Umnov 1974), the condition $\frac{\partial P}{\partial \alpha} = F(\frac{\alpha}{T})$ is sufficient for validating the following Taylor approximation:

$$\bar{x}_k(T,u) = x_k^*(u) + A_k(u)T + o(T) \quad ,$$

where $abs(A_k(u)) \le C < +\infty$ for those u, where $x_k^*(u)$, the exact solution of (6), is uniquely defined. The rest term $o(T)$ is treated here in the usual sense:

$$\lim_{T\to+0} \frac{o(T)}{T} = 0 \quad .$$

From the above, we have:

$$|\bar{x}_k(T,u) - x_k^*(u)| \le \max_k \{abs(A_k(u))\}\, T + abs(o(T)) \quad ,$$

which proves the fact of uniform convergence.

Note that not all the most frequently used penalty functions satisfy this condition. For example, from the following set of functions:

$$\frac{1}{2T}\left(\frac{\alpha - \text{abs}(\alpha)}{2}\right)^2 \quad ; \quad \begin{cases} -T\ln\alpha & , \ \alpha > 0 \\ +\infty & , \ \alpha \le 0 \end{cases} \quad ; \quad \begin{cases} \frac{\alpha}{T} & , \ \alpha > 0 \\ +\infty & , \ \alpha \le 0 \end{cases} \quad ;$$

$$\exp(-\tfrac{\alpha}{T}) \quad ; \quad T\exp(-\tfrac{\alpha}{T}) \quad ;$$

only the first, second, and fifth have the property.

Now, let us consider the problem of differentiability with respect to u of the functions $\bar{x}_k(T,u)$. Let us suppose that functions $f_k(x,u)$, $y_s(x,u)$, and $P(T,\alpha)$ are twice continuously differentiable. Then $\bar{x}_k(T,u)$ will be implicitly defined by the equation:

$$\text{grad}_x \ E_k(T,\bar{x}_k(T,u),u) = 0 \quad . \tag{10}$$

In applying the known 'implicit functions theorem' to (10), we can prove the continuous differentiability of the function $\bar{x}_k(T,u)$.

As a result, the functions $-E_k(T,\bar{x}\,(T,u),u)$ are continuously differentiable for all u and are close to $f_k^*(u)$ in the domain of its definition in the sense of uniform convergence.

2.3. The General Scheme of Parametric Analysis for Multicriteria Models

It is natural to use SPFM to calculate $\mu(u)$. However, the direct use of this method in (7) does not give an approximation of $\mu(u)$ with desirable properties, because the statement of (7) contains (in contrast to (6)) nonsmooth functions $f_k^*(u)$. Hence, the implicit functions theorem cannot be applied here.

This difficulty can be overcome by changing the statement of (7). Namely, we should substitute $-E_k(T,\bar{x}_k(T,u),u)$ for $f_k^*(u)$ in the R functions. The implicit functions theorem can now be applied in so far as $-E_k(T,\bar{x}_k(T,u),u)$ satisfies all the requirements of the theorem.

Naturally, the influence of the error produced by SPFM should be taken into account. The problem of accuracy will be considered in detail in section 2.5. Here, we only note that this small disturbance in (7) does not give any additional difficulties.

To simplify the notation, we substitute $\bar{E}_k(u)$ for $E_k(T,\bar{x}_k(T,u),u)$. Let us apply SPFM to the above modification of (7). We should then minimize the following auxiliary function:

$$\varepsilon = \mu + \sum_{s=1}^{m} P(T,y_s) + \sum_{k=1}^{N} P(T,V_k) \quad , \tag{11}$$

where

$$V_k = f_k(x,u) + \mu \operatorname{abs}(\bar{E}_k(u)) + \bar{E}_k(u) \quad .$$

We will denote the minimum point of function (11) as

$$\| \hat{\mu} \ ; \ \hat{x} \| \qquad \text{and}$$

$$\min_{\| \mu \ ; \ x \|} \varepsilon(\mu,x,u) = \varepsilon(\hat{\mu}, \hat{x}, u) = \hat{\varepsilon}(u) \quad .$$

According to assumptions about the smoothness of functions f_k, y_k, $\bar{E}_k$, and P, the point $\| \hat{\mu} ; \hat{x} \|$ should satisfy the equation of stationarity:

$$\operatorname{grad}_{\| \mu \ ; \ x \|} \varepsilon(\mu,x) = 0 \quad . \tag{12}$$

The first component of the vector $\| \hat{\mu} ; \hat{x} \|$ may be used to analyze the dependence of the equilibrium state on the vector of exogenous parameters of the model. However, for practical purposes it is more convenient to use $\hat{\varepsilon}(u)$ as the desirable approximation of $\mu^*(u)$, rather than $\hat{\mu}(u)$. An explanation is given below. At first, the difference between $\hat{\varepsilon}(u)$ and $\mu^*(u)$ is also small.

According to a property of SPFM, the absolute value of the sum

$$\sum_{s=1}^{m} P(T,y_s) + \sum_{k=1}^{N} P(T,V_k)$$

calculated at the point $\| \hat{\mu} ; \hat{x} \|$ should be small for all those u at which the model is still feasible. Moreover, it is possible to ensure uniform proximity of $\hat{\varepsilon}(u)$ to $\hat{\mu}(u)$ in the set of feasibilities of the model.

The main advantage of using $\hat{\varepsilon}(u)$ is that it has much greater values outside the domain of feasibility than $\hat{\mu}(u)$. As a result, the process of minimizing $\hat{\varepsilon}(u)$ is simpler than that of minimizing $\hat{\mu}(u)$. This advantage is well demonstrated in Figure 5. In this case the quadratic penalty function

$$P(T,\alpha) = \frac{1}{2T} \left(\frac{\alpha - \text{abs }(\alpha)}{2}\right)^2$$

was used with T = 0.1. Analogous curves with T = 0.01 are shown in Figure 6. $\mu(u)$ was defined as equal to 1 outside the domain of its definition.

In the proposed approach there are no constraints on the use of a scheme for minimizing $\hat{\varepsilon}(u)$ as long as this scheme is based on a Taylor approximation. Methods of linear and quadratic approximation are frequently used in practice. In these cases, in addition to the value of $\hat{\varepsilon}(u)$, we must calculate values of the first and second partial derivatives at each point u.

A specific feature of this calculation is that $\hat{\varepsilon}$ depends on u both explicitly and implicitly:

$$\hat{\varepsilon}(u) = \varepsilon(u, \{\bar{E}_k(u,\bar{x}_k(u)), k = [1,N]\}, \hat{x}(u), \hat{\mu}(u)) \quad .$$

According to the 'chain rule':

$$\frac{\partial\hat{\varepsilon}}{\partial u_r} = \frac{\partial\varepsilon}{\partial u_r} + \sum_{k=1}^{N} \frac{\partial\varepsilon}{\partial\bar{E}_k} \left(\frac{\partial\bar{E}_k}{\partial u_r} + \sum_{i=1}^{n} \frac{\partial\bar{E}_k}{\partial x_i} \frac{\partial\bar{x}_i}{\partial u_r}\right)$$

$$+ \frac{\partial\varepsilon}{\partial\mu} \frac{\partial\hat{\mu}}{\partial u_r} + \sum_{j=1}^{n} \frac{\partial\varepsilon}{\partial x_j} \frac{\partial\hat{x}_j}{\partial u_r} \ , \qquad \text{for all } r = [1,L] \quad .$$

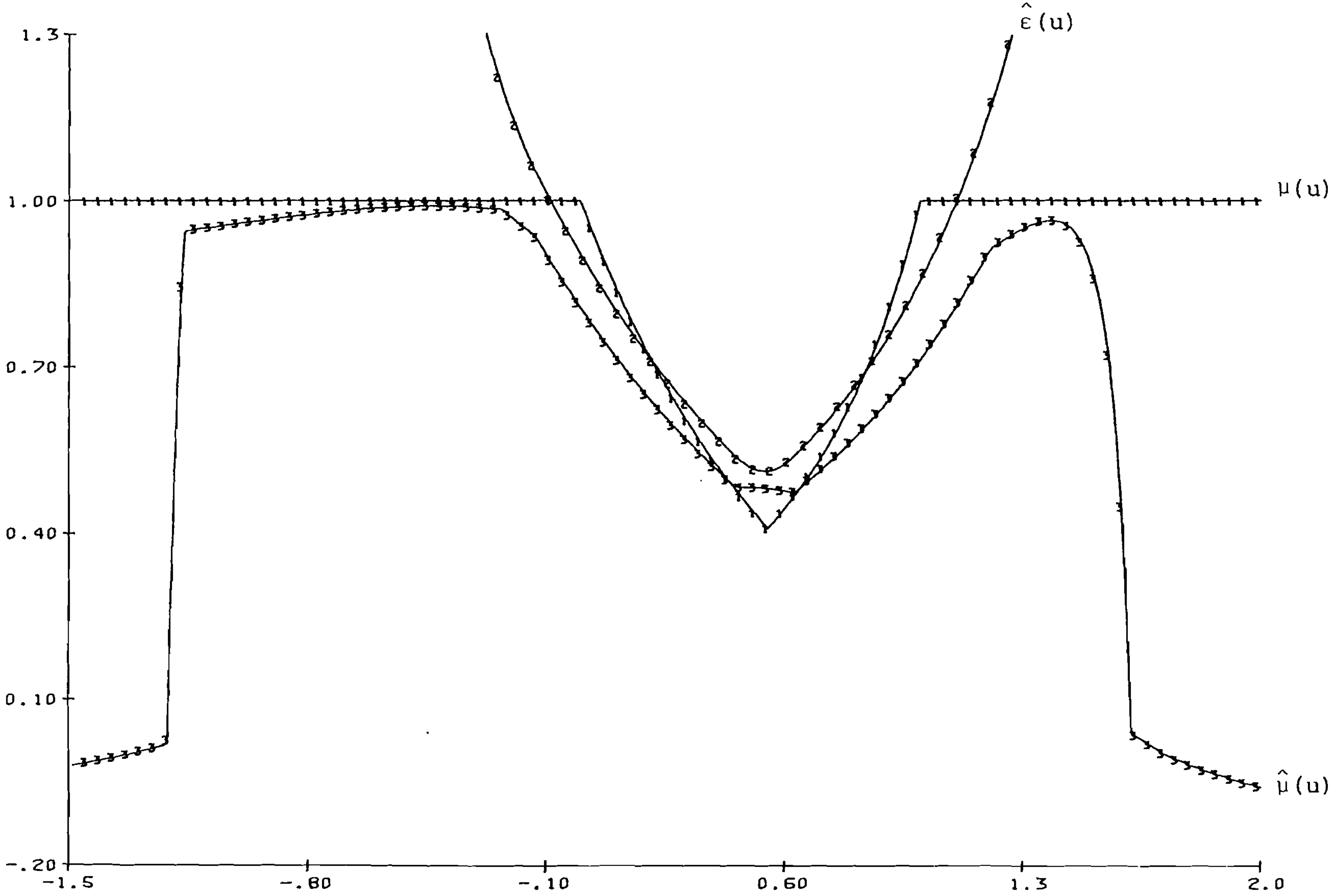

Figure 5. The dependence of μ, $\hat{\varepsilon}$, and $\hat{\mu}$ on u (T = 0.1).

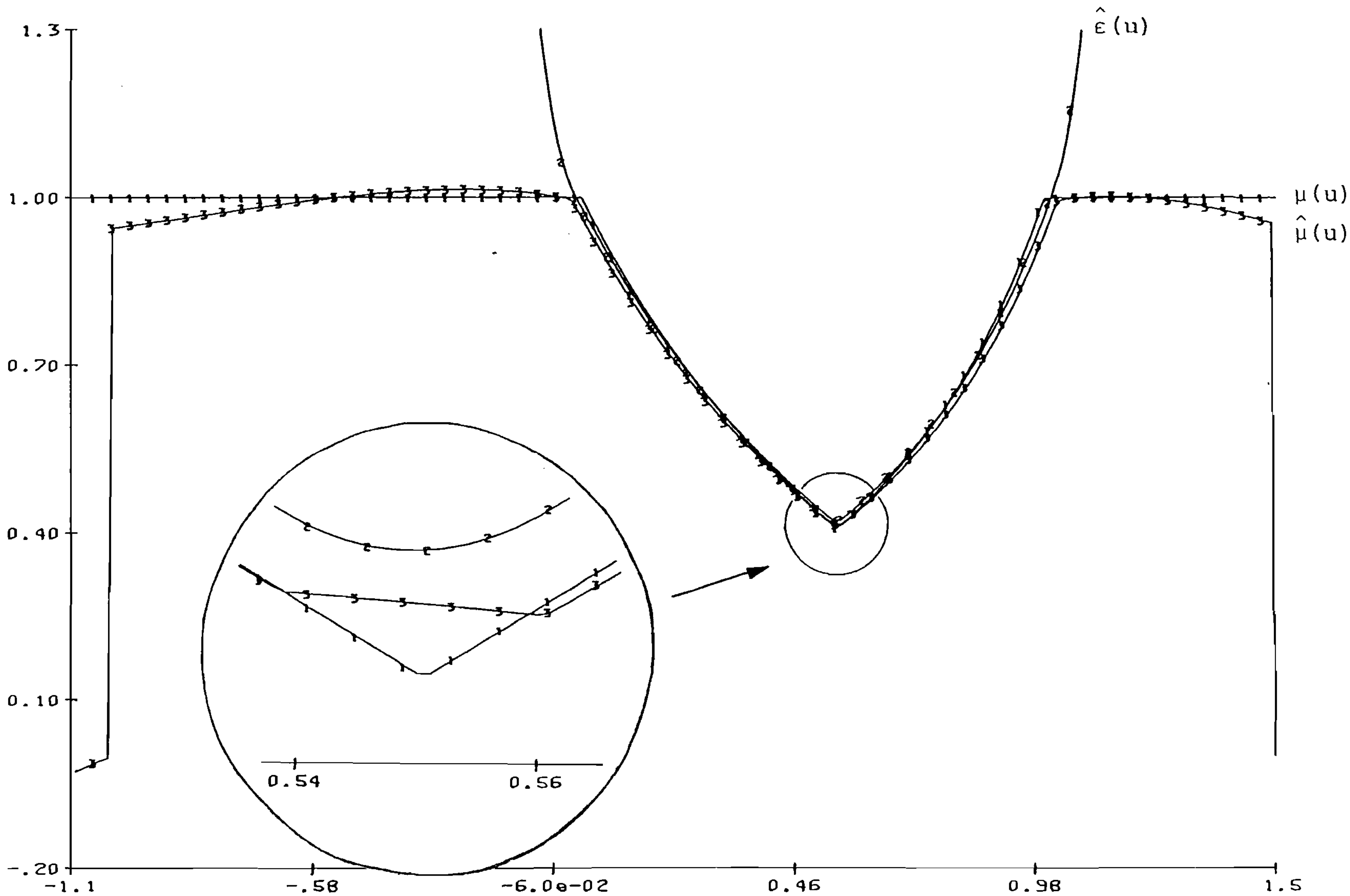

Figure 6. The dependence of μ, $\hat{\varepsilon}$, and $\hat{\mu}$ on u (T = 0.01).

