URBAN SYSTEM POPULATION DYNAMICS: INCORPORATING NON-LINEARITIES

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FOREWORD

Many large urban agglomerations in the developed countries are either experiencing population decline or are growing at rates lower than those of middle-sized and small settlements. This trend is in direct contrast to the one for large cities in the less developed world, which are growing rapidly. Urban contraction and decline is generating fiscal pressures and fueling interregional conflicts in the developed nations; explosive city growth in the less developed world is creating problems of urban absorption. These developments call for the reformulation of urban policies based on an improved understanding of the dynamics that have produced the current patterns.

During the period 1979-1982, the former Human Settlements and Services Area examined patterns of human settlement transformation as part of the research efforts of two tasks: the Urban Change Task and the Population, Resources, and Growth Task. This paper was written as part of that research activity. Its publication was delayed, and it is therefore being issued now a few months after the dissolution of the HSS Area.

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former Chairman
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INTRODUCTION

The purpose of this paper is to examine the population dynamics in a system of cities where the probability of migration between any pair of cities is not constant, but depends on the population distribution. In this sense, I shall consider a generalization of the models pioneered by Rogers (1975). However attention will be restricted to the simplest possible generalization, of the form:

$$m_{ij}(t) = n_{it}n_{jt}f_{ij} \ / \ \sum_k n_{kt}f_{ik} \tag{1}$$

where at time t, $m_{ij}(t)$ is the migration rate from city i to city j; n_{jt} is the population of city j; and f_{ij} is a time independent constant parametrizing the rate of migration.

This version is analyzed for the simple reason that some analytical results are obtainable. Although it is possible in principle to write down the dynamic equations for any functional relationship between $m_{ij}(t)$ and the vector of populations at time t, only elementary cases have been analyzed with any degree of success (Weidrich and Haag 1980).

This problem has been previously analyzed by others: notably Ledent (1978), Okabe (1979), and De Palma (1982). However each of these authors solved a different, and in Okabe's case an inconsistent, version of this model. Appendix A of this paper demonstrates how the formulation chosen here is superior to these other attempts.

Three cases of population dynamics will be considered: the redistribution of population under the assumption of zero population growth; population dynamics incorporating natural increase; and change when exogenous limits are imposed on the population of cities. The next section discusses the formulation of these three cases as non-linear continuous time models, and subsequent sections analyze the stability and equilibrium properties of each case in turn. The equilibrium points may be interpreted as possible long run city-size distributions, and in each case I shall attempt to demonstrate how these equilibrium points and their stability may be computed, suggesting in turn how the model may be used to analyze empirical data. Throughout, the effect of age distributions will be ignored; once again an introduction of this qualitatively increases the complexity of the analysis (Gurtin and McCamey 1974).

1. BASIC FORMULATIONS

Consider first the case of zero population growth. Then the population at time $t+\Delta t$ is the sum of non-migrants, migrants staying within the city, and inmigrants:

$$n_{j,t+\Delta t} = (1 - r_j\Delta t)n_{jt} + r_jM_{jj}(t,\Delta t) + \sum_{i\neq j} r_iM_{ij}(t,t+\Delta t) \quad (2)$$

where r_j is the instantaneous rate of mobility of the population in city j (the proportion migrating at any instant of time), and $M_{ij}(t,t+\Delta t)$ is the proportion of the migrant population in city i that migrate to city j between t and $t+\Delta t$. The introduction of r_j implies that a mover-stayer model is being analyzed, but cases will also be examined where the proportion of stayers is an endogenously determined function.

