VARIATIONAL INEQUALITIES REVISITED

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VARIATIONAL INEQUALITIES REVISITED

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Let $K$ be a closed convex subset of a reflexive Banach space $X$ and $A$ be a set-valued map from $K$ to $X^*$ satisfying

\begin{align*}
A \text{ is finitely upper-semicontinuous} \quad &\quad \text{(1)} \\
\text{with nonempty closed convex bounded images.}
\end{align*}

Our purpose is to solve variational inequalities (or generalized equations)

\begin{align*}
\begin{cases}
&\begin{align*}
&i) \quad \bar{x} \in K \\
&ii) \quad 0 \in A(\bar{x}) + N_K(\bar{x})
\end{align*}
\end{cases}
\end{align*}

where $N_K(x) := \{ p \in X^* | \sup_{y \in K} \langle p, x-y \rangle \geq 0 \}$ is the normal cone to $K$ at $x \in K$, by balancing

a) the lack of boundedness of $K$, measured by its "barrier cone" $b(K)$, defined by

\begin{align*}
b(K) := \{ p \in X^* | \sup_{x \in K} \langle p, x \rangle < +\infty \}
\end{align*}

(because the larger $b(K)$, the lesser is $K$ unbounded)
b) with the degree of monotonicity of $A$, measured by a nonnegative proper lower semicontinuous function $\beta$ from $X$ to $\mathbb{R} \cup \{+\infty\}$ satisfying

\[ \mathcal{W}(x,p), (y,q) \in \text{graph}(A), \langle p-q, x-y \rangle \geq \beta(x-y). \]

We shall say that such a set-valued map $A$ is $\beta$-monotone. We denote by $\beta^*$ its conjugate function \eqref{eq:conjugate}.

For instance, we can take

\[
\begin{align*}
\text{i)} & \quad \beta(z) := 0 \quad \text{(and thus, } \beta^* = \psi_{\{0\}}, \text{Dom } \beta^* = \{0\}) \quad \text{(3)} \\
\text{ii)} & \quad \beta(z) := \|z\| \quad \text{(and thus, } \beta^* = \psi_{\beta^*}, \text{Dom } \beta^* = \beta^*) \quad \text{(4)} \\
\text{iii)} & \quad \beta(z) := \frac{1}{\alpha} \|z\|^\alpha \quad \text{(and thus, } \beta^* = \frac{1}{\alpha^*} \|z\|^\alpha^*, \frac{1}{\alpha} + \frac{1}{\alpha^*} = 1, \text{Dom } \beta^* = X^*) \end{align*}
\]

In the following theorems, we shall measure the degree of monotonicity of $G$ through the size of the domain of $\beta^*$: the larger $\text{Dom } \beta^*$, the more "monotone" is $G$.

**Theorem 1.** We posit assumptions \eqref{eq:assumptions}. Assume that $A$ is $\beta$-monotone and that

\[ 0 \in \text{Int } (b(K) + A(K) + \text{Dom } \beta^*). \]

Then there exists a solution $\bar{x} \in K$ to the variational inequality $0 \in A(\bar{x}) + N_K(\bar{x})$. \hfill \qed

Assumption \eqref{eq:assumptions} shows how the lack of boundedness of $K$ is compensated by the degree of monotonicity of $A$. We point out that \eqref{eq:assumptions} is satisfied when one of the following instances is satisfied.

\[
\begin{align*}
i) & \quad K \text{ is bounded } (b(K) = X^*) \\
\text{ii)} & \quad A \text{ is surjective } (A(K) = X^*) \\
\text{iii)} & \quad A \text{ satisfies } \eqref{eq:beta} \text{ with } \beta(z) := \frac{c}{\alpha} \|z\|^\alpha, \quad c > 0, \quad \alpha > 1 \\
\text{iv)} & \quad A \text{ satisfies } \eqref{eq:beta} \text{ with } \beta(z) := c\|z\|, \quad c > 0 \quad \text{and} \quad A(K) \cap -b(K) \neq \emptyset.
\end{align*}
\]

\[ \blacksquare \]
Naturally, these examples are known (see Brézis (1968), Lions (1969) and Browder (1976)). The novelty lies in the introduction of the function $\beta$ as a parameter in assumption (6).