Taking into account that the vectors $\bar{x}_k(u)$ and $\| \hat{\mu} ; \hat{x} \|$ are points of stationarity of the auxiliary functions E_k and ε, respectively, we obtain as a result of (10) and (12), which are in scalar form:

$$\frac{\partial E_k}{\partial x_i} = 0 \quad , \quad i = [1,n] \quad , \quad k = [1,N] \quad ,$$

$$\frac{\partial \varepsilon}{\partial x_j} = 0 \quad , \quad j = [1,n] \quad , \quad \frac{\partial \varepsilon}{\partial \mu} = 0 \quad , \tag{13}$$

the following expression for components of $\text{grad}_u \hat{\varepsilon}$:

$$\frac{\partial \hat{\varepsilon}}{\partial u_r} = \frac{\partial \varepsilon}{\partial u_r} + \sum_{k=1}^{N} \frac{\partial \varepsilon}{\partial \bar{E}_k} \frac{\partial \bar{E}_k}{\partial u_r} \quad , \quad r = [1,L] \quad . \tag{14}$$

Substituting this expression for E_k and ε, we find

$$\frac{\partial \bar{E}_k}{\partial u_r} = - \frac{\partial f_k}{\partial u_r} + \sum_{t=1}^{m} \frac{\partial P}{\partial y_t} \frac{\partial y_t}{\partial u_r} \quad , \quad r = [1,L] \quad , \; k = [1,N] \quad ,$$

where all derivatives are calculated at point $\bar{x}_k$. Similarily, for $\| \hat{\mu} ; \hat{x} \|$,

$$\frac{\partial \varepsilon}{\partial u_r} = \sum_{s=1}^{m} \frac{\partial P}{\partial y_s} \frac{\partial y_s}{\partial u_r} + \sum_{k=1}^{N} \frac{\partial P}{\partial V_k} \frac{\partial f_k}{\partial u_r} \quad ,$$

and finally

$$\frac{\partial \varepsilon}{\partial \bar{E}_k} = \frac{\partial P}{\partial V_k} (1 + \mu \,\text{sign}(\bar{E}_k)) \quad ,$$

where

$$\text{sign}\,(\alpha) = \begin{cases} 1, & \text{for } \alpha > 0 \\ 0, & \text{for } \alpha = 0 \\ -1, & \text{for } \alpha < 0 \end{cases} \quad .$$

Following from (14), it is not necessary to know the sensitivity matrices

$$\left\| \frac{\partial \bar{x}_{ki}}{\partial u_r} \right\| \quad ; \quad \left\| \frac{\partial \hat{x}_i}{\partial u_r} \right\| \quad ; \quad \left\| \frac{\partial \hat{\mu}}{\partial u_r} \right\|$$

to calculate the first partial derivatives of $\hat{\varepsilon}(u)$, but we must predetermine points $\bar{x}_k$ and $\| \hat{\mu} ; \hat{x} \|$.

By means of similar modifications, we can obtain expressions for the second partial derivatives of the function $\hat{\varepsilon}(u)$. We present the results without a detailed explanation.

$$\begin{aligned} \frac{\partial^2 \hat{\varepsilon}}{\partial u_r \partial u_q} = {} & \frac{\partial^2 \varepsilon}{\partial u_r \partial u_q} + \sum_{j=1}^{n} \frac{\partial^2 \varepsilon}{\partial u_r \partial x_j} \frac{\partial \hat{x}_j}{\partial u_q} + \frac{\partial^2 \varepsilon}{\partial u_r \partial \mu} \frac{\partial \hat{\mu}}{\partial u_q} \\ & + \sum_{l=1}^{N} \frac{\partial^2 \varepsilon}{\partial u_r \partial \bar{E}_l} \frac{\partial \bar{E}_l}{\partial u_q} + \sum_{k=1}^{N} \left\{ \frac{\partial \bar{E}_k}{\partial u_r} \left[\frac{\partial^2 \varepsilon}{\partial \bar{E}_k \partial u_q} + \frac{\partial^2 \varepsilon}{\partial \bar{E}_k \partial \mu} \frac{\partial \hat{u}}{\partial u_q} \right. \right. \\ & \left. + \sum_{i=1}^{n} \frac{\partial^2 \varepsilon}{\partial \bar{E}_k \partial x_i} \frac{\partial \hat{x}_i}{\partial u_q} + \sum_{t=1}^{N} \frac{\partial^2 \varepsilon}{\partial \bar{E}_k \partial \bar{E}_t} \frac{\partial \bar{E}_t}{\partial u_q} \right] \\ & \left. + \frac{\partial \varepsilon}{\partial \bar{E}_k} \left(\frac{\partial^2 \bar{E}_k}{\partial u_r \partial u_q} + \sum_{i=1}^{n} \frac{\partial^2 \bar{E}_k}{\partial u_r \partial x_i} \frac{\partial \bar{x}_{ki}}{\partial u_q} \right) \right\} , \end{aligned} \tag{15}$$

for all $r = [1,L]$ and $q = [1,L]$.

This means that we must have sensitivity matrices for $\bar{x}_k(u)$, $\hat{\mu}(u)$, and $\hat{x}(u)$ in order to evaluate components of the hessian of $\hat{\varepsilon}(u)$, but only those of the first order.

Elements of these matrices can be found from systems of linear equations resulting from the differentiation of (10) and (12) with respect to components of u. For example, for $\left\| \frac{\partial \bar{x}_{ki}}{\partial u_r} \right\|$,

$$\sum_{j=1}^{n} \frac{\partial^2 E_k}{\partial x_i \partial x_j} \frac{\partial \bar{x}_{kj}}{\partial u_r} = - \frac{\partial^2 E_k}{\partial x_i \partial u_r} , \qquad i = [1,n] , \quad r = [1,L] , \quad k = [1,N] , \tag{16}$$

where point $x = \bar{x}_k$.

Analogously, for $\left\| \frac{\partial \hat{\mu}}{\partial u_r} ; \frac{\partial \hat{x}_i}{\partial u_r} \right\|$,

$$\begin{cases} \frac{\partial^2 \varepsilon}{\partial x_i \partial \mu} \frac{\partial \hat{\mu}}{\partial u_r} + \sum_{j=1}^{n} \frac{\partial^2 \varepsilon}{\partial x_i \partial x_j} \frac{\partial \hat{x}_j}{\partial u_r} = - \frac{\partial^2 \varepsilon}{\partial x_i \partial u_r} , & i = [1,n] , \\ \frac{\partial^2 \varepsilon}{\partial \mu^2} \frac{\partial \hat{\mu}}{\partial u_r} + \sum_{j=1}^{n} \frac{\partial^2 \varepsilon}{\partial \mu \partial x_j} \frac{\partial \hat{x}_j}{\partial u_r} = - \frac{\partial^2 \varepsilon}{\partial \mu \partial u_r} , & r = [1,L] , \end{cases} \qquad (17)$$

where point $\| \hat{\mu} ; \hat{x} \|$.

Since it is possible to calculate for all points u the values of $\hat{\varepsilon}(u)$, the components of its gradient, and its hessian, we are able to implement any constrained optimization algorithm to solve the problem: minimize $\hat{\varepsilon}(u)$, subject to $u \in \Omega$.

The method for tackling this problem is chosen on the basis of its specific features, i.e. on the properties of functions $f_k(x,u)$ and $y_s(x,u)$.

2.4. An Example

Let us demonstrate the approach described above for the case of the simple model (1)-(2). Auxiliary functions (9), constructed by means of the quadratic penalty function, are

$$E_1 = -2x_1 + x_2 + W(T,x_1,x_2) ,$$

$$E_2 = x_1 - 3x_2 + W(T,x_1,x_2) ,$$

where

$$W(T,x_1,x_2) = \frac{1}{2T}\left(\frac{x_1-\mathrm{abs}(x_1)}{2}\right)^2 + \frac{1}{2T}\left(\frac{x_2-\mathrm{abs}(x_2)}{2}\right)^2 + \frac{1}{2T}\left(\frac{x_1-u+\mathrm{abs}(x_1-u)}{2}\right)^2 + \frac{1}{2T}\left(\frac{x_2-1+u-\mathrm{abs}(x_2-1+u)}{2}\right)^2 .$$

It is easy to prove that, for $u \in (0,1)$, $\bar{E}_1(u) = -2u - 2.5T$ and $\bar{E}_2(u) = -3 + 3u - 5T$ take place.

The iterative procedure for minimizing $\hat{\varepsilon}(u)$ may be started at a feasible point, u = 0.1, for example. Auxiliary function (11) at this value of the parameter is

$$\varepsilon = \mu + \frac{1}{2T}(x_1 - u)^2 + \frac{1}{2T}(2x_1 - x_2 + (1-\mu)\bar{E}_1(u))^2 + \frac{1}{2T}(-x_1 + 3x_2 + (1-\mu)\bar{E}_2(u))^2 ,$$

the penalty coefficient T will be taken as 0.01 (see Figure 6).

Having completed all the necessary calculations, we find that for u = 0.1:

$$\bar{x}_1 = \|0.12 ; -0.01\| ; \quad \bar{x}_2 = \|-0.01 ; 0.93\| ;$$

$$\bar{E}_1 = -0.225 ; \quad \bar{E}_2 = -2.75 ;$$

$$\hat{x}_1 = 0.1146 ; \quad \hat{x}_2 = 0.1984 ; \quad \hat{\mu} = 0.8242 ;$$

and, finally, $\hat{\varepsilon} = 0.8391$.

The derivative at this point is

$$\frac{\partial\hat{\varepsilon}}{\partial u} = -\frac{1}{T}(\hat{x}_1 - u) + \frac{1}{T}(1-\hat{\mu})(2\hat{x}_1 - \hat{x}_2 + (1-\hat{\mu})\bar{E}_1(u))\frac{\partial\bar{E}_1}{\partial u} + \frac{1}{T}(1-\hat{\mu})(-\hat{x}_1 + 3\hat{x}_2 + (1-\hat{\mu})\bar{E}_2(u))\frac{\partial\bar{E}_2}{\partial u} .$$

Substituting specific values, we have

$$\frac{\partial\hat{\varepsilon}}{\partial u} \approx -1.4599 + 0.1758(-0.8756)(-2) + 0.1758(-0.2920)3 = -1.3060 ,$$

because

$$\frac{\partial\bar{E}_1}{\partial u} = -2 \quad \text{and} \quad \frac{\partial\bar{E}_2}{\partial u} = 3 .$$

This indicates that we should increase u to minimize $\hat{\varepsilon}(u)$.

Let us take a new approximation of u, such that the structure of the set of active model constraints will be changed. First, this occurs at $u \approx 0.54$, when the constraint $x_2 \le 1 - u$ becomes active. Hence $u = 0.54$ may be a new test case. At this point

$$\bar{x}_1 = \| 0.56 \, ; -0.01 \| \quad ; \quad \bar{x}_2 = \| -0.01 \, ; 0.49 \| \quad ;$$

$$\bar{E}_1 = -1.105 \quad ; \quad \bar{E}_2 = -1.430 \quad .$$

Auxiliary function (11)

$$\varepsilon = \mu + \frac{1}{2T}(x_1 - u)^2 + \frac{1}{2T}(x_2 - 1 + u)^2 + \frac{1}{2T}(2x_1 - x_2 + (1-\mu)\bar{E}_1)^2$$
$$+ \frac{1}{2T}(-x_1 + 3x_2 + (1-\mu)\bar{E}_2)^2 \quad .$$

Minimizing (11) with respect to $\| \mu ; x \|$, we get

$$\hat{x}_1 = 0.5499, \quad \hat{x}_2 = 0.4608 \quad , \quad \hat{\mu} = 0.4162 \quad , \quad \hat{\varepsilon} = 0.4233 \, .$$

The first partial derivative of $\hat{\varepsilon}$ at $u = 0.54$ will be

$$\frac{\partial \hat{\varepsilon}}{\partial u} = -\frac{1}{T}(\hat{x}_1 - u) + \frac{1}{T}(\hat{x}_2 - 1 + u) + \frac{1}{T}(1-\hat{\mu})(2\hat{x}_1 - \hat{x}_2 + (1-\hat{\mu})\bar{E}_1(u))\frac{\partial \bar{E}_1}{\partial u}$$
$$+ \frac{1}{T}(1-\mu)(-\hat{x}_1 + 3\hat{x}_2 + (1-\hat{\mu})\bar{E}_2(u))\frac{\partial \bar{E}_2}{\partial u} \quad . \qquad (18)$$

In numerical terms this will be

$$\frac{\partial \hat{\varepsilon}}{\partial u} = -0.9879 + 0.0788 + 0.5838(-0.6085)(-2)$$
$$+ 0.5838(-0.2291)3 = -0.5999 \quad .$$

At this iteration, we use the Newton method to increase the accuracy. We should make a preliminary calculation of elements of the hessian using (15).

