NOT FOR QUOTATION
WITHOUT PERMISSION
OF THE AUTHOR

# EVALUATION OF DANGER OR HOW KNOWLEDGE TRANSFORMS HAZARD RATES

A.I.Yashin

October 1983
WP-83-101



INTERNATIONAL INSTITUTE FOR APPLIED SYSTEMS ANALYSIS
2361 Laxenburg, Austria

**FOREWORD**

The perception of real or potential risks by individuals shapes the way in which society responds to various opportunities for development. However, this perception often bears little relation to the real danger, being affected not only by different social and cultural traditions but also by the amount of information available.

In this paper, Anatoli Yashin of the Core Concepts group of the System and Decision Sciences Area examines how changes in information about the risks associated with possible future events formally transform the chances of these events occurring. He describes an analytical tool for the probabilistic analysis of hazard rates under various assumptions concerning the available information.

Andrzej Wierzbicki
Chairman
System and Decision Sciences

# EVALUATION OF DANGER OR HOW KNOWLEDGE TRANSFORMS HAZARD RATES

A.I.Yashin

## INTRODUCTION

Decision making under conditions of uncertainty is based on the analysis of risk. The primary objective of this analysis is to increase the manager's ability to respond effectively and appropriately to the problems facing him.

The particular problems that are identified will depend on the manager's knowledge of the subject, the discrepancies between his wishes and reality, and his own perception of the real risks. The priority ranking of the problems will depend on the position of the manager in the hierarchy and on his responsibilities.

The perception of real or potential risks by individuals in the social environment shapes the social response to real problems. This response also plays an important role in determining the preference structure of the manager, through external pressures.

However, the social perception of risk sometimes bears little relation to the real danger. The various social principles that guide behavior affect the judgement of what danger or problem should be most feared and what risk is worth taking. These social and behavioral aspects of the decision making process may introduce additional restrictions into the control strategies.

The continued use of tobacco, alcohol and other drugs provides a good example. Everybody knows that smoking increases the risk of lung cancer, that alcohol and drug abuse can lead to alcohol and drug addiction, make accidents more likely, and so on. However, in spite of the success of causal analysis and quantitative calculation in identifying these and many other potentially dangerous activities, many people still continue to ignore the warnings. This means that health and social authorities have to resort to various indirect methods of decreasing the hazard rates associated with problems of this type by trying to change the social perception of the real dangers.

Differences in social principles and traditions are not the only causes of differences in individual perceptions of danger. Differences in knowledge about a particular situation and the related factors are also important in evaluating risk. Often people have different perceptions of the risk of a given course of action simply because they have different information about it. In actual fact, most of the people tend to be unaware of most of the dangers most of the time.

Thus, more exact knowledge creates a better background for the accurate perception of risk. Dissemination of this knowledge can change human preferences in the evaluation of risk and make people more aware of the real dangers that they face.

The study of risk requires two different stages of analysis. The first involves the quantitative determination of the real risk, calculated from specific information on the technological or environmental hazards. The second phase involves social and individual decisions about whether the risks are acceptable and how best to manage them.

An important intermediate stage is concerned with the *analysis* and *comparison* of formal risk assessments made on the basis of *different information* about the dangers. In other words, it is necessary to know how *changes in the information* about the risks associated with possible future events formally *transform the chances* of these events occurring. The subsequent evaluation of risk perception in different social groups with different cultural and other traditions should then be based on the results of this formal analysis of differences in information.

This paper is an attempt to provide an analytical tool for the probabilistic analysis of hazard rates under various assumptions concerning the available information.

## PROBABILISTIC DESCRIPTION OF RISK

The formal analysis of hazardous situations is based on probability theory. To deal with the dynamic aspects of risk evaluation requires the use of *hazard rates*, which are employed quite widely in the applied sciences and are often used in the description of mechanisms generating unexpected changes or unpredictable events, such as death or disaster, famine or failure [1,2].

*Random hazard rates* are used to characterize changes with a high degree of uncertainty, such as mortality in heterogeneous populations and transition rates in multistate demography [3,4]. These rates can also help to describe discontinuous changes in particular components of multidimensional (e.g., industrial) processes or failures of technical equipment [5,6,7,8,9], and are useful in analyzing causal changes in the social or medical status of individuals [4,10,11,12].