We recall that $N_K(x)$, the normal cone to $K$ at $x$, is the subdifferential of the indicator $\psi_K$. Therefore, variational inequalities are particular cases of inclusions of the form

$$f \in A(x) + \partial V(x)$$

when $V: X \to \mathbb{R} \cup \{+\infty\}$ is a proper lower semicontinuous convex function and $A$ is a set-valued map from the Banach space $X$ to $X^*$, which were studied by Brézis-Haraux (1976), when $A$ is maximal monotone, for solving Hammerstein equations (see Brézis-Browder (1976)). We shall extend Theorem 1 to this case. To this end, we assume once and for all that

$$\text{Dom } V \subset \text{Dom } A$$

and we observe that a necessary condition for the existence of a solution $x$ to (8) is that

$$f \in \text{Dom } V^* + A \text{ Dom } V.$$  

We shall prove that this condition is "almost sufficient".

Theorem 2. We posit assumptions (1). Assume moreover that $A$ is $\beta$-monotone. Then there exists a solution $x$ to the inclusion (8) when

$$f \in \text{Int } (\text{Dom } V^* + A \text{ Dom } V + \text{ Dom } \beta^*)$$

Remark. The size of $\text{Dom } \beta^*$ balances the interiority condition in assumption (11), as the following corollary shows.

Corollary 3. We posit assumptions (1).

a) If $A$ is monotone, (i.e., $\beta=0$), then

$$\text{Int } (\text{Dom } V^* + A \text{ Dom } V) \subset \text{Im } (A + \partial V) \subset \text{Dom } V^* + A \text{ Dom } V.$$
b) If there exists \( c > 0 \) such that

\[
\forall (x,p), (y,q) \in \text{graph} (A), \langle p-q, x-y \rangle \geq c\|x-y\|
\]

then

\[
\text{Im (} A + \beta V \text{)} = \text{Dom } V^* + A \text{ Dom } V
\]

c) If there exist \( c > 0 \) and \( \alpha > 1 \) such that

\[
\forall (x,p), (y,q) \in \text{graph} (A), \langle p-q, x-y \rangle \geq \frac{c}{\alpha} \|x-y\|^\alpha
\]

then

\[
\text{Im (} A + \beta V \text{)} = \text{Dom } V^* + A \text{ Dom } V = X^*
\]

Before proving Theorem 2, we shall characterize problem (8) by equivalent problems. For that purpose, we associate to the function \( V \), to the map \( A \) and to an element \( f \in X \) the function \( \phi \) defined on \( \text{Dom } V \) by:

\[
\phi(y) := V(y) + \inf_{u \in A(y)} (V^*(f-u) - \langle f-u, y \rangle)
\]

We observe that

\[
\forall y \in \text{Dom } V, \quad \phi(y) \geq 0
\]

since, for all \( u \in A(y) \), \( V(y) + V^*(f-u) - \langle f-u, y \rangle \geq 0 \), thanks to the Fenchel inequality.

We can also characterize the set-valued map \( A \) by the function \( \gamma \) defined on \( \text{Dom } A \times \text{Dom } A \) by

\[
\gamma(x,y) := \inf_{p \in A(x)} \langle p, x-y \rangle
\]
Proposition 4.