From (16):

$$\left\| \frac{\partial \bar{x}_{1i}}{\partial u} \right\| = \| 1 \, ; 0 \| \quad ; \quad \left\| \frac{\partial \bar{x}_{2i}}{\partial u} \right\| = \| 0 \, ; 1 \| \quad ;$$

and therefore all expressions

$$\frac{\partial^2 E_k}{\partial u_r \partial u_q} + \sum_{i=1}^{n} \frac{\partial^2 E_k}{\partial u_r \partial x_i} \frac{\partial \bar{x}_{ki}}{\partial u_q}$$

are equal to zero. From (17) we find that

$$\left\| \frac{\partial \hat{\mu}}{\partial u} ; \frac{\partial \hat{x}}{\partial u} \right\| = \| -0.0829;\ 0.5091;\ -0.4192 \| \quad .$$

Substituting in (15), we have

$$\frac{\partial^2 \hat{\varepsilon}}{\partial u^2} = 62.90 \text{ at the point } u = 0.54 \quad .$$

According to the Newton method, a better approximation of $\operatorname{argmin}_u \hat{\varepsilon}(u)$ is given by

$$u = u_o - \left(\frac{\partial \hat{\varepsilon}}{\partial u} \Big/ \frac{\partial^2 \hat{\varepsilon}}{\partial u^2}\right) \quad ,$$

where u_o is the test point. Therefore,

$$u = 0.54 - \frac{(-0.5999)}{62.90} = 0.5495 \quad .$$

To check the error made here, calculate the gradient of $\hat{\varepsilon}$ for the new u, using formula (18). We then have

$$\hat{x}_1(u) = 0.5548 \ ; \ \hat{x}_2(u) = 0.4568 \ ; \ \hat{\mu}(u) = 0.4154 ;$$

$$\text{and } \frac{\partial \hat{\varepsilon}}{\partial u} \simeq 0.022 \quad .$$

Finally, notice that the minimum of $\hat{\varepsilon}(u)$ is approximately at $\hat{u} = 0.54956$, i.e. the result might be acceptable.

2.5. Accuracy of the Approach

In the proposed approach we minimize $\hat{\varepsilon}(u)$ instead of $\mu(u)$, subject to $u \in \Omega$. However, $\hat{\varepsilon}(u)$ differs slightly from $\mu(u)$, because of the principal properties of SPFM. We should evaluate the results of this error.

It should be noted that subject to the use of the penalty function $P(T,\alpha)$, which satisfies the property described in section 2.2, page 12, the value of the error will be proportional to T. This provides us with a basis for making an initial evaluation.

Figure 7 presents values of $\mu - \hat{\mu}$ (curve 1), $\mu - \hat{\varepsilon}$ (curve 2) for $T = 0.01$ and $\mu - \hat{\mu}$ (curve 3), $\mu - \hat{\varepsilon}$ (curve 4) for $T = 0.001$.

This error has an influence on the iterative procedure for minimizing $\hat{\varepsilon}(u)$ and distorts the final results, i.e. produces the difference between u^* and $\hat{u}$, where

$$u^* = \underset{u \in \Omega}{\text{argmin}}\ \mu(u) \quad ,$$

and

$$\hat{u} = \underset{u \in \Omega}{\text{argmin}}\ \hat{\varepsilon}(u) \quad .$$

Its influence on the procedure for minimizing $\hat{\varepsilon}(u)$ is only interesting from a theoretical point-of-view because of the iterative nature of the procedure. In other words, there is no accumulation of errors in the stepwise process. Only a few characteristics of the process will depend on T: for example, the total number of iterations required. For some cases, it may be reasonable to increase the value of $\text{abs}(\hat{\varepsilon}(u) - \mu(u))$ in order to achieve a better convergence for the process of minimizing $\hat{\varepsilon}(u)$.

It is more important to determine the value of $\|\hat{u} - u^*\|$. In practice, the level of computational effort required to do this should be reasonable. For example, it may take into account the accuracy of the initial data in the model.

The approach permits us, at least theoretically, to find u^* using $\hat{u}$ and by analyzing the dependence of the auxiliary function (11) on T.

In the general case, we have

$$\hat{\varepsilon}(T,u) = \varepsilon(T,\hat{u},\{\bar{E}_k, k=[1,N]\},\hat{x}(T,\hat{u}),\hat{\mu}(T,\hat{u})) \quad .$$

The equation of stationarity may be used to define the implicit function $\hat{u}(T)$:

$$\text{grad}_u\ \hat{\varepsilon}(T,\hat{u}) = 0 \quad , \qquad (19)$$

which, by virtue of the assumptions made and the properties of SPFM, satisfies

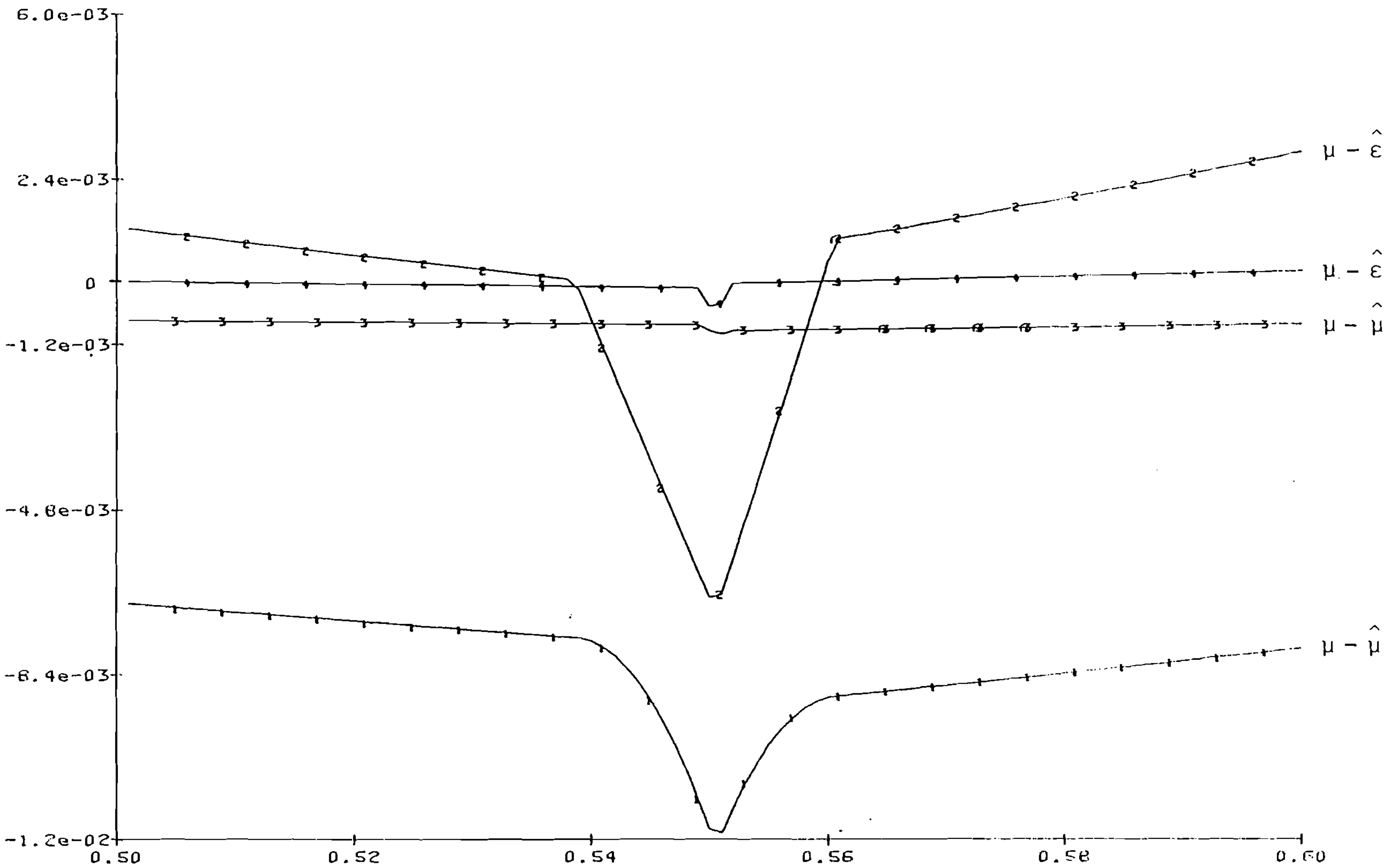

Figure 7. The dependence of $\mu - \hat{\mu}$ and $\mu - \hat{\varepsilon}$ on u (for curves 1 and 2, T = 0.01; for curves 3 and 4, T = 0.001).

$$\lim_{T \to +0} \hat{u}(T) = u^* \quad .$$

On the other hand, functions f_k, y_s, and P are such that $\hat{u}(T)$ can be described by means of the Taylor formulae:

$$\hat{u}(T+\Delta T) = \hat{u}(T) + \Delta T \hat{u}'_T + o(\Delta T) \quad .$$

Proceeding to the limit $\Delta T \to -T$, we find that

$$u^* = \hat{u}(T) - T\hat{u}'_T + o(T) \quad .$$

Naturally point

$$u = \hat{u}(T) - Tu'_T \qquad (20)$$

may be taken as a new approximation of u^*.

According to the principal property of the Taylor approximation, value $\mathrm{abs}(u_1 - u^*)$ is proportional to T^2, which means an improvement in the accuracy. Umnov (1974) demonstrates that the sequential use of (20) produces a series of points {u} that convergences to u^*.

In practice the use of (20) is equivalent to the problem of finding u'_T, i.e. the derivative of the implicit function $\hat{u}(T)$ with respect to T. From the implicit functions theorem it follows that components of u'_T satisfy the system of linear equations:

$$\sum_{r=1}^{L} \frac{\partial^2 \hat{\varepsilon}}{\partial u_r \partial u_q} (\hat{u}'_T)_r = - \frac{\partial^2 \hat{\varepsilon}}{\partial u_q \partial T} \quad , \qquad q = [1,L] \quad . \qquad (21)$$

The matrix of the system has elements defined by (15). Below, we give expressions for the right-hand-side only.

$$\frac{\partial^2 \hat{\varepsilon}}{\partial u_r \partial T} = \frac{\partial^2 \varepsilon}{\partial u_r \partial T} + \sum_{j=1}^{n} \frac{\partial^2 \varepsilon}{\partial u_r \partial x_j} \frac{\partial \hat{x}_j}{\partial T} + \frac{\partial^2 \varepsilon}{\partial u_r \partial \mu} \frac{\partial \hat{\mu}}{\partial T}$$

$$+ \sum_{l=1}^{N} \frac{\partial^2 \varepsilon}{\partial u_r \partial E_l} \frac{\partial \bar{E}_l}{\partial T} + \sum_{k=1}^{N} \left\{ \frac{\partial \bar{E}_k}{\partial u_r} \left[\frac{\partial^2 \varepsilon}{\partial E_k \partial T} + \right.\right. \qquad (22)$$

$$+ \frac{\partial^2 \varepsilon}{\partial E_k \partial \mu} \frac{\partial \hat{\mu}}{\partial T} + \sum_{i=1}^{n} \frac{\partial^2 \varepsilon}{\partial E_k \partial x_i} \frac{\partial \hat{x}_i}{\partial T} + \sum_{t=1}^{N} \frac{\partial^2 \varepsilon}{\partial E_k \partial E_t} \frac{\partial \bar{E}_t}{\partial T} \Bigg]$$

$$+ \frac{\partial \varepsilon}{\partial E_k} \left(\frac{\partial^2 E_k}{\partial u_r \partial T} + \sum_{i=1}^{n} \frac{\partial^2 E_k}{\partial u_r \partial x_i} \frac{\partial \bar{x}_{ki}}{\partial T} \right) \Bigg\} .$$

Partial derivatives $\frac{\partial \bar{x}_{ki}}{\partial T}$, $\frac{\partial \hat{x}_j}{\partial T}$, and $\frac{\partial \hat{\mu}}{\partial T}$ can be found from the system of linear equations, which are formed by differentiating (10) and (12) with respect to T.

$$\sum_{j=1}^{n} \frac{\partial^2 E_k}{\partial x_i \partial x_j} \frac{\partial \bar{x}_{kj}}{\partial T} = - \frac{\partial^2 E_k}{\partial x_i \partial T} , \qquad i = [1,n] , \qquad (23)$$

where all coefficients are calculated at point $\bar{x}_k$.

Analogously

$$\begin{cases} \frac{\partial^2 \varepsilon}{\partial x_i \partial \mu} \frac{\partial \hat{\mu}}{\partial T} + \sum_{j=1}^{n} \frac{\partial^2 \varepsilon}{\partial x_i \partial x_j} \frac{\partial \hat{x}_j}{\partial T} = - \frac{\partial^2 \varepsilon}{\partial x_i \partial T} \\ \frac{\partial^2 \varepsilon}{\partial \mu^2} \frac{\partial \hat{\mu}}{\partial T} + \sum_{j=1}^{n} \frac{\partial^2 \varepsilon}{\partial \mu \partial x_j} \frac{\partial \hat{x}_j}{\partial T} = - \frac{\partial^2 \varepsilon}{\partial \mu \partial T} \end{cases} , \quad i = [1,n] , (24)$$

subject to all derivatives being calculated at point $\| \hat{\mu} ; \hat{x} \|$.

Let us demonstrate the procedure for model (1) - (2). We take $\hat{u} = 0.5495$, which was found in section 2.4. At this point we have

$$\hat{x}_1 = 0.5548 , \quad \hat{x}_2 = 0.4568 , \quad \hat{\mu} = 0.4154 .$$

From (23)

$$\left\| \frac{\partial \bar{x}_{ki}}{\partial T} \right\| = \begin{Vmatrix} 2 & 0 \\ 0 & 3 \end{Vmatrix} ,$$

and, from (24),

$$\frac{\partial \hat{x}_1}{\partial T} = 0.4619 \quad , \quad \frac{\partial \hat{x}_2}{\partial T} = 0.6749 \quad , \quad \frac{\partial \hat{\mu}}{\partial T} = 0.7122 \quad .$$

Since the set of active constraints of model (1) - (2) does not change during the step from $u = 0.54$ to $\hat{u} = 0.5495$, and function $\hat{\varepsilon}(T,u)$ is piecewise-quadratic with respect to u, we may take $\frac{\partial^2 \hat{\varepsilon}}{\partial u^2} = 62.90$.