The most convenient probabilistic models of the dynamics of rapid unexpected changes or unpredictable events are random point processes or random jumping processes [13,14,15,16]. The combination of rapid jumps with the relatively slow evolution of systems variables observed in many real situations may be described by a general random process model with piecewise-continuous sampling paths. Several such models have been developed in the framework of the "martingale approach" [17,18,19].

Stochastic intensities or compensators or, more generally, dual predictable projections of integer-valued random measures may be taken as stochastic models of hazard rates and can be used in conjunction with martingale theory to formulate many interesting results. Among these are: conditions for the absolute continuity and singularity of probabilistic measures corresponding to piecewise-continuous processes [20], formulas for filters [16], consistency conditions for Bayesian parameter estimation [21], and weak convergence properties [22,23]. Note that to obtain such results it is only necessary to know that random intensities exist, not to know their internal structure.

To apply these results in practice requires the detailed structuring of the random intensities. It is usually most convenient to represent intensities in terms of probability distributions, or more exactly, in terms of conditional probability distributions.

Some results have already been obtained using this type of representation [24,25,26]. The more general of these use Jacod's formula for the dual predictable projection of integer-valued random measures [15]. However, such results cannot be used in some situations where the observer (statistician) has to deal with an increasing volume of information, as is usually the case for recursive estimation and control in a situation with incomplete information.

In this paper we will give a representation of random intensity processes in this more general situation. In a certain sense this may be seen as an attempt to formalize the relations between the abstract results of martingale theory and the classical approach to the analysis of random phenomena. We will also compare the hazard rates perceived by two observers, one of whom has some information about the environmental factors and processes influencing the chance of the random events occurring, while the other knows only about past events of the same kind.

The algorithms used to estimate the hazard rates depend on the dynamics (stable, continuously evolving, or jumping) of the factors influencing the environment. However, all of these algorithms are similar in that they are generated by the nonlinear filter approach using random observations. This approach is gradually becoming popular in technical fields such as reliability analysis [6,7] and communication theory [13] as well as being used for social and medical research in areas such as event history analysis [27,28].

**PRELIMINARIES**

In order to gain a better understanding of the role of martingale theory in the analysis of random intensities, we shall consider the following situation.

Assume that somebody is affected by a sequence of unfavorable (favorable) events occurring at random times. This person may have observations or measurements of environmental parameters at various times between successive events; these observations or measurements give him additional information about the possible timing of the events and scale of the damage (benefits). Denote by $H_t$ the information available to the person up to time $t$. At any time the person can either save some money "for a rainy day" or else spend it. The question is how much money he should save (spend) at time $t$ if he wants to compensate exactly for the damage (benefits) expected at some time in the future.

If we denote by $Y_t$ the cumulative total of the random damage (benefits) experienced at random times up to time $t$ and by $C_t$ the total sum of money that the person has saved (spent) up to this time, then the process

$$M_t = Y_t - C_t, \quad t \geq 0$$

should have the property

$$\mathbf{E}(M_t \mid H_u) = M_u$$

for any $t \geq u$, and if $M_0 = 0$, then

$$\mathbf{E}\, M_t = 0 \quad .$$

These equalities mean that process $C_t$, $t \geq 0$, may be considered as a *compensator* of the discontinuous changes caused by process $Y_t$, $t \geq 0$. It turns out that processes like $C_t$, $t \geq 0$, may be compared to cumulative intensities, and processes such as $M_t$, $t \geq 0$, have the martingale property with respect to information flow $H_t$, $t \geq 0$.

Notice that the process $C_t$ satisfying the above conditions is not unique. However, it is possible to find a unique process corresponding in some sense to the information available up to the current time $t$. A formal way of constructing such a process is given below.

## DETERMINISTIC HAZARD RATES

We will start with the conventional definition of a hazard rate for a continuously distributed random time of occurrence of the events under consideration. If $F(t)$ is the time-of-occurrence distribution function, then the local hazard rate $\lambda(t)$ is equal to minus the logarithmic derivative of the function

$\mathbf{P}(\mathrm{T} > t) = 1 - F(t)$:

$$\lambda(t) = -\frac{1}{\mathbf{P}(\mathrm{T} > t)} \frac{\mathrm{d}\mathbf{P}(\mathrm{T} > t)}{\mathrm{d}t} \quad .$$