Assume that the images A(x) are nonempty, closed, convex and bounded for all x ∈ Dom V. The following problems are equivalent:

\begin{itemize}
  \item[i)] \exists \overline{x} ∈ Dom V such that \( f ∈ A\overline{x} + \partial V(\overline{x}) \)
  \item[ii)] \exists \overline{p} ∈ Dom V such that \( f ∈ \overline{p} + A\partial_v^*(\overline{p}) \)
\end{itemize}

\text{(20)}
\begin{itemize}
  \item[iii)] \exists \overline{x} ∈ Dom V such that ∀y ∈ Dom V,
    \[ γ(\overline{x}, y) - \langle f, \overline{x} - y \rangle + V(\overline{x}) - V(y) \leq 0 \]
  \item[iv)] \exists \overline{x} ∈ Dom V such that \( φ(\overline{x}) = 0 \) (\( = \min_{y ∈ Dom V} φ(y) \))
\end{itemize}

Proof.

a) Let \( \overline{x} \) be a solution to (20)i): then there exists \( \overline{p} ∈ \partial V(\overline{x}) \) such that \( f - \overline{p} ∈ A\overline{x} \cap A\partial_v^*(\overline{p}) \). Conversely, let \( \overline{p} \) be a solution to (20)ii). Then there exists \( \overline{x} ∈ \partial_v^*(\overline{p}) \) such that \( f ∈ \overline{p} + A\overline{x} \). Since \( \overline{p} ∈ \partial V(\overline{x}) \), then \( f \) belongs to \( \partial V(\overline{x}) + A\overline{x} \).

b) Let \( \overline{x} \) be a solution to (20)i). There exists \( \overline{u} ∈ A(\overline{x}) \) such that \( f ∈ \partial V(\overline{x}) + \overline{u} \), i.e., such that ∀y ∈ Dom V,

\[ \langle \overline{u}, \overline{x} - y \rangle - \langle f, \overline{x} - y \rangle + V(\overline{x}) - V(y) \leq 0 \]

By taking the infimum on \( A(\overline{x}) \), we deduce inequalities (20)iii).

c) Inequality (20)iii) can be written

\[ \inf_{y ∈ Dom V} \inf_{u ∈ A(\overline{x})} [V(x) - V(y) - \langle f - u, x - y \rangle] \leq 0 \]

Since Dom V is convex, \( A(\overline{x}) \) is convex weakly compact, the lop-sided minimax theorem implies that

\[ \sup_{u ∈ A(\overline{x})} \sup_{y ∈ Dom V} [V(\overline{x}) - V(y) - \langle f - u, \overline{x} - y \rangle] \]
\[ \sup_{u ∈ A(\overline{x})} \sup_{y ∈ Dom V} [V(\overline{x}) - V(y) - \langle f - u, \overline{x} - y \rangle] \]
\[ = \inf_{u ∈ A(\overline{x})} [V(\overline{x}) + \partial_v^*(f - u) - \langle f - u, \overline{x} \rangle] = φ(\overline{x}) \]
Hence \( \phi(\bar{x}) \leq 0 \).

d) Let \( \bar{x} \in \text{Dom} \ V \) satisfy \( \phi(\bar{x}) = 0 \). Since \( A(\bar{x}) \) is weakly compact and \( V \) is weakly lower semicontinuous, there exists \( \bar{u} \in A(\bar{x}) \) such that

\[
\phi(\bar{x}) := V(\bar{x}) + V^*(f-\bar{u}) - \langle f-\bar{u}, \bar{x} \rangle = 0
\]

This is equivalent to saying that: \( f-\bar{u} \in \partial V(\bar{x}) \), i.e. that \( \bar{x} \) solves (20)i).

The equivalence between (20)i) and (20)iv) allows to interpret the solutions to problem (8) as a solution to a minimization problem (minimization of the functional \( \phi \)) and provides a variational principle. The equivalence between (20)i) and (20)iii) allows to solve problem (8), (and, in particular, variational inequalities) by applying minimax inequalities to the function defined by

\[
(21) \quad \phi(x,y) := \gamma(x,y) - \langle f, x-y \rangle + V(x) - V(y)
\]

We observe that

\[
\text{i) } \forall x, y \to \phi(x,y) \text{ is concave}
\]

\[
\text{ii) } \forall y, \phi(y,y) = 0
\]

that \( \phi \) is "monotone" in the sense that

\[
(23) \quad \forall x, y \in \text{Dom} \ (V), \ \phi(x,y) + \phi(y,x) \geq 0
\]

and that

\[
(24) \quad \forall y \in X, \ x \to \phi(x,y) \text{ is lower semicontinuous for the finite topology}^{(1)}.
\]