From (22), we find $\frac{\partial^2 \hat{\varepsilon}}{\partial u \partial T} = 6.4904$. Note that this is not very difficult when the expressions within parentheses in (22) equal zero. Substituting these data in (21), we get $u'_T = -0.1032$, and, hence, a new approximation of u^*:

$$u = \hat{u} - Tu'_T = 0.5495 - 0.01(-0.1032) = 0.5505 \quad ,$$

which is close enough to $u^* = 3-\sqrt{6} \approx 0.5505$. If necessary, process (21) - (22) may be repeated several times until a desirable level of accuracy is achieved.

2.6. Generalization of the Approach

The proposed method for optimizing the vector of exogenous parameters of a multicriteria model has been considered for the case where it is necessary to find the best consistency of the criteria. But this approach can also be applied to other schemes for finding the equilibrium point. For example, it is possible to introduce weight coefficients to evaluate the relative importance of the unit of inconsistency of each criterion for the decision makers.

In this case the second group of constraints for problem (7) may be formulated as follows:

$$R_k(\mu, x, u) = f_k(x,u) + \mu w_k \mathrm{abs}(f_k^*(u)) - f_k^*(u) \geq 0 \quad ,$$

where $w_k > 0$ are the weight coefficients .

The model can also be supplied with additional constraints on the feasible values of criteria at the point of equilibrium.

In a more general case, it would be possible to state problem (7) in terms of a vector coefficient of inconsistency:

$$\mu = \{\mu_1, \mu_2, \ldots, \mu_N\} .$$

The equilibrium point will be defined, for example, as an extreme point of the functional $\phi(\mu_1, \mu_2, \ldots, \mu_N)$, subject to, perhaps, several additional constraints on components of μ. Note that in this case the equilibrium point might not be a Pareto one. Hence, the geometrical interpretation of the approach will be different, but its theoretical basis will not be changed.

The practical use of the parametric analytical scheme is not so constrained by the need to avoid the use of those models in which at least one of f_k^* is close to zero. For such a case a reconstruction of the model is recommended. But this should be done carefully because the addition of a constant in $f_k(x,u)$ may change the equilibrium point.

Finally, note that there is an opportunity to use the approach to find consistent solutions for a system of single criterion mathematical models. Not only does the criterion of each model differ, but the sets of constraining functions also differ.

Let us suppose that the k^{th} model is

$$\begin{array}{ll} \text{maximize} & f_k(x_k,u) \quad , \\ \text{with respect to} & x_k \in E^{n_k} \quad , \\ \text{subject to} & y_{ks}(x_k,u) \geq 0 \ , \ s = [1,m_k] \quad , \end{array}$$

where n_k is the number of variables for the k^{th} model, and u is the vector of exogenous parameters, which is common to all N models.

This problem may be reduced to the problem under analysis by linking all x_k into a new common vector of variables. The increase in the dimensions of the model that takes place will not introduce difficulties, since all auxiliary functions are separable with respect to x_k and they can be minimized independently.

A more detailed description of the problem is given in Umnov (1980).

3. PARAMETRICAL ANALYSIS OF LINEAR MULTICRITERIA MODELS

3.1. Preliminary Notes

Despite its theoretical simplicity, the approach is rather difficult to apply. The main obstacle is that SPFM is relatively ineffective for solving problems such as (6). To overcome this difficulty we can take advantage of the fact that in the approach only the result of using SPFM is important, but not its convergence properties. Therefore, we may try to replace SPFM by a more effective algorithm (or a combination of algorithms), which allows us to achieve the same results.

In this section we will consider a special case of linear models with exogenous parameters in free terms of constraints (5) only. Moreover, the application of the approach will be described for a specific model, a regional model of water resources allocation for Skåne, Sweden. Thus, we can simplify the evaluation of the usefulness of the method.

3.2. Regional Model of Water Resources Allocation

The proposed approach is used to find states with the best consistency for the Model of Water Resources Allocation (MWRA). This model, which was developed for the south-west Skåne region in Sweden (Andersson et al. 1979) by the 'Resources and Environment Area' of IIASA, is used to evaluate the impacts of different water supply policies, subject to certain economic and environmental constraints.

A scheme of the water supply system for Skåne is shown in Figure 8. In MWRA it is assumed that the main source of water for the region is the Kävlinge River, which flows through Lake Vomb to the Baltic Sea. Lake Vomb also serves as a partially

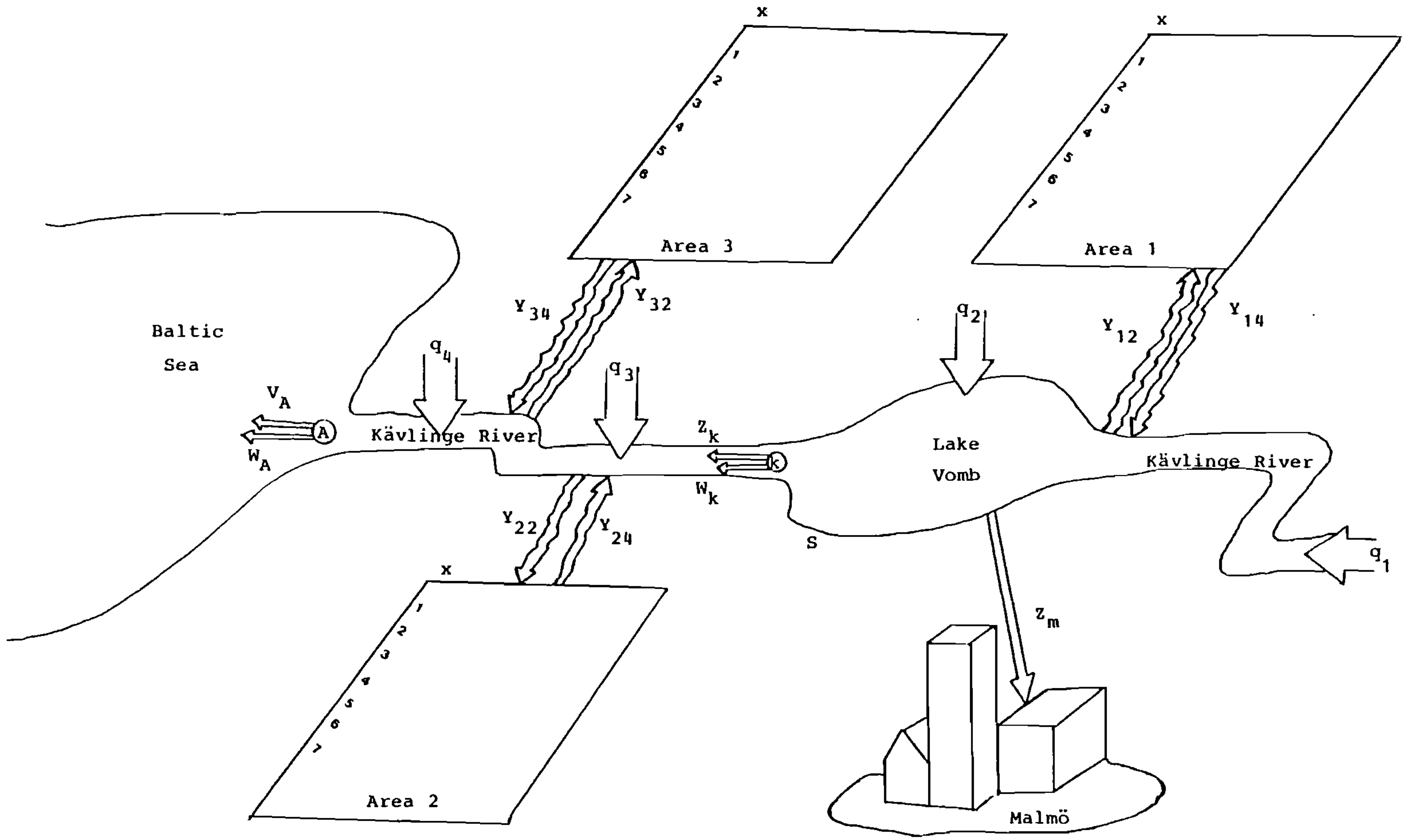

Figure 8. The water supply system for the Skåne region. (For an explanation of the notation, see pp. 65-68.)

controlled water reservoir. In considering the general water balance of the region, ground water and precipitation are also included. The regional water supply is needed for servicing the town of Malmö, for agricultural production in three areas of the region, and for maintaining ecological equilibrium in the regional water system. From an environmental point-of-view, the state of the water supply system is defined by its pollution level.

Non-natural sources of pollution are Malmö and the agricultural areas. The pollution level depends both on the technologies and area of land used in the production process.

The state of the system is characterized by the values of two water flows at points k and A (see Figure 8), by the volumes of water used for agricultural purposes and stored in the reservoir, by the pollution concentration, and finally by the volume of agricultural production. The following constraints, which define the feasible states of the model, are included:

-- the balance conditions for nodes of the water supply system;
-- the need to satisfy upper and lower bounds for water flows to the sea and the town, and for the volume of water stored in the reservoir;
-- the technological relations between the volume of fertilizer, the volume of water, and the areas of land used for agricultural production;
-- water circulation in the natural system, i.e. water inputs and losses;
-- constraints on the total area of land available for agriculture in all areas of the region.

The quality of the state of the system is evaluated using the following criteria:

-- maximization of the volume of agricultural production for all areas;
-- maximization of the volume of water remaining at the end of the growing season;

-- maximization of the water flow to the town;

-- maximization of the water flow to the sea;

-- minimization of the pollution flow to the sea.

Formally, MWRA is described in terms of the following data:

A_j - the area of land available for agriculture in area j (j = [1,3]);

x_{ij} - the area of land in area j in which technology i is used (in the version of MWRA described in this paper seven types of technologies were considered);

a_{kij} - k^{th} normative coefficient, related to a unit of land characterizes technology i in area j (five types of coefficients were used:

- real yields per unit of land (k=1),
- required water volume (k=2),
- volume of fertilizer required (k=3),
- water loss (k=4),
- fertilizer loss (k=5));

S_o - volume of water in the reservoir at the beginning of the growing season;

S - volume of water in the reservoir at the end of the growing season;

τ - the length of the growing season;

Z_m - the volumetric flow rate of water supply to the town;

Z_k - the volumetric flow rate of water from Lake Vomb;

V_A - the volumetric flow rate of water to the sea;

W_k - the volumetric flow rate of pollutants from the lake;

W_A - the volumetric flow rate of pollutants to the sea;

α_A - the maximal acceptable level of pollution in the water flow to the sea;

Y_{j1} - the volume of agricultural production from area j;

Y_{j2} - the volume of water used in area j for agricultural purposes;

Y_{j3} - the volume of fertilizers used in area j;

Y_{j4} - the volume of water returned to the system from area j;

Y_{j5} - the volume of fertilizers entering the water system from area j;

q_n - the difference between the water volumes entering and leaving the water system by means of natural exchange at control point n (see Figure 8);

ϕ - the coefficient representing the removal of pollutants through natural processes within the lake;

ψ_n - the concentration of natural pollutants in the water at control point n.

The constraints defining feasible states of the model are given below.

Land-use constraints in area j are

$$\sum_{i=1}^{m_t} x_{ij} = A_j \quad ,$$

where m_t is the number of technologies in use.

The dependence of agricultural production on the technologies used is described by

$$Y_{jk} = \sum_{i=1}^{m} a_{kij} x_{ij} \quad .$$

The water balance constraints for all nodes of the system are:

-- for the lake

$$S = S_o + \tau(q_1 + q_2) - \tau(Z_k + Z_m) + Y_{14} - Y_{12} \quad ,$$

$$\underline{S} \le S \le \bar{S} \quad , \text{ and}$$

$$\underline{Z}_m \le Z_m \le \bar{Z}_m \quad ,$$

where $\underline{S}$, $\bar{S}$, $\underline{Z}_m$, and $\bar{Z}_m$ are the lower and upper bounds for S and Z_m, respectively;

-- for the river

$$V_A = Z_k + q_3 + \quad _4 + \frac{1}{\tau}(Y_{24} + Y_{34}) - \frac{1}{\tau}(Y_{22} + Y_{32}) \quad ,$$

$$\underline{V}_A \le V_A \quad ,$$

where $\underline{V}_A$ is the lower bound for V_A.

Here the following relations should be valid:

$$Y_{12} \le \tau q_1 \quad ,$$

$$Y_{22} + Y_{32} \le \tau(Z_k + q_3) \quad .$$

Finally, there is a set of constraints defining the environmental characteristics of the system:

-- the flow of pollutants from the lake,

$$W_k = (1 - \phi)(\psi_1 q_1 + \psi_2 q_2 + \frac{1}{\tau} Y_{15}) ;$$

-- the flow of pollutants to the sea,

$$W_A = \frac{1}{\tau}(Y_{25} + Y_{35}) + \psi_3 q_3 + \psi_4 q_4 + W_k$$

and

-- the limiting constraint representing the concentration of pollutants in the sea,

$$W_A \le \alpha_A V_A \quad .$$

The set of criteria can also be formulated as follows:

maximize Y_{11} ,

maximize Y_{21} ,

maximize Y_{31} ,

maximize S ,

maximize Z_m ,

maximize V_A ,

minimize W_A .