The migrating population is taken to be:

$$M_{ij}(t,t+\Delta t) = m_{ij}(t)\Delta t \tag{3}$$

where

$$m_{ij}(t) = n_{it}n_{jt}g(y_j)d_{ij} \Big/ \sum_k n_k(t)g(y_k)d_{ik} \tag{4}$$

Here $g(y_j)$ is a time independent measure of the *in situ* attractiveness of city j for migration, and d_{ij} is a measure of the ease of migration from i to j. Defining:

$$f_{ij} = g(y_j)d_{ij} \tag{5}$$

then:

$$n_{j,t+\Delta t} = \sum_i r_i n_{it} \frac{n_{jt}f_{ij}}{\sum_k n_{kt}f_{ik}} \Delta t + (1 - r_j\Delta t)n_{jt} \tag{6}$$

It is readily apparent from equation (6) that migration rates are given by a gravity-type of formulation. As discussed above, the reason for choosing equation (4) is detailed in Appendix A. To summarize, equation (4) guarantees that the initial population in a city equals the sum of stayers and all migrants:

$$n_{jt} = (1 - r_j\Delta t)n_{jt} + r_j \sum_k m_{jk}(t)\Delta t \tag{7}$$

By substituting (4) into the right-hand side of (7) it can be seen that (7) is true by definition. This is of course just an origin-constrained gravity formulation.

Defining A_{it} as equal to $\sum_k n_{kt}f_{ik}$, (6) becomes:

$$n_{j,t+\Delta t} = \Delta t \sum_i r_i n_{it} n_{jt} f_{ij} A_{it}^{-1} + (1 - r_j\Delta t)n_{jt} \tag{8}$$

This may be converted into a continuous time dynamic equation by subtracting n_{jt} from both sides dividing through by Δt and taking the limit as Δt tends to zero:

$$dn_{jt}/dt = n_{jt}\left[\sum_i r_i n_{it} f_{ij} A_{it}^{-1} - r_j\right] \tag{9}$$

Equation (9) represents the first model to be analyzed; population redistribution with zero population growth.

A natural generalization of (9) that remains simple enough for elementary analysis is to assume that there is a constant instantaneous rate of natural increase, β_i, in city i. As a counterpart to equation (7), the population dynamics of an expanding system must satisfy the accounting identity that the total population in city i at time t, plus those born to that population between t and $t+\Delta t$, should equal the stayers in i, plus all migrants, plus those born to the stayers and migrants between t and $t+\Delta t$:

$$\begin{aligned}(1 + \beta_j \Delta t)n_{jt} = {} & (1 - r_j \Delta t)(1 + \beta_j \Delta t)n_{jt} \\ & + r_j \Delta t(1 + \beta_j \Delta t)\sum_k m_{jk}(t)\end{aligned} \tag{10}$$

It may be seen that if $m_{ij}(t)$ is given by (4), then (10) is true. Thus a satisfactory model of population change with natural increase is:

$$\begin{aligned}n_{j,t+\Delta t} = {} & \Delta t \sum_i \left[1 + \beta_j \Delta t) r_i n_{it} n_{jt} f_{ij} A_{it}^{-1}\right] \\ & + (1 + \beta_j \Delta t)(1 - r_j \Delta t)n_{jt}\end{aligned} \tag{11}$$

Taking first differences of (11), and taking the limit as $\Delta t \to 0$, it is possible to ignore terms of the order of $(\Delta t)^2$. Thus under this presumption that the probability of two events (birth and migration) occurring simultaneously is negligible:

$$dn_{jt}/dt = n_{jt}\left[\sum_i r_i n_{it} f_{ij} A_{it}^{-1} - r_j + \beta_j\right] \qquad (12)$$

Equation (12) represents the second model to be analyzed. Note that in this continuous version it is not important whether in the discrete time case births were assumed to occur at the beginning or the end of the time period. Once second-order terms are ignored the continuous time version is identical in each case.

The model with natural increase and migration of equation (13) has a property that is unrealistic. Population growth and inmigration rates are strictly proportional to the size of the city. This implies that cities can in principle grow to an unlimited size, which is inconsistent with recent trends in Europe and North America (Korcelli 1980). The phenomenon of counter-urbanization, or reversed polarization, is of course not a purely demographic one. The rates of population growth of large cities depend on a complex of social, political, and economic forces which influence the locations at which jobs and opportunities for social advancement are available. A univariate population model is certainly inadequate to capture this phenomenon. However, one way of representing limits to growth in individual cities, while maintaining the relative simplicity of a purely demographic approach is to specify an upper limit, q_i, on the population of city i.