The cumulative intensity function $\Lambda(t) = \int_0^t \lambda(u)\mathrm{d}u$ is then

$$\Lambda(t) = -\int_0^t \frac{\mathrm{d}\mathbf{P}(\mathrm{T} > u)}{\mathbf{P}(\mathrm{T} > u)} \quad . \tag{1}$$

This means that the distribution function may be represented in the following form:

$$F(t) = 1 - \mathrm{e}^{\Lambda(t)} \quad . \tag{2}$$

Discontinuities in the time-of-occurrence distribution function modify the definition of the intensity function only slightly:

$$\Lambda(t) = -\int_0^t \frac{\mathrm{d}\mathbf{P}(\mathrm{T} > u)}{\mathbf{P}(\mathrm{T} \geq u)} \quad . \tag{1'}$$

However, they cause complications in equation (2), which represents the distribution function $F(t)$ as a function of the cumulative intensity function $\Lambda(t)$. It can be shown that in this case the analog of formula ( 2 ) is as follows:

$$F(t) = 1 - \mathrm{e}^{-\Lambda(t)} \prod_{u<t} (1 - \Delta\Lambda(u))\mathrm{e}^{-\Delta\Lambda(u)} \tag{2'}$$

which reverts to formula (2) if $\Lambda(t)$ is not discontinuous.

Discontinuous hazard rates are often produced by the estimation procedure: the Kaplan-Meier estimator is one well-known culprit [29,30].

**RANDOM INTENSITY FUNCTIONS**

The dependence of the random time of occurrence on various other random factors representing the state of the environment should also be included somehow in the formula for the hazard rate. The conventional way of doing this is to put the randomness into the intensity function. If this randomness is generated by a random variable **Z** which can be interpreted as an external environmental factor, we can represent the random intensity as a function of the conditional distribution function as follows:

$$\Lambda(t, \mathbf{Z}) = -\int_0^t \frac{d\mathrm{P}(\mathrm{T} > u \mid \mathbf{Z})}{\mathrm{P}(\mathrm{T} \geq u \mid \mathbf{Z})} \tag{3}$$

where $\mathbf{P}(\mathrm{T} > u \mid \mathbf{Z})$ is the conditional probability of the event $\{\mathrm{T} > u\}$ given random variable **Z**.

Let $\mathrm{I}(\mathrm{T} \leq t)$ be the indicator of event $\{\mathrm{T} \leq t\}$ and $H_t^z$ be the past history of process $X_t = \mathrm{I}(\mathrm{T} \leq t)$, $t \geq 0$, and random variable **Z** up to time $t$. It turns out that the process $M(t)$ defined by the equality

$$M(t) = \mathrm{I}(\mathrm{T} \leq t) - \Lambda(t \Delta \mathrm{T}, \mathbf{Z}) \tag{4}$$

is martingale with respect to the family of histories $H_t^z$, $t \geq 0$ and consequently

$$\mathbf{E}(M(t) \mid H_u^z) = M(u)$$

for any $t \geq u$.

## COMPENSATOR FOR POINT PROCESSES

For a sequence of random times of occurrence (a random point process) with random variable $\mathbf{Z}$ influencing the flow of random times, the probabilistic representation of the random intensity is as follows:

$$\Lambda(t,\mathbf{Z}) = \sum_{n=0}^{\infty} \mathbf{I}(\mathbf{T}_n < t \le \mathbf{T}_{n+1}) \sum_{p=0}^{n} \int_{\mathbf{T}_p\Delta t}^{\mathbf{T}_{p+1}\Delta t} \frac{\mathrm{d}_u \mathbf{P}(\mathbf{T}_{p+1} \le u \mid H^{\mathbf{N}}_{\mathbf{T}_p}, \mathbf{Z})}{\mathbf{P}(\mathbf{T}_{p+1} \ge u \mid H^{\mathbf{N}}_{\mathbf{T}_p}, \mathbf{Z})} \tag{5}$$

where $H^{\mathbf{N}}_{\mathbf{T}_p}$ is the history of the counting process $\mathbf{N}_t$ defined by the equality

$$\mathbf{N}_t = \sum_{n=0}^{\infty} \mathbf{I}(\mathbf{T}_n \le t)$$

up to random time $\mathbf{T}_p$. It turns out that the process

$$M_t^{\mathbf{N}} = \mathbf{N}_t - \Lambda(t, \mathbf{Z}), \quad t \ge 0 ,$$

is martingale with respect to the family of histories $H_t^{\mathbf{NZ}}$, $t \ge 0$, generated by the values of the random process $\mathbf{N}_t$ and random variable $\mathbf{Z}$. The process $\Lambda(t, \mathbf{Z})$, $t \ge 0$, is called the *compensator* of the random point process $(\mathbf{T}_n)_{n \ge 0}$.