3.3. Some Examples of Problems Solved with the Use of MWRA

Numerical data for one version of MWRA are given in Table 1. In this version it is assumed that the values of A_j, a_{kij},

Table 1. The parameter values of MWRA.

k	j \ i	1	2	3	4	5	6	7	Dimension
1	1	0	4	5.5	8	4.5	6.8	9.5	Ton/ha
	2	0	4	5.5	9.2	4.5	6.8	10.8	
	3	0	4	5.5	9.2	4.5	6.8	10.8	
2	1	0	0.3	0.3	0.3	0.55	0.55	0.55	$10^3 m^3$/ha
	2	0	0.3	0.3	0.3	0.55	0.55	0.55	
	3	0	0.3	0.3	0.3	0.55	0.55	0.55	
3	1	0	0	80	150	0	80	150	kg/ha
	2	0	0	80	180	0	80	180	
	3	0	0	80	180	0	80	180	
4	1	0	0.06	0.06	0.06	0.11	0.11	0.11	$10^3 m^3$/ha
	2	0	0.03	0.03	0.03	0.055	0.055	0.055	
	3	0	0.06	0.06	0.06	0.11	0.11	0.11	
5	1	0	0	12	22.5	0	12	22.5	kg/ha
	2	0	0	12	27	0	12	27	
	3	0	0	12	27	0	12	27	

A_1 = 3,000 ha

A_2 = 2,500 ha

A_3 = 2,300 ha

$$\left.\begin{aligned} q_1 &= 1.8 \\ q_2 &= 1.5 \\ q_3 &= 0.8 \\ q_4 &= 0.7 \end{aligned}\right\} (m^3/sec \equiv 10^3 m^3/10^3 sec)$$

$\psi_1 = \psi_2 = 1 (kg/10^3 m^3)$

$\psi_3 = 2 (kg/10^3 m^3)$

$\psi_4 = 1.5 (kg/10^3 m^3)$

$\tau = 2{,}590\ (10^3\ sec)$

$\phi = 0.9$

$\bar{Z}_m = 2 (10^3 m^3/10^3\ sec)$

$\bar{S} = 30{,}000\ (10^3 m^3)$

$S_o = 30{,}000\ (10^3 m^3)$

$\alpha_A = 10 (g/m^3 = kg/10^3 m^3)$

$\bar{V}_A = 6 (10^3 m^3/10^3\ sec^3)$

S_o, $\underline{S}$, $\bar{S}$, τ, $\underline{Z}_m$, $\bar{Z}_m$, V_A, ψ_n, q_n, ϕ, and α_A are predetermined (i.e. they are parameters of the model). We have tried to find values for some of the above parameters, so that there is an acceptable degree of consistency among the criteria.

Table 2 gives the results of calculations made with the help of MWRA for the scheme (6) - (8). The case considered was for a growing season with a normal level of precipitation and with the following exogenous parameters:

$$\underline{S} = 21 \cdot 10^6 m^3 \quad ,$$

$$\underline{Z}_m = 1 \ m^3/sec \quad ,$$

$$\underline{V}_A = 7 \ m^3/sec \quad .$$

The consistency achieved was 23.1%, Table 2 presents the optimal values for criteria $f_k^*(u)$ (see the column 'Opt value') and $f_k(x^*,u)$ (see the column 'Value'). The values of the model variables for the equilibrium point are also given in Table 2. If the exogenous parameter values are changed, the equilibrium point for the model will change. For example, for values

$$\underline{S} = 15 \cdot 10^6 m^3 \quad ,$$

$$\underline{Z}_m = 1.5 \ m^3/sec \quad ,$$

$$\underline{V}_A = 6 \ m^3/sec \quad ,$$

the consistency equals 0.45. The variables used in this case are shown in Table 3. These two examples demonstrate that a variation in the exogenous parameters of MWRA may significantly change both the equilibrium point and the consistency of the criteria.

A more interesting example of the dependence of the consistency on the exogenous parameters is shown in Figure 9. In this figure a piece-wise linear approximation of the dependence of the coefficient of consistency on both the volume of water S in the lake at the end of the period and the average level of precipitation q during this period. In examining Figure 9, we find

Table 2. State of MWRA for $\underline{S} = 21$, $\underline{Z}_m = 1$, and $\underline{V}_A = 7$.

EXOGENOUS CONSTRAINTS:

Final quantity of water in the lake	=	21000.00(1000cub.m)
Capacity of natural water sources	=	1.00(to norm.)
Minimal flow to Malmo	=	1.00(cub.m/sec)
Minimal flow to the sea	=	7.00(cub.m/sec)
Concent.level of poll. in the sea	=	10.00(g/cub.m)

List of criteria: yields, water in the lake, flow to Malmo;
flow to the sea, flow of pollution to the sea

RESULTS: Min.inconsistency = 76.9%

# of criterion		Value	Opt.value	Consist.%
1	max	5477.82497	23733.30477	23.1
2	max	5369.29934	23263.10502	23.1
3	max	5179.31893	22439.99315	23.1
4	max	21000.00000	21711.99914	96.7
5	max	1.00000	1.27490	78.4
6	max	7.00000	7.27490	96.2
7	min	9.94286	5.62000	23.1

Technology #	Area 1 ha	Area 2 ha	Area 3 ha
1	2315.27	1622.57	1005.17
2	0.	519.81	1294.83
3	0.	0.	0.
4	684.73	357.61	0.
5	0.	0.	0.
6	0.	0.	0.
7	0.	0.	0.

	Area 1	Area 2	Area 3
Yield (ton)	5477.825	5369.299	5179.319
Water in. (1000cub.m)	205.418	263.228	388.449
Water out.	41.084	26.323	77.690
Fertil.in.(ton)	102.709	64.370	0.
Fertil.out.	15.406	9.656	0.

Water in the lake	21000.000(1000cub.m)
Flow out the lake	5.711(cub.m/sec)
Flow into the sea	7.000(cub.m/sec)
Flow to Malmo	1.000(cub.m/sec)
Pollution out the lake	3.565(g/sec)
Pollution to the sea	9.943(g/sec)

Table 3. State of MWRA for $\underline{S} = 15$, $\underline{Z}_m = 1.5$, and $\underline{V}_A = 6$.

EXOGENOUS CONSTRAINTS:

Final quantity of water in the Lake	=	15000.00(1000cub.m)
Capacity of natural water sources	=	1.00(to norm.)
Minimal flow to Malmo	=	1.50(cub.m/sec)
Minimal flow to the sea	=	6.00(cub.m/sec)
Concent.level of poll. in the sea	=	10.00(g/cub.m)

List of criteria: yields, water in the lake, flow to Malmo, flow to the sea, flow of pollution to the sea

RESULTS: Min.inconsistency = 54.7%

# of criterion		Value	Opt.value	Consist.%
1	max	12897.23203	28500.00000	45.3
2	max	12218.43035	27000.00000	45.3
3	max	11240.95592	24840.00000	45.3
4	max	19678.60633	23006.99914	85.5
5	max	1.50000	2.00000	75.0
6	max	6.00000	9.09151	66.0
7	min	8.69676	5.62000	45.3

Technology #	Area 1 ha	Area 2 ha	Area 3 ha
1	0.	0.	0.
2	1205.54	0.	0.
3	0.	0.	0.
4	0.	0.	0.
5	1794.46	2346.28	2158.58
6	0.	0.	0.
7	0.	153.72	141.42

	Area 1	Area 2	Area 3
Yield (ton)	12897.232	12218.430	11240.956
Water in. (1000cub.m)	1348.616	1375.000	1265.000
Water out.	269.723	137.500	253.000
Fertil.in.(ton)	0.	27.669	25.456
Fertil.out.	0.	4.150	3.818

Water in the lake	19678.606(1000cub.m)
Flow out the lake	5.369(cub.m/sec)
Flow into the sea	6.000(cub.m/sec)
Flow to Malmo	1.500(cub.m/sec)
Pollution out the lake	2.970(g/sec)
Pollution to the sea	8.697(g/sec)

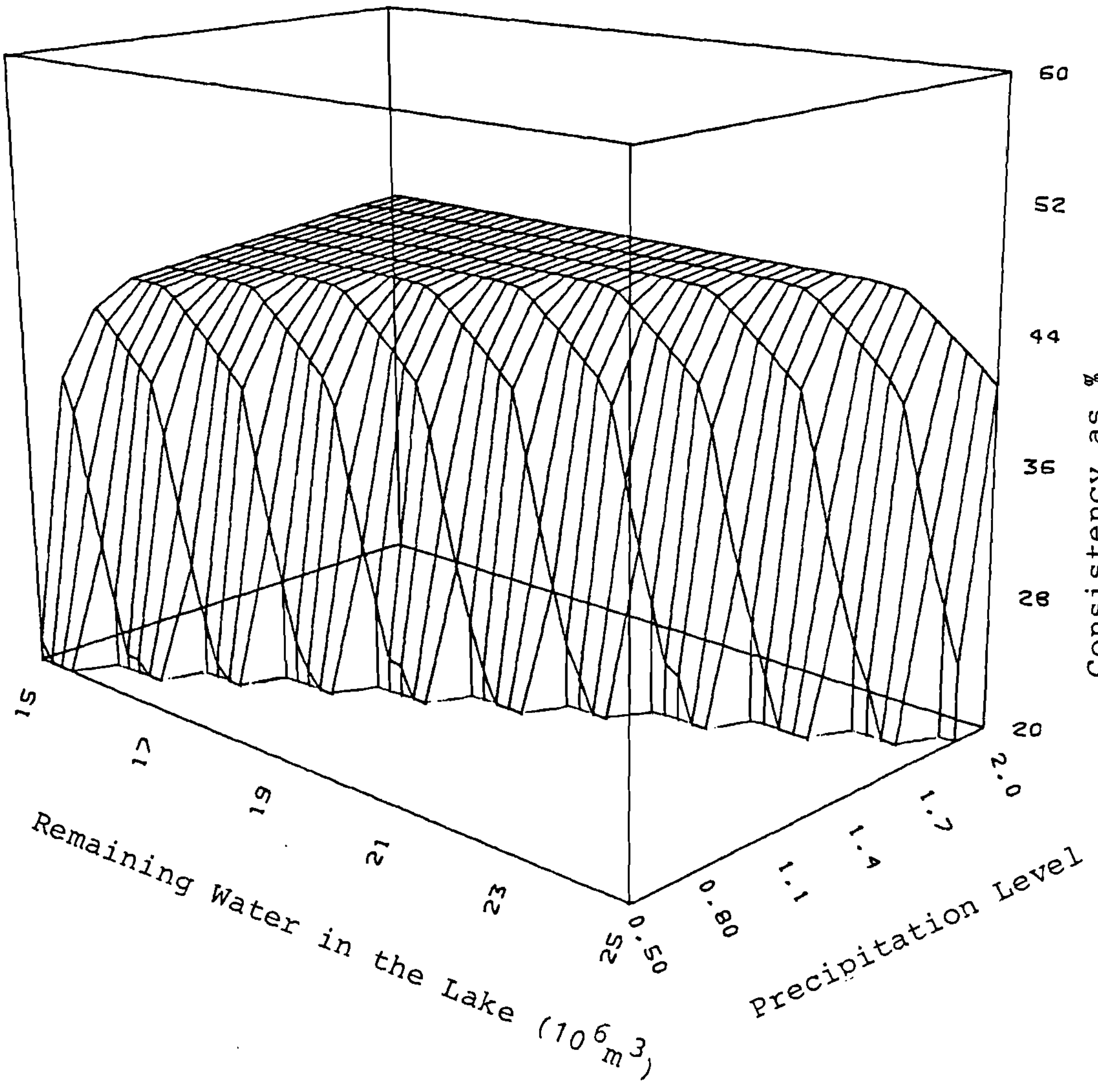

Figure 9. The dependence of the consistency on the level of precipitation and the remaining water in the lake.

that for values of $\underline{S}$ and q satisfying the set of constraints

$$0.5 \le q \le 2.0 \quad ,$$
$$15.0 \le \underline{S} \le 25 \quad ,$$
$$\underline{S} \le 20.9q - 18.8 \quad ,$$

where q is defined as the ratio of the current level of precipitation to the average level, the consistency is relatively high: 45% - 50%. For the other $\underline{S}$ and q, the consistency is much lower, or MWRA is infeasible. In other words, the model forecasts that there exists a minimal value for the remaining water in the lake, when a consistency of about 40% can be achieved for a given q. This minimal value of $\underline{S}$ is equalled approximately by

$$S_{min} = 20.9q - 18.8 \quad .$$

Finally, let us consider a graphic interpretation of another problem of optimizing the consistency. Suppose that we choose values for the three exogenous parameters of the model:

- -- the lower limit of the flow to the town;
- -- the lower limit of the flow to the sea; and
- -- the lower level of the remaining water in the lake at the end of the growing season.

The chosen values provide MWRA with the best consistency for the given criteria, subject to

$$\tau(\underline{Z}_m + \underline{V}_A) + \underline{S} = C \quad , \tag{25}$$

where C is a given constant. In this case we will solve the problem

$$\min_{u \in \Omega} \{ \min_{x} \mu(x) \} \quad ,$$

using the notation of (8).

In Figure 10 a piecewise linear approximation of the value of $(1-\min_x \mu(x))100\%$ is shown for $C = 4.10^7$ m^3. The exogenous parameters $\underline{V}_A$, $\underline{Z}_m$, and $\underline{S}$ are linked by equation (25). Hence, it is sufficient to consider the dependence of $\hat{\mu}$ on $\underline{V}_A$ and $\underline{Z}_m$. In Figure 10 only feasible points are considered. It can be concluded that the consistency changes within a range of approximately 22% to 33%. Hence, the search for the optimal points may provide useful information for decision-making processes. This can be also proved by comparing Tables 4, 5, and 6, which contain the solutions of MWRA for different values of $\underline{S}$, $\underline{V}_A$, $\underline{Z}_m$.

3.4. Determining the Optimal Values for the Exogenous Parameters of MWRA

Let us consider the procedure for determining the optimal values for the exogenous parameters of linear models, which are described by functions $f_k(x,u)$ and $y_s(x,u)$ of the following form

$$f_k(x,u) = \sum_{i=1}^{n} c_{ki}x_i \quad , \quad k = [1,N] \quad , \quad \text{and}$$

$$y_s(x,u) = \rho_{so} + \sum_{j=1}^{L} \rho_{sj}u_j + \sum_{i=1}^{n} a_{si}x_i \quad , \quad s = [1,m] \quad . \tag{26}$$

Note that by using the free term of $y_s(x,u)$ as a linear function of u, there will not be a significant loss of equality.