To construct a model of urban population change with limits to growth, the following two concepts are sufficient:

$$\beta_{it} = \gamma_i(q_i - n_{it}) \qquad (13)$$

$$M_{ij}(t,t+\Delta t) = \sum_i N_{it}(q_j - n_{jt})n_{jt}f_{ij}A_i^{-1}(n)\Delta t \qquad (14)$$

where $N_{it} = (1 + \beta_{it}\Delta)n_{it}$ and $A_i(n) = \sum_k (q_k - n_{kt})n_{kt}f_{ik}$. Equation (13) states that the rate of population increase in city i is bounded above by its capacity, q_i. Equation (14) states that the number of migrants from city i to city j is the product of: the population of city i including natural increase (N_{it}); and the fraction of migrants moving to city j. The latter term depends logistically on the size of city j relative to its maximum size. Thus as a city grows it becomes initially more and more attractive to migrants, but as it approaches its maximum size this attractiveness reduces again to zero. The constraint imposed by $A_i(n)$ ensures that total migration originating in i equals the mobile population of i:

$$r_i N_{it} = r_i \sum_j M_{ij}(t,t+\Delta t) \tag{15}$$

Population change in city i is then the sum of migration [equation (15)], and natural increase in the immobile population:

$$\begin{aligned} n_{j,t+\Delta t} &= \sum_i r_i N_{it}(q_j - n_{jt}) n_{jt} f_{ij} A_{it}^{-1}(n)\Delta t \\ &\quad + (1 + \beta_j \Delta t)(1 - r_j \Delta t) n_{jt} \end{aligned} \tag{16}$$

Neglecting the second order terms

$$\begin{aligned} n_{j,t+\Delta t} &= \sum_i r_i n_{it}(q_j - n_{jt}) n_{jt} f_{ij} A_{it}^{-1}(n)\Delta t \\ &\quad + (1 + \beta_j \Delta t - r_j \Delta t) n_{jt} \end{aligned} \tag{17}$$

Taking differences, dividing through by Δt, recalling equation (13), and taking the limit as $\Delta t \to 0$:

$$\begin{aligned} dn_{jt}/dt &= \sum_i r_i n_{it}(q_j - n_{jt}) n_{jt} f_{ij} A_{it}^{-1}(n) \\ &\quad - (r_j - \gamma_j (q_j - n_{jt})) n_{jt} \end{aligned} \tag{18}$$

Equation (18) represents the third model to be analyzed. Unfortunately, equation (18) as it stands is inconsistent. If for all i, n_i equals q_i (all cities have grown to full capacity) then dn_j/dt equals $-r_j n_j$ in each city. Population is migrating into thin air and aggregate urban population is decreasing.

This result could be interpreted as an urban to rural population flow that results when cities reach their upper limit. But there is no rural sector in the model, and to include a rural sector with a growth limit would eventually lead to the same inconsistency. In fact to assume that the rate of mobility, r_i, is independent of the urbanization pattern is of course a simplification; and it is this that leads to the inconsistency. A method for overcoming this will be introduced in section 4. A further problem with this model is the choice of upper limits q_i. To choose these *a priori* is nothing less than an imposition of a city size distribution to be reached in the limit. But the size to which cities can profitably grow does not just depend on internal size considerations; it rather depends on the location of the city in the urban hierarchy and on urban development patterns (Sheppard 1982). However, the urban dynamics depend in turn on q_i; thus it is circular to choose values for these parameters in order to investigate a process from which the same values should be an output. These represent significant problems for future investigation.

2. POPULATION DYNAMICS WITH ZERO POPULATION GROWTH

2.1 The Existence of Equilibrium

Study of the dynamics of non-linear processes typically starts with a classification of any equilibrium points in the system. This need not imply that the process itself is equilibrating; such points simply serve as reference points with respect to which different regimes with varying dynamic behavior may be traced out (Hirsch and Smale 1974). In the ZPG model it can be determined that such an equilibrium exists and that it is probably unique in empirical applications. Furthermore, its location and stability properties may be computed.