Therefore, if \( \text{Dom} \ V \) were compact, we could apply the generalization of the Ky Fan inequality (1972) due to Brézis-Nirenberg-Stampacchia (1972), which would imply the existence of a solution \( \bar{x} \in \text{Dom} \ V \)}
to the inequalities (20)\textsuperscript{iii)}, i.e., a solution $\bar{x}$ to problem (8).

When \text{Dom} V is not compact, we shall prove by approximation that assumption (11) is sufficient for the existence of a solution to inequalities (20)\textsuperscript{iii}).

**Proof of Theorem 2.**

We set $K_n := \{x \in \text{Dom} V | V(x) \leq n \text{ and } \|x\| \leq n\}$. The subsets $K_n$ are weakly compact and convex and $\text{Dom} V = \bigcup_{n=1}^{\infty} K_n$ because $X$ is reflexive. Since $K_n$ is weakly compact and convex, Ky Fan's inequality for monotone functions implies that, for all $n \geq 1$, there exists $x_n \in K_n$ solution to

\begin{equation}
\forall y \in K_n, \quad \phi(x_n, y) \leq 0
\end{equation}

thanks to properties (22), (23) and (24).

We shall now use assumption (11) for proving that $x_n$ remains in a weakly compact subset of $X$. For that purpose, thanks to the uniform boundedness theorem, it is sufficient to prove that

\begin{equation}
\forall p \in X^*, \quad \exists n(p) \text{ such that } \sup_{n \geq n(p)} \langle p, x_n \rangle < +\infty .
\end{equation}

By assumption (11), there exist $\eta > 0$, $r \in \text{Dom} \beta^*$, $q \in \text{Dom} V^*$, $y \in \text{Dom} V$, $u \in A(y)$ such that

\begin{equation}
f + \frac{n(p)}{\|p\|} = r + q + u .
\end{equation}

We choose $n(p)$ to be the smallest $n$ such that $y \in K_n$. By taking the duality product with $x_n$ we get

\begin{equation}
\frac{n}{\|p\|} \langle p, x_n \rangle = \langle r, x_n - y \rangle + \langle q, x_n \rangle + \langle u, x_n - y \rangle - \langle f, x_n - y \rangle + \langle r + u - f, y \rangle .
\end{equation}

We use Fenchel's inequalities $\langle r, x_n - y \rangle \leq \beta(x_n - y) + \beta^*(r)$ and $\langle q, x_n \rangle \leq V(x_n) + V^*(q)$. We obtain
Since $A$ is $\beta$-monotone, we deduce that

$$
\gamma(x_n, y) - \langle u, x_n - y \rangle = \inf_{p \in A(x_n)} \langle p - u, x_n - y \rangle \geq \beta(x_n - y).
$$

Therefore, inequality (29) becomes

$$
\frac{n}{\|p\|} \langle p, x_n \rangle \leq (\gamma(x_n, y) - \langle f, x_n - y \rangle + V(x_n) - V(y)) + \beta^*(r) + V^*(q) + V(y) + \langle r + u - f, y \rangle.
$$

Consequently, for all $n \geq n(p)$, we deduce from (25) that

$$
\frac{n}{\|p\|} \langle p, x_n \rangle \leq \frac{\|p\|}{n} (\beta^*(r) + V^*(q) + V(y) + \langle r + u - f, y \rangle).
$$

The right-hand side is finite because $r \in \text{Dom } \beta^*$, $q \in \text{Dom } V^*$ and $y \in \text{Dom } V$. Hence the sequence is bounded and thus, weakly relatively compact.