## JACOD'S FORMULA

A more general form of the random intensity function can be derived for a process involving a sequence of random times of occurrence and random variables $(\mathbf{T}_n, \mathbf{Z}_n)_{n \ge 0}$. If $Y$ is some known random variable which influences the sequence $(\mathbf{T}_n, \mathbf{Z}_n)_{n \ge 0}$, then the formula for the random intensity $\nu^y((0,t], \Gamma)$, $t \ge 0$, of the process

$$\mu((0,t], \Gamma) = \sum_{n=0}^{\infty} \mathbf{I}(\mathbf{T}_n \le t, \mathbf{Z}_n \in \Gamma), \quad t \ge 0$$

where $\Gamma$ is some subset of the space of values of the random variables $(\mathbf{Z}_n)_{n \geq 0}$, is as follows:

$$\nu^y((0,t],\Gamma) = \sum_{n=0}^{\infty} \mathbf{I}(\mathbf{T}_n < t \leq \mathbf{T}_{n+1}) \sum_{p=0}^{n} \int_{\mathbf{T}_p \Delta t}^{\mathbf{T}_{p+1}\Delta t} \frac{\mathrm{d}_u \mathbf{P}(\mathbf{T}_{p+1} \leq u, \mathbf{Z}_{p+1} \in \Gamma \mid H^{\mu}_{\mathbf{T}_p}, Y)}{\mathbf{P}(\mathbf{T}_{p+1} \geq u \mid H^{\mu}_{\mathbf{T}_p}, Y)} \tag{6}$$

where $H^{\mu}_{\mathbf{T}_p}$ is the history of the process $(\mathbf{T}_n, \mathbf{Z}_n)_{n \geq 0}$ up to random time $\mathbf{T}_p$ [15].

## HAZARD RATES IN A STABLE ENVIRONMENT

The most interesting results are obtained from an analysis of the intensity processes that correspond to different levels of knowledge about the random factors influencing the sequence of random times of occurrence and random variables. If, for instance, one observer knows the value of the random variable $\mathbf{Z}$ influencing the sequence of random times $\mathbf{T}_n$, $n \geq 0$, while another does not, they will construct different representations of the intensity functions. Denoting by $\lambda(t, \mathbf{Z})$ and $\bar{\lambda}(t)$ the intensities perceived by the two observers, we consider the natural question: what is the relation between $\lambda(t,\mathbf{Z})$ and $\bar{\lambda}(t)$? It turns out that this relation is as follows:

$$\bar{\lambda}(t) = \mathbf{E}(\lambda(t, \mathbf{Z}) \mid H_t^{\mathbf{N}}) \tag{7}$$

where $H_t^{\mathbf{N}}$ is the history of the counting process $\mathbf{N}_t$ corresponding to the sequence of random times of occurrence $\mathbf{T}_n$, $n \geq 0$, up to time $t$.

**FILTERING FORMULA**

Equation (7) shows that when the observable process is discontinuous it is necessary to use some sort of estimation algorithm to calculate $\bar{\lambda}(t)$. Various estimation procedures of this kind based on [13] and [14] have been developed. The general formula for the a posteriori mathematical expectation of some arbitrary integrable function $f(\mathbf{Z})$ of random variable $\mathbf{Z}$ when the observations are taken from the sequence (point process) $\mathbf{T}_n$, $n \geq 0$, of random occurrence times or, equivalently, from the counting process $\mathbf{N}_t$, $t \geq 0$, is as follows:

$$\mathbf{E}(f(\mathbf{Z}) \mid H_t^{\mathbf{N}}) = \mathbf{E}(f(\mathbf{Z}) \mid H_0^{\mathbf{N}}) + \int_0^t \mathbf{E}(f(\mathbf{Z})(\frac{\lambda(u, \mathbf{Z})}{\bar{\lambda}(u)} - 1) \mid H_u^{\mathbf{N}})(d\mathbf{N}_u - \bar{\lambda}(u)du)$$

where

$$\bar{\lambda}(t) = \mathbf{E}(\lambda(t, \mathbf{Z}) \mid H_t^{\mathbf{N}}) \quad .$$