To demonstrate that an equilibrium exists, it is first necessary to show that negative populations cannot occur in the model. This is easily done. Divide through equation (9) by n_{jt}:

$$n_{jt}^{-1}dn_j / dt = d \log n_j / dt \tag{19}$$

$$d \log n_{jt} / dt = \sum_i r_i n_i f_{ij} A_i^{-1} - r_j \tag{20}$$

or

$$n_{jt} = \exp\left\{ \int_a^t \left[\sum_i r_i n_{is} f_{ij} A_i^{-1} - r_j \right] ds + C \right\} \geq 0 \tag{21}$$

where C is a constant of integration. Therefore n_{jt} is always non-negative. Thus the dynamics of population growth in a system of H cities occurs in the positive quadrat of N-dimensional space bounded by the hyperplanes $n_i = 0$ for all i. Indeed under ZPG we can go further and state that the dynamics are restricted to those locations where $\sum_i n_{it} = N$; N being the total population. In the case of three cities, this restricts the process to a bounded plane in three-space (Figure 1). In general, the process occurs on a bounded hyperplane of H-1 dimensions.

An equilibrium point on this surface is defined by:

$$\dot{n}_j = 0 \qquad \text{for all } j \tag{22}$$

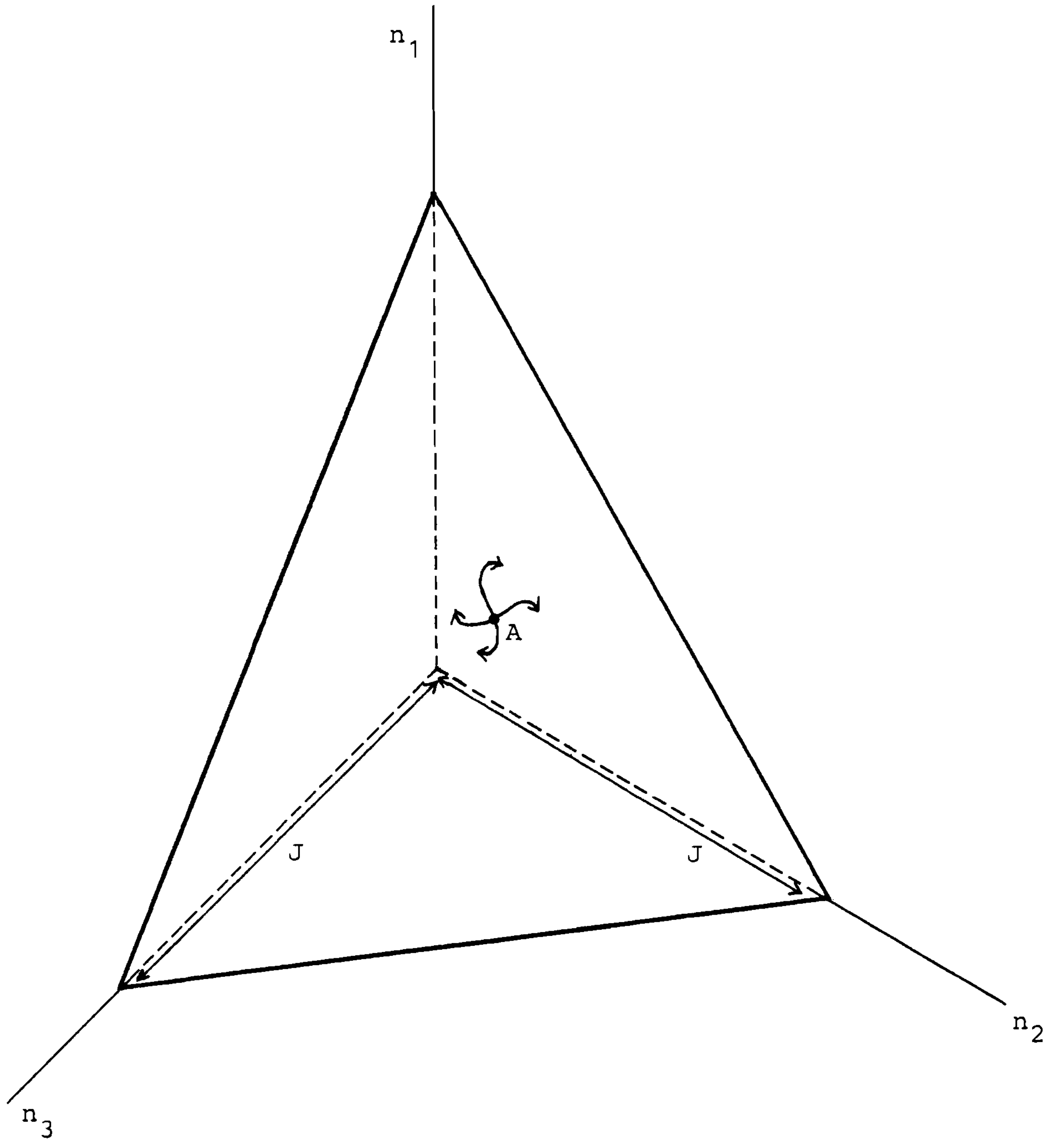
where $\dot{n}_j = dn_j/dt$. This occurs when [from equation (9)]:

$$n_{jt} r_j = n_{jt} \sum_i r_i n_{it} f_{ij} A_{it}^{-1} \qquad \text{for all } j \tag{23}$$

Note that if $\dot{n}_j = 0$ for all j, $\dot{A}_j = 0$ for all j. Such an equilibrium point is shown in Figure 1. The fact that such an equilibrium always exists may be shown in the following way (Papageorgiou 1982).

Figure 1 illustrates that the dynamics of population change are confined to a closed, bounded, convex set of points. Equation (23) may be summarized as:

$$\underline{n}_t = f(\underline{n}_t) \tag{24}$$



A - equilibrium point on population plane (unstable)

Figure 1. The plane of feasible population vectors for ZPG, with an unstable equilibrium point.

where $\underline{n}_t'$ is the vector $[n_{1t}, n_{2t}, \ldots]$. This is a continuous mapping from the bounded N-1 dimensional hyperplane into the same hyperplane. Under these conditions Brouwer's fixed point theorem tells us that there exists a point where $\underline{n}_t = f(\underline{n}_t)$; i.e., there exists an equilibrium point satisfying equation (23) with $n_{jt} \geq 0$. Note that this generalizes the results of McGinnis and Henry (1973) and Ledent (1978).

2.2 Computing the Equilibrium Point

The condition for equilibrium [equation (23)] may be rewritten as:

$$\underline{r}' = \underline{n}^{*\prime}(\underset{\sim}{R})(\underset{\sim}{A})^{-1}\underset{\sim}{F} \tag{25}$$

where $\underline{n}^{*\prime}$ is the vector of equilibrium populations being sought; $\underline{r}' = (r_1, \ldots, r_N)$; $(\underset{\sim}{R})$ is a diagonal matrix with k-th diagonal entry equal to r_k; $(\underset{\sim}{A})$ is a diagonal matrix with k-th diagonal entry equal to A_k; and $\underset{\sim}{F}$ is the N by N matrix containing entries f_{ij}. From equation (25):

$$\underline{n}^{*\prime} = \underline{r}'\underset{\sim}{G}(\underset{\sim}{R})^{-1}(\underset{\sim}{A}) \tag{26}$$

where $\underset{\sim}{G}$ is equal to the inverse of $\underset{\sim}{F}$. In simple algebra, from (26):

$$n_i^* = \sum_{k=1}^{N} r_k g_{ki} \sum_{j=1}^{n} r_i^{-1} f_{ij} n_j^* \tag{27}$$