So, a subsequence of elements $x_n$, converges weakly to some $\overline{x} \in X$. Since $V$ is lower semicontinuous, we deduce from the monotonicity of $A$ and from the variational inequalities that

$$
V(\overline{x}) \leq \liminf_n V(x_n) \leq \liminf_n [(V(y) + \langle f, x_n - y \rangle + \gamma(y, x_n)) - \gamma(y, x_n) - \gamma(x_n, y)] \\
\leq \limsup_n [V(y) + \langle f, x_n - y \rangle + \gamma(y, x_n)] \leq V(y) + \langle f, \overline{x} - y \rangle + \gamma(y, \overline{x}).
$$

Therefore, $\overline{x}$ belongs to $\text{Dom } V$ and

$$
\forall y \in \text{Dom } V, \quad 0 \leq \phi(y, \overline{x}).
$$
d) We deduce from properties (22) and (23) that

\[(32) \quad \forall z \in \text{Dom } V, \quad \phi(\bar{x}, z) \leq 0 \; .\]

Indeed, if the conclusion is false, there would exist \(z \in \text{Dom } V\) such that \(0 < \phi(\bar{x}, z)\) and by (24) there would exist \(\xi \in ]0,1[\) such that

\[0 < \phi(\bar{x} + \xi(z-\bar{x}), z) \; .\]

By taking \(y = \bar{x} + \xi(z-\bar{x})\), inequality (31) implies that

\[0 \leq \phi(\bar{x} + \xi(z-\bar{x}), \bar{x}) \; .\]

Hence, the concavity of \(\phi\) with respect to the second variable yields that

\[(33) \quad 0 < \phi(\bar{x} + \xi(z-\bar{x}), \bar{x} + \xi(z-\bar{x}))\]

a contradiction to (22)ii). Then Proposition 4 implies that the solution \(\bar{x}\) of (32) is a solution to the problem (8).
Notes

(1) The finite topology on a convex subset $N$ of a vector space is the topology for which the maps $\beta_K$ from the simplex $S^n := \{\lambda \in \mathbb{R}_+^n \mid \sum_{i=1}^n \lambda_i = 1\}$ to $N$ defined by

$$\beta_K(\lambda) := \sum_{i=1}^n \lambda_i x_i$$

are continuous for all finite subsets $K := \{x_1, \ldots, x_n\}$ of $N$. It is stronger than any vector space topology and any affine map is continuous for the finite topology (see Aubin (1979), §7.1.3). A finitely upper semicontinuous map from $K$ to $X^*$ is a set-valued map upper semicontinuous from $K$ supplied with the finite topology to $X^*$ supplied with the weak $*$-topology. When $A$ is finitely upper semicontinuous, then the map $x \mapsto \inf\{u, x-y\}$ is lower semicontinuous on $K$ for the finite topology (see Aubin (1979), §13.2.4).

(2) The conjugate function $\beta^*$ of a function $\beta: X \to \mathbb{R} \cup \{+\infty\}$ is defined on $X^*$ by

$$\beta^*(p) := \sup_{x \in X} \{\langle p, x \rangle - \beta(x)\}.$$

A function $\beta$ is convex and lower semicontinuous if and only if $\beta = \beta^*$. It satisfies the Fenchel inequality

$$\langle p, x \rangle \leq \beta(x) + \beta^*(p).$$

(3) The indicator of a subset $K$ is the function $\psi_K$ defined by $\psi_K(x) = 0$ when $x \in K$ and $\psi_K(x) = +\infty$ if not.

(4) $B_*$ denotes the unit ball of the dual.

(5) The subdifferential of a convex function $V$ is the subset

$$\partial V(x) := \{p \in X^* \mid \langle p, x \rangle = V(x) + V^*(p)\}.$$

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of gradients of the affine functions \( x + \langle p, x \rangle - V^*(p) \) below \( V \) and passing through \( (x, V(x)) \). When \( V \) is Gâteaux-differentiable at \( x \), then \( \partial V(x) = \{ VV(x) \} \). The set of points \( x \in X \) for which \( \partial V(x) \neq \emptyset \) is dense in \( \text{Dom} \, V \).

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