If $\mathbf{Z}$ is the finite state random variable $\mathbf{Z} = (\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_K)$ with a priori probabilities $p_i$, $i = 1,2,\ldots,K$, and the observations are taken from the counting process $\mathbf{N}_t$ with intensity $\lambda(t, \mathbf{Z})$, the formula for the intensity function $\bar{\lambda}(t)$ will be as follows:

$$\bar{\lambda}(t) = \sum_{i=1}^{K} \lambda(t, \mathbf{Z}_i)\, \pi_i(t) \tag{8}$$

where the $\pi_i(t)$, $i = 1,2,\ldots,K$, are given by the filtering formula:

$$d\pi_i(t) = \pi_i(t)(\frac{\lambda(t, \mathbf{Z}_i)}{\bar{\lambda}(t)} - 1)(d\mathbf{N}_t - \bar{\lambda}(t)dt), \quad \pi_i(0) = p_i \quad .$$

In the simplest case, in which there is only one random time of occurrence (e.g., time of death or failure), this relation can be transformed to the equality:

$$\bar{\lambda}(t) = \mathbf{E}(\lambda(t, \mathbf{Z}) \mid \mathbf{T} > t) \quad . \tag{7'}$$

If $\mathbf{Z} = Y^2$ where $Y$ is a Gaussian random variable with mean $a$ and variance $\sigma^2$, and $\lambda(t, \mathbf{Z}) = \mathbf{Z}\,\lambda(t)$, then one can write

$$\bar{\lambda}(t) = \bar{Z}(t)\, \lambda(t)$$

where $\bar{Z}(t)$ is given by the formula

$$\bar{Z}(t) = m^2(t) + \gamma(t) \tag{9}$$

and $m(t)$ and $\gamma(t)$ are the solutions of the ordinary differential equations

$$\frac{dm(t)}{dt} = -2\, \lambda(t)\, m(t)\, \gamma(t), \quad m(0) = a \tag{10}$$

$$\frac{d\gamma(t)}{dt} = -2\, \lambda(t)\, \gamma^2(t), \quad \gamma(0) = \sigma^2 \quad . \tag{11}$$

These equations show that even when $\lambda(t)$ is a constant and the environmental factors do not change over time (the hazard rate perceived by one observer is constant), the hazard rate perceived by another observer may still be time-dependent.

**THE GENERAL FORM OF HAZARD RATES IN A DYNAMIC ENVIRONMENT**

In many cases the environmental factors that influence the hazard rate can change over time. This section is concerned with results based on the assumption of a randomly changing environment. Let $X_t$ , $t \geq 0$, be the random process available for observation and suppose that it includes the sequence of times and events as well as the additional environmental factors. Introduce the auxiliary process $X_{n,t}$ which coincides with the process $X_t$ up to time $\mathbf{T}_n$ and does not contain the random occurrence times $\mathbf{T}_p$ and random variables $\mathbf{Z}_p$ after time $\mathbf{T}_n$ . Denoting by $H_t^{x_n}$ the history of the auxiliary process up to time $t$ and introducing some additional conditions it is possible to prove the following formula for the random intensity $\nu^x((0,t], \Gamma)$ of the process $\mu((0,t], \Gamma)$ introduced earlier:

$$\nu^{x}((0,t],\Gamma) = \sum_{n=0}^{\infty} \mathbf{I}(\mathbf{T}_n < t \le \mathbf{T}_{n+1}) \sum_{p=0}^{n} \int_{\mathbf{T}_p \Delta t}^{\mathbf{T}_{p+1}\Delta t} \frac{\mathrm{d}_u \mathbf{P}(\mathbf{T}_{p+1} \le u, \mathbf{Z}_{p+1} \in \Gamma \mid H^{x_p}_{\mathbf{T}_p})}{\mathbf{P}(\mathbf{T}_{p+1} \ge u \mid H^{x_p}_{\mathbf{T}_p})} \quad (12)$$