A quadratic penalty function $P(T,\alpha)$ is chosen:

$$P(T,\alpha) = \frac{1}{2T}\left(\frac{\alpha - \text{abs}(\alpha)}{2}\right)^2 \quad . \tag{27}$$

It has continuous derivatives for all $T > 0$. Let us assume that all auxiliary functions are minimized by an algorithm that uses only the values of the functions and their gradients.

As has already been shown, determination of the optimal values for the exogenous parameters consists in minimizing the

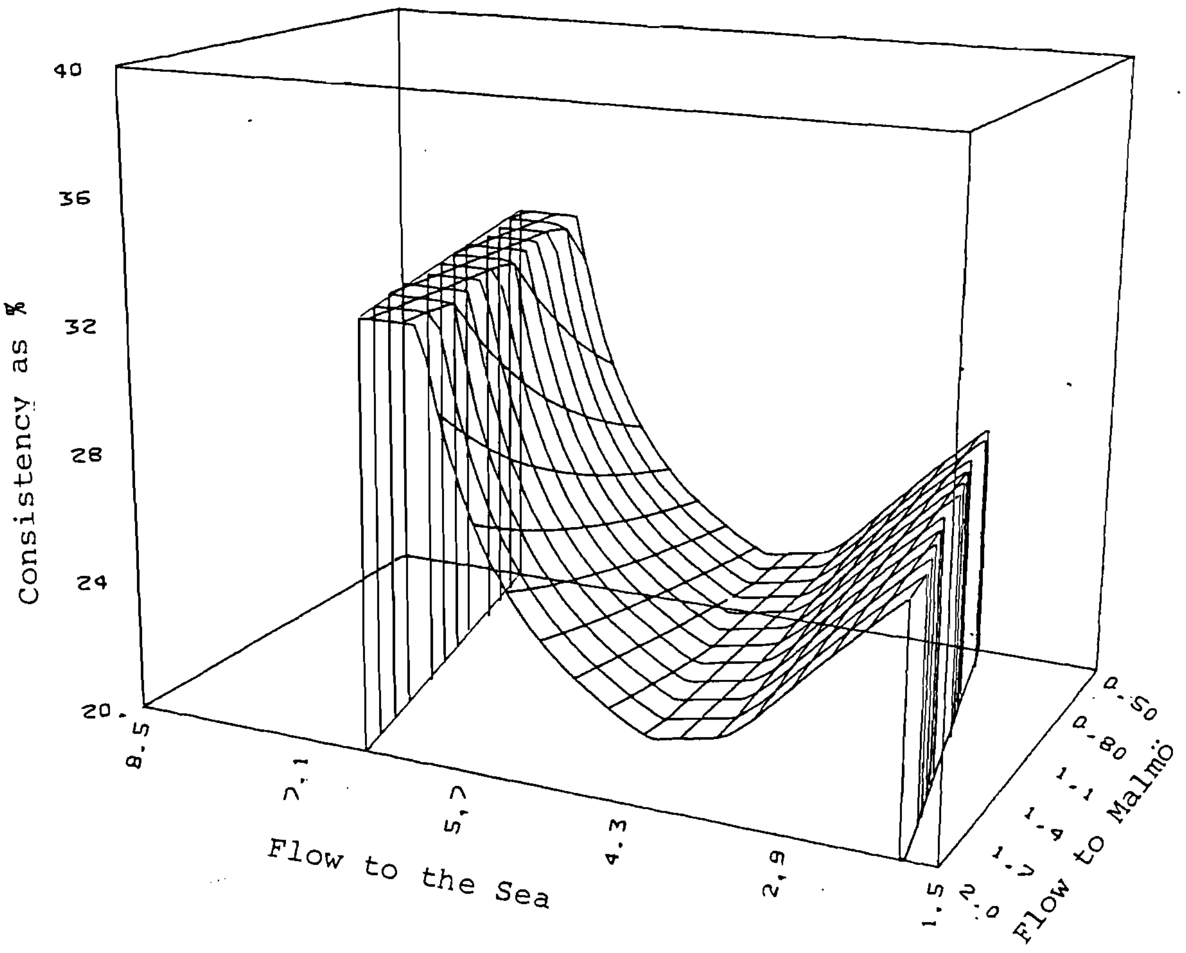

Figure 10. The dependence of the consistency on $\underline{V}_A$ and $\underline{Z}_m$. (All flows are in m^3/sec.)

Table 4. State of MWRA for $\underline{V}_A = 4$ and $\underline{Z}_m = 1$.

EXOGENOUS CONSTRAINTS:

Final volume of water in the lake	= 28550.00(1000cub.m)
Capacity of natural water sources	= 1.00(to norm.)
Minimal flow to Malmo	= 1.00(cub.m/sec)
Minimal flow to the sea	= 4.00(cub.m/sec)
Concent.level of poll. in the sea	= 10.00(g/cub.m)

List of criteria: yields, water in the lake, flow to Malmo, flow to the sea, flow of pollution to the sea

RESULTS: Min.inconsistency = 73.2%

# of criterion		Value	Opt.value	Consist.%
1	max	6866.85981	25589.99357	26.8
2	max	6662.26434	24827.54946	26.8
3	max	6493.86487	24199.99315	26.8
4	max	28550.00000	30000.00000	95.2
5	max	1.00000	1.35985	73.5
6	max	4.00000	4.35985	91.7
7	min	9.73192	5.62000	26.8

Technology #	Area 1 ha	Area 2 ha	Area 3 ha
1	2141.64	1254.22	676.53
2	0.	922.87	1623.47
3	0.	0.	0.
4	858.36	322.91	0.
5	0.	0.	0.
6	0.	0.	0.
7	0.	0.	0.

	Area 1	Area 2	Area 3
Yield (ton)	6866.860	6662.264	6493.865
Water in. (1000cub.m)	257.507	373.735	487.040
Water out.	51.501	37.373	97.408
Fertil.in.(ton)	128.754	58.124	0.
Fertil.out.	19.313	8.719	0.

Water in the lake	28550.000(1000cub.m)
Flow out the lake	2.780(cub.m/sec)
Flow into the sea	4.000(cub.m/sec)
Flow to Malmo	1.000(cub.m/sec)
Pollution out the lake	3.716(g/sec)
Pollution to the sea	9.732(g/sec)

Table 5. State of MWRA for $\underline{V}_A = 5$ and $\underline{Z}_m = 1$.

EXOGENOUS CONSTRAINTS:

Final volume of water in the lake	= 26260.00(1000cub.m)
Capacity of natural water sources	= 1.00(to norm.)
Minimal flow to Malmo	= 1.00(cub.m/sec)
Minimal flow to the sea	= 5.00(cub.m/sec)
Concent.level of poll. in the sea	= 10.00(g/cub.m)

List of criteria: yields, water in the lake, flow to Malmo, flow to the sea, flow of pollution to the sea

RESULTS: Min.inconsistency = 77.6%

# of criterion		Value	Opt.value	Consist.%
1	max	4727.09371	21066.63811	22.4
2	max	4832.14024	21534.78562	22.4
3	max	4891.64953	21799.99315	22.4
4	max	26260.00000	26891.99914	97.6
5	max	1.00000	1.24402	80.4
6	max	5.00000	5.24402	95.3
7	min	9.97894	5.62000	22.4

Technology #	Area 1 ha	Area 2 ha	Area 3 ha
1	2409.11	1771.53	1077.09
2	0.	359.58	1222.91
3	0.	0.	0.
4	590.89	368.89	0.
5	0.	0.	0.
6	0.	0.	0.
7	0.	0.	0.

	Area 1	Area 2	Area 3
Yield (ton)	4727.094	4832.140	4891.650
Water in. (1000cub.m)	177.266	218.542	366.874
Water out.	35.453	21.854	73.375
Fertil.in.(ton)	88.633	66.401	0.
Fertil.out.	13.295	9.960	0.

Water in the lake	26260.000(1000cub.m)
Flow out the lake	3.689(cub.m/sec)
Flow into the sea	5.000(cub.m/sec)
Flow to Malmo	1.000(cub.m/sec)
Pollution out the lake	3.483(g/sec)
Pollution to the sea	9.979(g/sec)

Table 6. State of MWRA for $\underline{V}_A = 6.5$ and $\underline{Z}_m = 1$.

EXOGENOUS CONSTRAINTS:

Final volume of water in the lake	= 22825.00(1000cub.m)
Capacity of natural water sources	= 1.00(to norm.)
Minimal flow to Malmo	= 1.00(cub.m/sec)
Minimal flow to the sea	= 6.50(cub.m/sec)
Concent.level of poll. in the sea	= 10.00(g/cub.m)

List of criteria: yields, water in the lake, flow to Malmo, flow to the sea, flow of pollution to the sea

RESULTS: Min.inconsistency = 70.3%

# of criterion		Value	Opt.value	Consist.%
1	max	1799.19675	6066.63811	29.7
2	max	1839.17890	6201.45229	29.7
3	max	2069.07626	6976.63382	29.7
4	max	22825.00000	23006.99914	99.2
5	max	1.00000	1.07027	93.4
6	max	6.50000	6.57027	98.9
7	min	9.57326	5.62000	29.7

Technology #	Area 1 ha	Area 2 ha	Area 3 ha
1	2775.10	2300.09	1991.47
2	0.	0.	147.96
3	0.	0.	0.
4	224.90	199.91	160.57
5	0.	0.	0.
6	0.	0.	0.
7	0.	0.	0.

	Area 1	Area 2	Area 3
Yield (ton)	1799.197	1839.179	2069.076
Water in. (1000cub.m)	67.470	59.973	92.559
Water out.	13.494	5.997	18.512
Fertil.in.(ton)	33.735	35.984	28.902
Fertil.out.	5.060	5.398	4.335

Water in the lake	22825.000(1000cub.m)
Flow out the lake	5.049(cub.m/sec)
Flow into the sea	6.500(cub.m/sec)
Flow to Malmo	1.000(cub.m/sec)
Pollution out the lake	3.165(g/sec)
Pollution to the sea	9.573(g/sec)

function $\hat{\varepsilon}(u)$, which can be obtained from (11) by substituting $\hat{\mu}$ and $\hat{x}$ for μ and x.

Components of the gradient of $\hat{\varepsilon}$ in the general case are defined by (14). But this formula can be simplified for MWRA by taking into account (26) and (27). At first, we have

$$\frac{\partial f_k}{\partial u_r} = 0 \qquad \text{and} \qquad \frac{\partial y_s}{\partial u_r} = \rho_{sr} \quad .$$

Hence,

$$\frac{\partial \hat{\varepsilon}}{\partial u_r} = \sum_{s=1}^{m} \frac{\hat{y}_s - \text{abs}(\hat{y}_s)}{2T} \rho_{sr}$$

$$+ \sum_{k=1}^{N} \frac{\hat{V}_k - \text{abs}(\hat{V}_k)}{2T} (1 + \hat{\mu}\,\text{sign}\,\bar{E}_k) \frac{\partial \bar{E}_k}{\partial u_r} \quad ,$$

where $\hat{y}_s = y_s(\hat{x},u)$ and $\hat{V}_k = V_k(\hat{\mu},\hat{x},u)$.

$$\frac{\partial \bar{E}_u}{\partial u_r} = \frac{\partial E_k}{\partial u_r}(\bar{x}_k) = \sum_{s=1}^{m} \frac{\bar{y}_{sk} - \text{abs}(\bar{y}_{sk})}{2T} \rho_{sr} \quad ,$$

where $\bar{y}_{sk} = y_s(\bar{x}_k,u)$.

Finally, we find

$$\frac{\partial \hat{\varepsilon}}{\partial u_r} = \sum_{s=1}^{m} \frac{\hat{y}_s - \text{abs}(\hat{y}_s)}{2T} \rho_{sr}$$

$$+ \sum_{k=1}^{N} \frac{\hat{V}_k - \text{abs}(\hat{V}_k)}{2T} (1 + \hat{\mu}\,\text{sign}\,\bar{E}_k) \sum_{t=1}^{m} \frac{\bar{y}_{tk} - \text{abs}(\bar{y}_{tk})}{2T} \rho_{tk} \, , \tag{28}$$

$$\text{for all } r = [1,L] \quad .$$

Therefore, in order to use the algorithm for minimizing $\hat{\varepsilon}(u)$, we must find vectors $\bar{x}_k$ and $\|\hat{\mu};\hat{x}\|$. The values of $\bar{y}_{sk}$, $\bar{E}_k$, $\hat{y}_s$, $\hat{V}_k$, and $\hat{\varepsilon}$ are easily calculated because their form is explicit.

3.5. Modeling the Smooth Penalty Function Method

A method for solving the problem is considered to be effective if it allows the solution to be found with a reasonable level of computational effort on the part of the user. SPFM, which forms the theoretical basis for the proposed approach, is not very effective for most practical problems. For the models described by functions (26), more effective algorithms can be used. In the case of MWRA a scheme based on the combined use of two algorithms was implemented. This scheme is outlined below.

The problem of minimizing each of the auxiliary functions (9) and (11) is divided into two stages. First, the linear problem (6), or (7), is solved by means of a standard simplex procedure. At the second stage the following quadratic problem is solved.

Minimize with respect to $\| \bar{x}_k ; \bar{w}_k \|$

$$- \sum_{i=1}^{n} c_{ki} \bar{x}_{ki} + \frac{T}{2} \sum_{s \in A} \left(\frac{\bar{w}_{ks} + \mathrm{abs}(\bar{w}_{ks})}{2} \right)^2 ,$$

subject to (29)

$$-\infty < \bar{w}_{ks} < +\infty ,$$

$$T\bar{w}_{ks} + \rho_{so} + \sum_{r=1}^{L} \rho_{sr} u_r + \sum_{i=1}^{n} a_{si} \bar{x}_{ki} = 0 ,$$

for all $s \in A$.