In the case of a point process (a sequence of random times of occurrence $\mathbf{T}_n$, $n \ge 0$) the formula for the random intensity $\Lambda(t, X_0^t)$ corresponding to observable process $X_t$, $t \ge 0$, will be:

$$\Lambda(t, X_0^t) = \sum_{n=0}^{\infty} \mathbf{I}(\mathbf{T}_n < t \le \mathbf{T}_{n+1}) \sum_{p=0}^{n} \int_{\mathbf{T}_p \Delta t}^{\mathbf{T}_{p+1}\Delta t} \frac{\mathrm{d}_u \mathbf{P}(\mathbf{T}_{p+1} \le u \mid H^{x_p}_{\mathbf{T}_p})}{\mathbf{P}(\mathbf{T}_{p+1} \ge u \mid H^{x_p}_{\mathbf{T}_p})} \quad (13)$$

Assuming that the distribution of random times is continuous and comparing the intensities corresponding to different levels of knowledge leads to the following formula for the intensity $\bar{\Lambda}(t)$:

$$\bar{\Lambda}(t) = \int_0^t \mathbf{E}(\lambda(u, X_0^u) \mid H_u^{\mathbf{N}})\mathrm{d}u$$

Assume that $\lambda(t, X_0^t) = f(X_t)$ and that $X_t$ is the solution of the following stochastic differential equation:

$$\mathrm{d}X_t = a(X_t)\mathrm{d}t + b(X_t)\mathrm{d}W(t), \quad X_0 = X$$

The formula for $\bar{\lambda}(t)$ will then be as follows:

$$\begin{aligned}\bar{\lambda}(t) = \mathbf{E}(f(X_0) \mid H_0^{\mathbf{N}}) &+ \frac{1}{2}\int_0^t \mathbf{E}((f(X_u)b^2(X_u))''_{x,x} \mid H_u^{\mathbf{N}})\mathrm{d}u \\ &+ \int_0^t \mathbf{E}((f(X_u)a(X_u))'_x \mid H_u^{\mathbf{N}})\mathrm{d}u \\ &+ \int_0^t \mathbf{E}(f(X_u)(\frac{\lambda(X_u)}{\bar{\lambda}(u)} - 1) \mid H_u^{\mathbf{N}})(\mathrm{d}\mathbf{N}_u - \bar{\lambda}(u)\mathrm{d}u)\end{aligned}$$

However, if instead of a sequence of random times and variables one has only a single random occurrence time $\mathbf{T}$ the formula for the random intensity $\Lambda(t, X_0^t)$

corresponding to the information given by observation $X_t$ will be

$$\Lambda(t, X_0^t) = \int_0^t \frac{dP(T \le u \mid H_u^{x_0})}{P(T \ge u \mid H_u^{x_0})}$$

and the formula for $\bar{\Lambda}(t)$ will be

$$\bar{\Lambda}(t) = \int_0^t \mathbf{E}(\lambda(u, X_0^u) \mid T > u) du$$

If $\lambda(t, X_0^t) = \lambda(X_t)$ then the formula for $\bar{\lambda}(t)$ will be as follows:

$$\bar{\lambda}(t) = \mathbf{E}(\lambda(X_0) \mid T \ge 0) + \int_0^t \mathbf{E}((\lambda(X_u) a(X_u))_x \mid T \ge u) du$$

$$+ \frac{1}{2}\int_0^t \mathbf{E}((\lambda(X_u) b^2(X_u))''_{x,x} \mid T \ge u) du - \int_0^t \mathbf{E}(\lambda(X_u)(\lambda(X_u) - \bar{\lambda}(u)) \mid T \ge u) du$$

These general formulas can be made more specific if a more detailed description of the processes is available.

## HAZARD RATES IN A CONTINUOUSLY EVOLVING ENVIRONMENT

Environmental factors that are changing continuously may be treated in the following way.

Let $Y(t)$, $t \ge 0$, be some process that satisfies the linear stochastic differential equation

$$dY(t) = a_0(t) + a_1(t)Y(t)\,dt + b(t)\,dW(t), \; Y(0) = Y_0$$

where $Y_0$ is a Gaussian random variable with mean $m_0$ and variance $\gamma_0$. Assume that random time of occurrence **T** is related to the process $Y(t)$ by the equality

$$P(T > t \mid H_t^y) = \exp\{-\int_0^t Y^2(u)\,\lambda(u) du\}$$

where $H_t^y$ is the history of the process $Y(t)$ up to time $t$. It turns out that the conditional distribution of $Y(t)$ given $\{T \geq t\}$ is Gaussian. The mean $m(t)$ and variance $\gamma(t)$ of this distribution are given by the following equations:

$$\frac{dm(t)}{dt} = a_0(t) + a_1(t)\, m(t) - 2m(t)\gamma(t)\mu(t)\,, \quad m(0) = m_0 \tag{14}$$

$$\frac{d\gamma(t)}{dt} = 2a_1(t)\gamma(t) + b^2(t) - 2\mu(t)\gamma^2(t)\,, \quad \gamma(0) = \gamma_0 \tag{15}$$

The relation between $\bar{\lambda}(t)$ and $\lambda(t)$ is as follows:

$$\bar{\lambda}(t) = (\, m^2(t) + \gamma(t)\,)\, \lambda(t)$$

This example shows that two observers may have quite different values for hazard rates. It should therefore come as no surprise that their perceptions of risk are different and that the decisions based on these perceptions may also be different.

## HAZARD RATES IN A DISCONTINUOUSLY CHANGING ENVIRONMENT

Environmental factors can sometimes change discontinuously, i.e., their values jump about at random. This can happen, for instance, when there are several correlated sequences of random occurrence times and variables. Examples of such correlated sequences are: changes in the place of residence or work of some particular individual and changes in his health; rapid changes in the weather and the survival chances of living organisms; discontinuous changes in the price or demand structure and structural change in organizations or firms.

Assume that $\mathbf{Z}_t$, $t \geq 0$, is a finite-state continuous-time jumping process with transition intensity matrix $r_{ij}(t)$, $i,j \in \overline{(1,K)}$, and $\lambda(t, \mathbf{Z}_t) = \lambda(t)\, \mathbf{Z}_t$ The relation between $\bar{\lambda}(t)$ and $\lambda(t)$ is then as follows:

$$\bar{\lambda}(t) = \sum_{i=1}^{K} \pi_i(t)\, \mathbf{Z}_i$$

where $\mathbf{Z}_i$ is the $i$-th state of process $\mathbf{Z}_t$, and processes $\pi_i(t)$ are solutions of the following equations:

$$\frac{\mathrm{d}\pi_i(t)}{\mathrm{d}t} = \sum_{k=1}^{K} \pi_k(t)\, r_{k,i}(t) + \pi_i(t)\left( \frac{\mathbf{Z}_i}{\bar{\mathbf{Z}}_t} - 1 \right)\left( \mathrm{d}\mathbf{N}_t - \lambda(t)\,\bar{\mathbf{Z}}_t \mathrm{d}t \right) \qquad (16)$$

where

$$\bar{\mathbf{Z}}_t = \sum_{i=1}^{K} \pi_i(t)\, \mathbf{Z}_i$$

If instead of a sequence of random occurrence times we have only one random time of occurrence $\mathbf{T}$, the formula for $\bar{\lambda}(t)$ remains the same but the equation for the $\pi_i(t)$ is simplified:

$$\frac{\mathrm{d}\pi_i(t)}{\mathrm{d}t} = \sum_{k=1}^{K} \pi_k(t)\, r_{k,i}(t) + \pi_i(t)\left( \mathbf{Z}_i - \sum_{k=1}^{K} \pi_k(t)\mathbf{Z}_k \right) \lambda(t), \quad i \in \overline{(1,K)}. \qquad (17)$$

These equations are useful in the analysis of population heterogeneity.

**REFERENCES**


1. G.G. Berg and H.D. Maillie, "Measurement of Risk". In *Environmental Science Research*, Vol. 21, Plenum Press, New York and London (1982).
2. W.J. Petak and A.A. Atkisson, *Natural Hazard Risk Assessment and Public Policy*, Springer-Verlag, New York, Heidelberg, and Berlin (1982).
3. N. Keyfitz and G. Littman, "Mortality in Heterogeneous Populations", *Population Studies*, 33, pp. 333-343 (1980).
4. J.W. Vaupel and A.I. Yashin, *The Deviant Dynamics of Death in Heterogeneous Populations*, RR-83-1, International Institute for Applied Systems Analysis, Laxenburg, Austria (1982).
5. R.E. Barlow and F. Proschan, *Statistical Theory of Reliability and Life Testing*, Holt, Rinehart and Winston, New York (1975).
6. H.W. Block and T.H. Savits, "Multivariate Classes of Life Distributions in Reliability Theory", *Mathematics of Operations Research*, 6(3), pp.453-461 (1981).