The set A contains indices of these constraints for problem (6), which are active at point x_k^*, i.e., for all $s \in A$,

$$y_s(x_k^*, u) \leq 0$$

must be valid independently of whether x_k^* is the optimal point of (6), or is the point at which the nfeasibility of (6) has been found.

It is evident that problems (10) and (29) are similar. This may be proved by substituting $y_s(\bar{x}_k, u)$ for $T\bar{w}_{ks}$ and the specific expression for $P(T,\alpha)$. The form of (29) is more suited to the calculations, since the components of the gradient of the auxiliary function have the simple form

$$\operatorname{grad}_{u_r} E_k = -\frac{1}{2} \sum_{s \in A} \rho_{sr} (\bar{w}_{ks} + \operatorname{abs}(\bar{w}_{ks})) \quad . \tag{30}$$

To find $\bar{E}_k$ and $\bar{y}_{ks}$ it is not always necessary to solve (29). From theorem 4 (Fiacco and McCormick 1968), it follows that if problem (6) is feasible and rows $\{a_{si}, i = [1,n]\}$ are linearly independent for $s \in A$, then the optimal solution of (29) is the optimal vector of the dual variables for problem (6), where T is sufficiently small.

For the case where it is possible to recognize the linear independency of $\{a_{si}, i = [1,n]\}$, $s \in A$, values of $\bar{E}_k$ and $\bar{y}_{ks}$ can be calculated by

$$\begin{aligned} \bar{y}_{ks} &= -Tw_{ks}^* \quad , \\ \bar{E}_k &= -\sum_{i=1}^{n} c_{ki} x_{ki}^* - \frac{T}{2} \sum_{s=1}^{m} (w_{ks}^*)^2 \quad , \end{aligned} \tag{31}$$

where w^* is the dual optimal vector for problem (6).

For all other cases problem (29) must be solved. The effort required to do this, however, will not be too great, if the point $\bar{x}_k$ is equal to x_k^* and if all $\bar{w}_{ks}$ are zeros at the first iteration. Moreover, problem (29) does not contain all functions $y_s(x,u)$, but only those for $s \in A$. All the above conclusions also apply to problem (12).

In practice, it is sometimes reasonable to extend the set A, in order to increase the reliability of the scheme. We will include in A all s for which

$$\varepsilon + y_s(x_k^*, u) \le 0$$

is valid, where $\varepsilon > 0$ is a small parameter. This avoids the introduction of errors. It is preferable to evaluate $\bar{E}_k$ and $\bar{y}_{ks}$ by means of (29) in some cases of uncertainty, since the statement of (29) is independent of whether or not (6) is feasible. This is well illustrated by Figure 11, which shows the dependence of $\hat{\varepsilon}$ and μ on $\underline{V}_A$ in MWRA, subject to

$$6.2 \le \underline{V}_A \le 7.2 \quad ,$$

$$\underline{Z}_m = 1 \quad ,$$

$$S = 40 - \tau(\underline{Z}_m + \underline{V}_A) \quad .$$

It was found that the solution of a modification of problem (7) is optimal for $\underline{V}_A \le 7.1$ and infeasible for $\underline{V}_A > 7.1$. However, from $\hat{\varepsilon}(\underline{V}_A)$ in Figure 11, we can see that the model is infeasible even for $\underline{V}_A = 6.65$, because inappropriate tolerance parameters were chosen. In this case the use of w_k^* in (31) instead of solution (29) to evaluate $\bar{E}_k$ and $\bar{y}_{ks}$ might destroy the convergence of the minimization of $\hat{\varepsilon}(u)$.

To conclude, it can be seen that the linkage described above enables us to create an effective method possessing all the desirable properties of SPFM. This method can be used for any model described by (26), or even for certain more complex models.

3.5. Some Practical Applications of the Approach Using MWRA

The analytical method proposed for the multicriteria model MRWA was implemented by the Regional Development Group of IIASA on the VAX/1178 computer, under the UNIX operating system. During development of the required software, the main effort was devoted to achieving maximal efficiency through the use of highly efficient standard subroutines. Since VAX/1178 is a virtual machine, it has the capacity for storing a large volume of

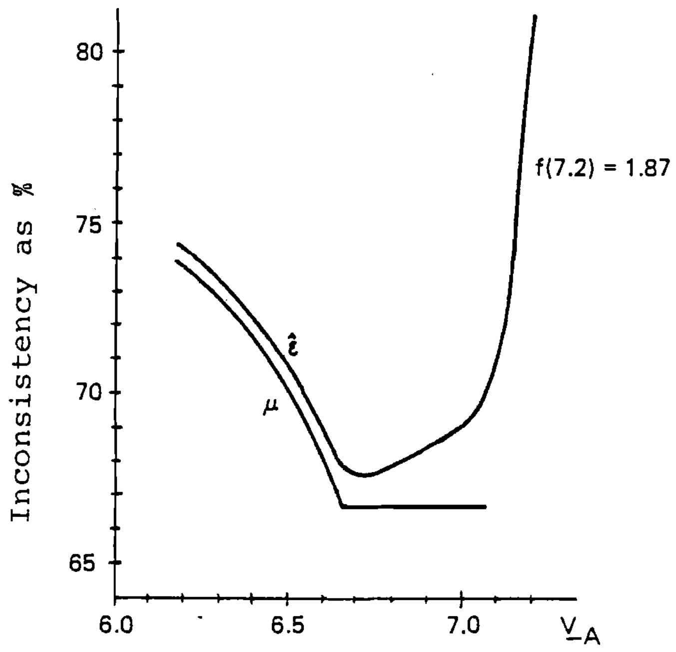

Figure 11. The dependence of $\hat{\varepsilon}$ and μ on $\underline{V}_A$.

intermediate data and the total operational time required may thus be reduced. The main element of the software system completing the approach is the MINOS program package (Murtagh and Saunders 1980) for solving the linear problems (6) and a modification of (7), and the nonlinear and, generally speaking, nonconvex problem:

$$\begin{array}{ll} \text{minimize} & \hat{\varepsilon}(u) \;, \\ \text{subject to} & u \in \Omega \;. \end{array} \tag{32}$$

The set Ω was defined in an explicit way as a system of constraints on the components of vector u. Therefore, in accordance with the procedures for using MINOS, it is sufficient for the normal run solving (32) to be able to calculate $\hat{\varepsilon}(u)$ and $\text{grad}_u\, \hat{\varepsilon}$ at any point u. It has been already shown that $\hat{\varepsilon}(u)$ and components of $\text{grad}_u\, \hat{\varepsilon}$ may be found from (6), (7), and (29).

We have seen that the recursive use of MINOS presents a difficult problem. For this reason we have decided to organize the solution procedure for (32) as an independent run supplied with all the necessary information. The solution procedures for (6), (7), and (29) consist of another run, which begins only when required. Having obtained values for $\hat{\varepsilon}(u)$ and the components of $\text{grad}_u\, \hat{\varepsilon}$, control is returned to the first procedure, which lies dormant during operation of the second. The amount of time required for operating the second procedure is the greatest, because the dimensions of problems (6), (7), and (29) are considerably larger than those of (32). To reduce user time, runs for (6), (7), and (29) commence with the optimal solution obtained at the preceding step. We illustrate the operational features of the method with three examples.

Table 7 presents the results of running the procedure for optimizing the exogenous parameters of model (1) - (2). These parameters were considered in this case to be independent. The penalty function $P(T,\alpha)$ was quadratic and the region of feasibility Ω was defined by the system of constraints:

$$\begin{cases} u \geq 0.1 \;, \\ v \geq 0.2 \;, \\ u + v \leq 1.5 \;. \end{cases} \tag{33}$$

Table 7. The numerical results of running the main procedure for model (1) - (2) using the square penalty function.

#p.	u	v	E	grad E(u)	grad E(v)
Iteration 1					
0	0.200000000	0.800000000	0.718931580	-0.869239520	0.236521180
1	1.069239520	0.563478820	1.376846797	13.452038706	12.926606841
2	0.634619760	0.681739410	0.464301275	-0.293817978	0.271295500
3	0.851929639	0.622609116	0.435709547	0.204558680	-0.284875032
4	0.792854923	0.638683421	0.419051122	0.169010419	-0.217169949
5	0.751737215	0.649871601	0.425172426	-0.231862062	0.263711706
6	0.783957636	0.641104384	0.417909810	0.019063932	-0.034147126
Iteration 2					
7	0.774652696	0.671742430	0.425609705	-0.225521343	0.255528537
8	0.783583500	0.642336286	0.417882957	-0.004726925	-0.005092015
Iteration 3					
9	0.788395014	0.646431070	0.417839759	-0.007070349	-0.002145127
10	0.807641070	0.662810206	0.417674855	-0.016468163	0.009666693
11	0.884625293	0.728326748	1.055047621	11.240801813	11.352448354
12	0.826887126	0.679189341	0.419368663	0.581744126	0.629161232
13	0.817264098	0.670999774	0.417597112	-0.021180952	0.015586327
14	0.822075612	0.675094557	0.417559412	-0.023540689	0.018549402
15	0.824481369	0.677141949	0.417672612	0.137610419	0.182363552
16	0.822731727	0.675652937	0.417554332	-0.023862640	0.018953621
17	0.823267443	0.676108852	0.417550194	-0.024125542	0.019283694
18	0.823670655	0.676452001	0.417547838	-0.012057811	0.031797759
19	0.823580182	0.676375004	0.417547783	-0.024279030	0.019476394
20	0.823625418	0.676413503	0.417547510	-0.020409129	0.023396372
Iteration 4					
21	0.830483446	0.680888533	0.423946700	1.130582527	1.135204454
22	0.823644213	0.676425766	0.417547462	-0.172548562	0.026443233
Iteration 5					
23	0.826629684	0.673335048	0.417704033	0.061322460	-0.085067610
24	0.824329900	0.675715907	0.417532156	-0.000006417	0.000008697
Iteration 6					
25	0.824330139	0.675715663	0.417532156	-0.000000003	0.000000003

Note that the problem described in section 2.1 has Ω as

$$\begin{cases} u \geq 0 \quad , \\ v \geq 0 \quad , \\ u + v = 1 \quad . \end{cases}$$

The quadratic penalty function was also employed to prevent violations of (33). The optimal solution was achieved in six iterations, during which twenty-five values for $\hat{\varepsilon}(u)$ $\mathrm{grad}_u\, \hat{\varepsilon}$ were calculated. In contrast to the results obtained for the case described in section 2.4, the function $\hat{\varepsilon}(u)$ was found to be nonconvex within Ω.

The above example is illustrated in Figure 12 in which the dots represent test points and circles are optimal points for the six iterations. In this example, all components of the gradient of $\hat{\varepsilon}(u)$ were continuous but not smooth functions. The problem was also solved using a 'cubic' penalty function, which grants smoothness to the gradient:

$$P(T,\alpha) = \frac{1}{3T}\left(\frac{|\alpha| - \alpha}{2}\right)^3 \quad .$$

Table 8 presents the numerical results of this run. Note that the objective function was also modified in (29).

In the third example an algorithm with a projection-gradient approach was used to retain $\| u \, ; \, v \|$ within Ω. The convergence achieved for this case was better than for the other two examples. The results are shown in Table 9.

Finally, let us consider MWRA. Figure 13 shows the results of solving problem (32) for MWRA. The test points marked were considered to be the best approximations for each iteration. In Figure 13 the trajectory for running the optimization procedure is shown. The value of the penalty coefficient T is equal to 0.1. The set Ω was defined for the case as

$$\begin{cases} 2.5 \leq \underline{V}_A \leq 8 \quad , \\ 0.5 \leq \underline{Z}_m \leq 2 \quad . \end{cases}$$

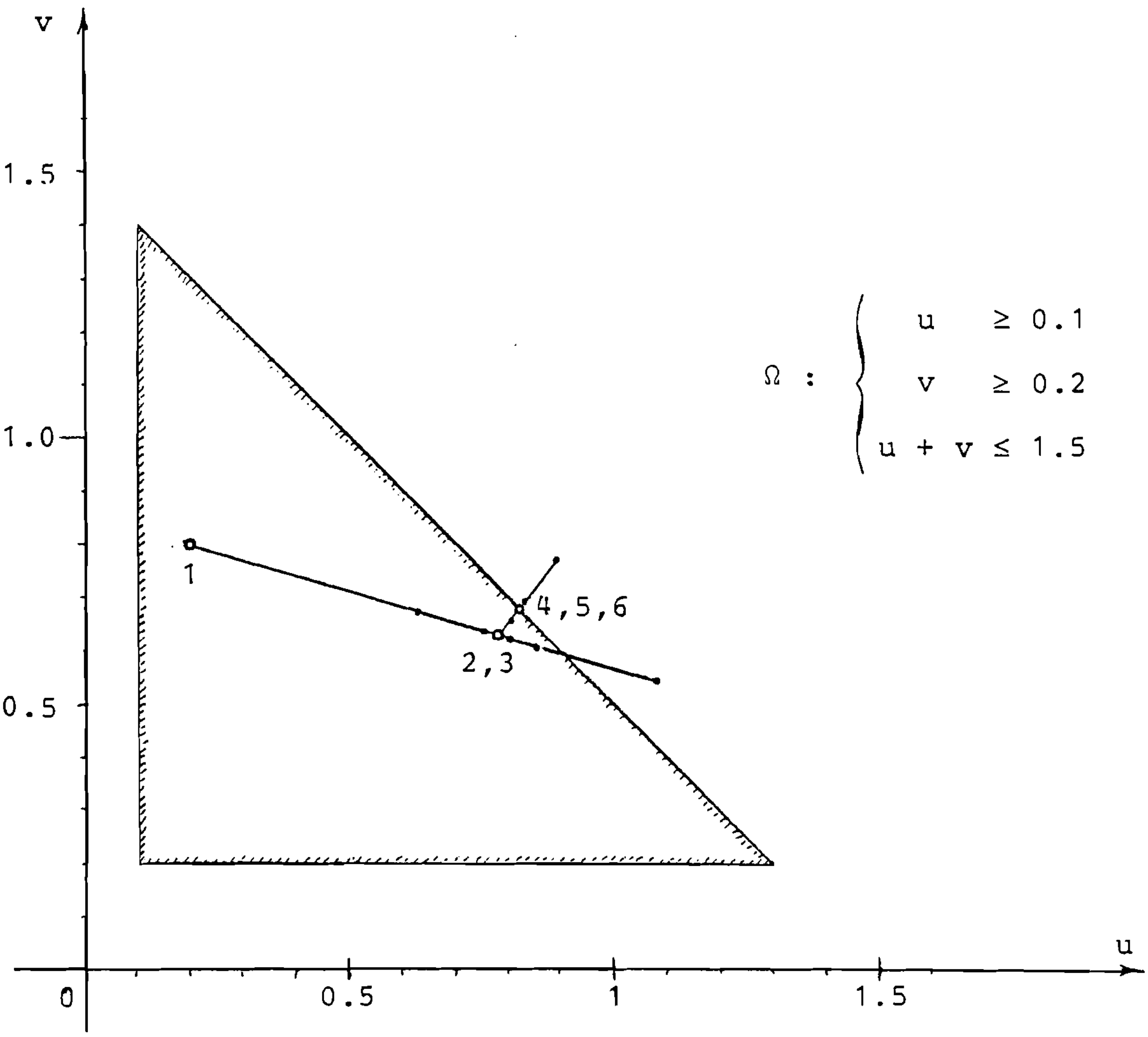

Figure 12. A graphic representation of the results of main procedure for model (1) - (2) using the square penalty function.

Table 8. The results of running the main procedure for model (1) - (2) using the cubic penalty function.

#p.	u	v	E	grad E(u)	grad E(v)
Iteration 1					
0	0.200000000	0.800000000	0.714869718	-0.871664578	0.236593116
1	1.071664578	0.563406884	1.404536861	13.688761696	13.164194548
2	0.635832290	0.681703442	0.460262307	-0.290699650	0.272851742
3	0.853748433	0.622555163	0.431586406	0.206177332	-0.282216329
4	0.794717145	0.638577836	0.414881332	0.182475868	-0.228146756
5	0.746294068	0.651721156	0.423470582	-0.232045511	0.265611462
6	0.785341157	0.641122730	0.413649544	0.002464120	-0.007038797
Iteration 2					
7	0.784260656	0.647756339	0.414246530	-0.133848672	0.159147681
8	0.785302345	0.641361007	0.413648634	-0.002786224	-0.000621506
Iteration 3					
9	0.787224642	0.642939971	0.413642330	-0.002901178	-0.000460874
10	0.794913828	0.649255826	0.413617171	-0.003405187	0.000172031
11	0.825670571	0.674519247	0.413519640	0.013583464	0.021665931
12	0.816732020	0.667177170	0.413546481	-0.004827848	0.001949246
13	0.823050180	0.672366872	0.413526215	-0.005227335	0.002472368
14	0.824881289	0.673870935	0.413520360	-0.005346919	0.002622518
15	0.825409326	0.674304662	0.413518673	-0.005381538	0.002662907
16	0.825539949	0.674411954	0.413518256	-0.005389930	0.002673517
17	0.825605260	0.674465601	0.413518298	0.001691918	0.009764884
18	0.825572604	0.674438777	0.413518158	-0.004253857	0.003814357
19	0.825577281	0.674442619	0.413518156	-0.003402252	0.004666640
20	0.825575978	0.674441549	0.413518156	-0.003639533	0.004429167
Iteration 4					
21	0.825879671	0.674504334	0.413524443	0.036525553	0.036776944
22	0.825593634	0.674445199	0.413518132	-0.001304509	0.006309910
Iteration 5					
23	0.845711252	0.649468299	0.422568334	0.204028337	-0.265666801
24	0.825704263	0.674307849	0.413517758	0.000341704	-0.001642334
25	0.825681411	0.674336220	0.413517731	0.000001197	0.000000935
Iteration 6					
26	0.825681402	0.674336219	0.413517731	-0.000000002	0.000000002

Table 9. The numerical results of running the main procedure for model (1) - (2) using the projection-gradient approach.

#p	u	v	E	grad E(u)	grad E(v)
Iteration 1					
1	0.200000000	0.800000000	0.718931580	-0.869239520	0.236521178
2	0.886908741	0.613091259	0.445553466	0.200549398	-0.294832486
3	0.763817028	0.646584670	0.421538945	-0.226466790	0.262839002
4	0.787197483	0.640222819	0.418119630	0.073715000	-0.100851265
5	0.783506835	0.641227049	0.417899312	0.011455157	-0.024860499
Iteration 2					
6	0.778194085	0.663999976	0.422831819	-0.223129076	0.256832079
7	0.783318279	0.642035291	0.417885811	-0.003433487	-0.006676667
8	0.783267131	0.642254533	0.417885166	-0.007472815	-0.001743353
9	0.788261865	0.646344146	0.417840891	-0.007413647	-0.001728078
10	0.808240798	0.662702597	0.417667325	-0.007203636	-0.001641986
11	0.824216631	0.675783369	0.417532505	-0.007065115	-0.001545963
Iteration 3					
12	0.824303922	0.675696078	0.417532261	-0.004609555	-0.004544872
Iteration 4					
13	0.824304957	0.675695043	0.417532261	-0.004580436	-0.004580434

The minimum water volume possible in the lake was defined as

$$\underline{S} = 40 - 2.29(\underline{Z}_m + \underline{V}_A) \quad .$$

From Table 10 we are able to evaluate the exact (local) solution to the problem:

$$\underline{V}_A^* = 2.5 \quad ,$$

$$\underline{Z}_m^* = 1.8668122 \quad ,$$

$$\underline{S}^* = 30 \quad .$$

In Table 11 the state of MWRA for the optimal values of the exogenous parameters is described.

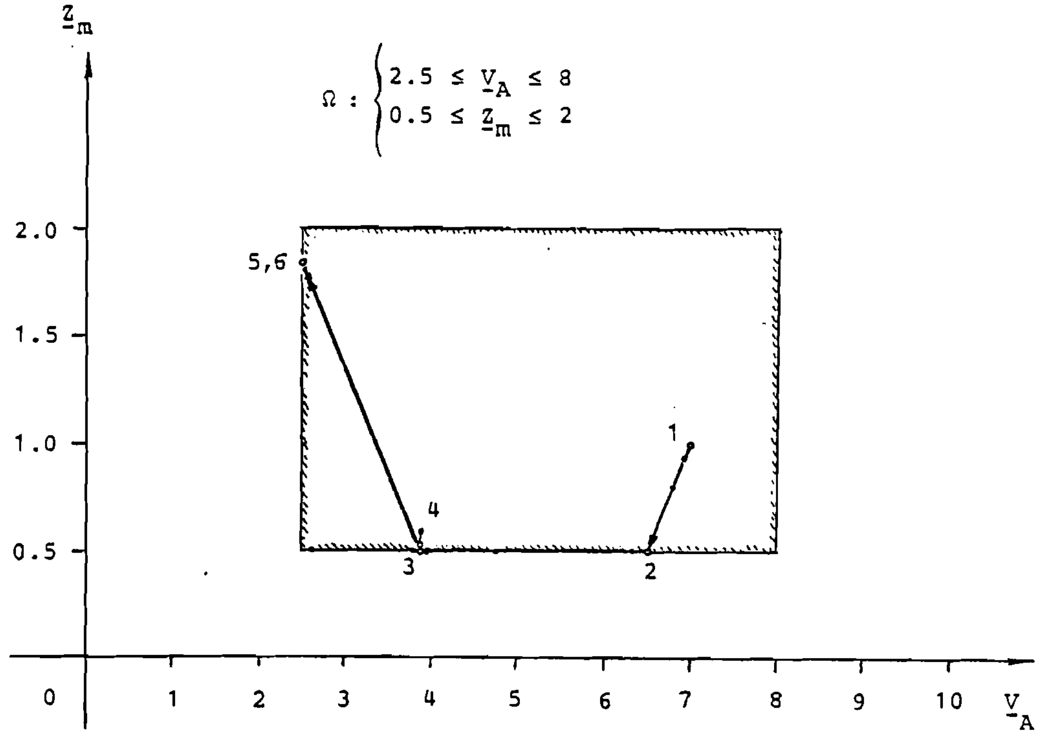

Figure 13. A graphic representation of the results of running the main procedure for MWRA (T = 0.1).

4. CONCLUDING REMARKS

In this paper the method discussed has been considered as a tool for improving the consistency of model criteria. However, the dependence of the equilibria states on the values of the exogenous parameters may also be used for improving the consistency of a set of mathematical subsystem models representing a complex system.

The problem of model linkage is based on an approach discussed in Umnov (1975) and (1979) for the case where there exists a common criterion for the system of models. The approach discussed here may be considered as a method for linking models with diverse criteria measured in different units. For example, MWRA can be divided into three submodels:

-- submodel of water dynamics,
-- submodel of environmental impacts,
-- submodel of agriculture.

Each submodel has the water input volume as an exogenous parameter and the criterion value measuring the quality of the model states as its output. Thus, we have a system of

Table 10. The results of running the main procedure for MWRA.

#p	u	v	E	grad E(u)	grad E(v)
Iteration 1					
1	0.70000000d+01	0.10000000d+01	0.94603224d+00	0.44425917d-01	0.43561843d-01
2	0.69555741d+01	0.95643816d+00	0.94215905d+00	0.44544142d-01	0.43529998d-01
3	0.67778704d+01	0.78219079d+00	0.92674064d+00	0.43750133d-01	0.42050302d-01
4	0.64900822d+01	0.50000000d+00	0.90210146d+00	0.45175721d-01	0.42007092d-01
Iteration 2					
5	0.64449065d+01	0.50000000d+00	0.90005500d+00	0.45426026d-01	0.42032535d-01
6	0.62642036d+01	0.50000000d+00	0.89174967d+00	0.46522249d-01	0.42160727d-01
7	0.55413921d+01	0.50000000d+00	0.85615808d+00	0.52068101d-01	0.42997965d-01
8	0.26501460d+01	0.50000000d+00	0.19356799d+02	-.31901598d+02	-.31901598d+02
9	0.48185805d+01	0.50000000d+00	0.81617291d+00	0.58982369d-01	0.44047242d-01
10	0.41926074d+01	0.50000000d+00	0.77677370d+00	0.67330543d-01	0.45220944d-01
11	0.37012981d+01	0.50000000d+00	0.10708367d+01	-.40924162d+01	-.41259009d+01
12	0.39949954d+01	0.50000000d+00	0.76342447d+00	0.68595045d-01	0.43627551d-01
13	0.38745399d+01	0.50000000d+00	0.75503600d+00	0.70705746d-01	0.43886321d-01
14	0.38023113d+01	0.50000000d+00	0.79281882d+00	-.14197579d+01	-.14488438d+01
15	0.38515702d+01	0.50000000d+00	0.75429374d+00	-.14408132d+00	-.17141897d+00
16	0.38580293d+01	0.50000000d+00	0.75390403d+00	0.23410870d-01	-.37065162d-02
17	0.38571265d+01	0.50000000d+00	0.75389346d+00	-.45364897d-07	-.27148199d-01
18	0.38571265d+01	0.50000000d+00	0.75389346d+00	-.10901585d-06	-.27148166d-01
Iteration 3					
19	0.38571292d+01	0.52714820d+00	0.75499502d+00	0.70961990d-01	0.43949549d-01
20	0.38571268d+01	0.50303895d+00	0.75393022d+00	0.72613044d-01	0.45524735d-01
21	0.38571266d+01	0.50097630d+00	0.75387935d+00	0.25352648d-01	-.17754190d-02
Iteration 4					
22	0.25000000d+01	0.18570871d+01	0.70181470d+00	0.63113968d-01	0.11580182d-01
Iteration 5					
23	0.25000000d+01	0.18566415d+01	0.70181212d+00	0.51564543d-01	0.13076689d-04
Iteration 6					
24	0.25000000d+01	0.18566410d+01	0.70181212d+00	0.51551339d-01	-.26549118d-07

Table 11. The state of MWRA for the optimal value of the exogenous parameters.

EXOGENOUS CONSTRAINTS:

Final volume of water in the lake	=	30000.00(1000cub.m)
Capacity of natural water sources	=	1.00(to norm.)
Minimal flow to Malmo	=	1.87(cub.m/sec)
Minimal flow to the sea	=	2.50(cub.m/sec)
Concent.level of poll. in the sea	=	10.00(g/cub.m)

List of criteria: yields, water in the lake, flow to Malmo, flow to the sea, flow of pollution to the sea

RESULTS: Min.inconsistency = 67.0%

# of criterion		Value	Opt.value	Consist.%
1	max	8921.24442	27014.51507	33.0
2	max	7499.10131	22708.10843	33.0
3	max	7370.43615	22318.49608	33.0
4	max	30000.00000	30000.00000	100.0
5	max	1.86682	2.00000	93.3
6	max	2.50000	2.93318	85.2
7	min	9.38406	5.62000	33.0

Technology #	Area 1 ha	Area 2 ha	Area 3 ha
1	1884.84	973.81	457.39
2	0.	1258.05	1842.61
3	0.	0.	0.
4	1115.16	268.14	0.
5	0.	0.	0.
6	0.	0.	0.
7	0.	0.	0.

	Area 1	Area 2	Area 3
Yield (ton)	8921.244	7499.101	7370.436
Water in. (1000cub.m)	334.547	457.858	552.783
Water out.	66.909	45.786	110.557
Fertil.in.(ton)	167.273	48.265	0.
Fertil.out.	25.091	7.240	0.

Water in the lake	30000.000(1000cub.m)
Flow out the lake	1.330(cub.m/sec)
Flow into the sea	2.500(cub.m/sec)
Flow to Malmo	1.867(cub.m/sec)
Pollution out the lake	3.939(g/sec)
Pollution to the sea	9.384(g/sec)

models with diverse criteria that describe various processes within the region. Hence, the linkage problem can be defined as one of finding optimal values for the exogenous parameters such that an optimal degree of consistency between the models can be achieved.

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