7. E. Arjas, "The Failure and Hazard Processes in Multivariate Reliability Systems", *Mathematics of Operations Research*, 6(4), pp. 551-562 (1981).

8. E. Arjas, "A Stochastic Process Approach to Multivariate Reliability Systems: Notions Based on the Conditional Stochastic Order", *Mathematics of Operations Research*, 6(2), pp. 263-276 (1981).

9. N. Langberg, F. Proschan, and A.J. Quinzi, "Estimating Dependent Life Length, with Applications to the Theory of Competing Risk", *The Annals of Statistics*, 9(1), pp. 157-167.

10. J.J. Heckman and B. Singer, "Population Heterogeneity in Demographic Models". In *Advances in Multiregional Demography*, Academic Press, New York (1982).

11. J.W. Vaupel, K. Manton, and E. Stallard, "The Impact of Heterogeneity in Individual Frailty on the Dynamics of Mortality", *Demography*, 16, pp. 439-454 (1979).

12. K. Manton, E. Stallard, and J.W. Vaupel, "Methods for Comparing the Mortality Experience of Heterogeneous Populations", *Demography*, 18, pp. 389-410 (1981).

13. D.L. Snyder, *Random Point Processes*, John Wiley and Sons, New York (1975).

14. A.I. Yashin, "Filtering of Jumping Processes", *Automatic and Remote Control*, 5, pp. 52-58 (1970).

15. J. Jacod, "Multivariate Point Processes: Predictable Projection, Radon-Nicodim Derivatives, Representation of Martingales", *Zeitschrift fuer Wahrscheinlichkeitstheory und Verwandte Gebiete*, 31, pp. 235-253 (1975).

16. P. Bremaud, *Point Processes and Queues*, Springer-Verlag, New York, Heidelberg, and Berlin (1980).

17. J. Jacod, "Calcules Stochastique et Probleme de Martingales", *Lecture Notes in Mathematics*, Vol. 714, Springer-Verlag, Heidelberg (1979).

18. P.A. Meyer, *Probability and Potentials*, Waltham, Blaisdell (1966).

19. C. Dellacherie, *Capacities et Processus Stochastiques*, Springer-Verlag, Berlin and New York (1972).

20. Yu.M. Kabanov, R.S. Liptzer, and A.N. Shiryaev, "Absolute Continuity and Singularity of Locally Absolutely Continuous Probability Distributions", *Math. Sbornik USSR* (in Russian), 35(5), pp. 631-680 (1979).

21. A.I. Yashin, "Convergence of Bayesian Estimations in Adaptive Control Schemas", *Proceedings of the Workshop on Adaptive Control*, October 27-29, 1982, Florence, Italy, pp. 51-75 (1982).

22. R. Rebolledo, "Central Limit Theorem for Local Martingales", *Zeitschrift fuer Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 51, pp. 269-286 (1980).

23. R.S. Liptzer and A.N. Shiryaev, "Functional Central Limit Theorem for Semimartingales", *Probability Theory and Applications* (in Russian), 25, pp. 667-688 (1980).

24. C.S. Chou and P.A. Meyer, "Sur la Representation des Martingales Comme Integrales Stochastiques dans les Processus Pounctuels", *Lecture Notes in Mathematics*, Vol. 465, Springer-Verlag, Berlin (1975).

25. R.S. Liptzer and A.N. Shiryaev, *Statistics of Random Processes*, Springer-Verlag, Berlin and New York (1978).

26. R. Boel, P. Varaja, and E. Wong, "Martingales on Jump Processes. II Representation Results", *SIAM Journal of Control*, 13(5), pp. 999-1060 (1975).

27. M.T. Hannan and G.R. Carroll, "Dynamics of Formal Political Structure: An

Event History Analysis", *American Sociological Review*, 46, pp. 19-35 (1981).

28. N.B. Tuma, M.T. Hannan, and L.P. Groenveld, "Dynamic Analysis of Event Histories", *American Journal of Sociology*, 84, pp. 820-854 (1980).

29. E.L. Kaplan and P. Meier, "Nonparametric Estimation from Incomplete Observations", *Journal of the American Statistical Association*, 53, pp. 457-481 (1958).

30. A.V. Peterson, "Expressing the Kaplan-Meier Estimator as a Function of Empirical Subsurvival Functions", *Journal of the American Statistical Association*, 72(360), pp. 854-858 (1977).