Equation (27) comes from applying the definition of (A) and the rules of matrix algebra to equation (26). But equation (27) can also be written as:

$$\underline{n}^* = (\underset{\sim}{H})(\underset{\sim}{R})^{-1}\underset{\sim}{F}\underline{n}^* \tag{28}$$

where $(\underset{\sim}{H})$ is a diagonal matrix with i-th diagonal entry equal to $\sum_k r_k g_{ki}$. Rearranging (28):

$$(\underset{\sim}{M} - \underset{\sim}{I})\underline{n}^* = 0 \tag{29}$$

or $\underline{n}^*$ is given as that eigenvector of $\underset{\sim}{M}$ which is associated with an eigenvalue of $\underset{\sim}{M}$ that is identically equal to one. $\underset{\sim}{M}$ is the matrix product $(\underset{\sim}{H})(\underset{\sim}{R})\underset{\sim}{F}$. The fact that an equilibrium must exist, proven above, is sufficient to show that $\underset{\sim}{M}$ must have at least one eigenvalue equal to one. This may be confirmed by direct investigation of $\underset{\sim}{M}$ (see Appendix B, theorem 1). The associated equilibrium vector $\underline{n}^*$ may then be computed from the appropriate eigenvector of $\underset{\sim}{M}$. Call this eigenvector, $\underline{x}^*$. Then employing the population constraint of ZPG, the equilibrium population vector is equal to the eigenvector scaled to sum up to the fixed total population:

$$\underline{n}^* = \underline{x}^* N X^{-1} \qquad (30)$$

where X is the scalar $(\underline{i}'\underline{x}^*)$; the sum of the elements in $\underline{x}^*$. By theorem 2 of Appendix B this equilibrium vector is the strictly positive right-hand eigenvector of $\underset{\sim}{M}$ that is associated with its largest eigenvalue (an eigenvalue of unity).

2.3 Uniqueness of Equilibrium

Theorem 2 of Appendix B shows that if $\underset{\sim}{M}$ is indecomposable and primitive then there is only one eigenvalue equal to one, and thus only one internal equilibrium vector. Indecomposability implies that migration streams occur between cities in such a way that each city is directly or indirectly connected to each other city. Given the fact that migration streams are highly dispersed, indecomposability is likely to be true of any empirical inter-urban migration matrix. If $\underset{\sim}{M}$ is indecomposable it will also be primitive if at least one of the entries on its main diagonal is non-zero (Solow 1952). But this is true for all diagonal entries, by equation (29). Thus we can expect $\underset{\sim}{M}$ to be indecomposable and primitive in practical applications, the equilibrium will be unique, and as a consequence the stability properties of this equilibrium point will be sufficient to characterize global stability conditions in the urban system.

2.4 Stability of Population Change

Define the vector of displacement of city sizes from the equilibrium point at time t:

$$\hat{\underline{n}}_t = \underline{n}_t - \underline{n}^* \tag{31}$$

Taking a Taylor expansion about $\underline{n}^*$ and retaining just linear terms because it is local variations that are of interest:

$$d\hat{\underline{n}}_t/dt = J_1 \hat{\underline{n}}_t \tag{32}$$

Here J_1 is the Jacobean matrix with i,j-th entry equal to $\partial F_i(\underline{n}^*) / \partial n_j$, and $d\hat{\underline{n}}_t/dt$ is a N by 1 vector of entries $d\hat{n}_{it}/dt$. Finally,

$$F_j(\underline{n}^*) = n_j^* \left(\sum_i r_i n_i^* f_{ij} A_i^{-1} - r_j \right) \tag{33}$$

is the dynamic model (9) evaluated at the equilibrium point. The local stability properties depend on $\underset{\sim}{J}$ which (see theorem 3 and lemma 1 of Appendix B) may be written as:

$$\underset{\sim}{J}_1 = [\underset{\sim}{I} - (\underset{\sim}{n}^*)\underset{\sim}{F}'\underset{\sim}{A}^{-1}](\underset{\sim}{R})(\underset{\sim}{n}^*)\underset{\sim}{A}^{-1}\underset{\sim}{F} - (Y) - (R) \tag{34}$$

where $(\underset{\sim}{n}^*)$ is a diagonal matrix with n_i^* as the i-th diagonal entry. $(\underset{\sim}{Y})$ is a diagonal matrix with j-th entry equal to $\sum_i r_i n_i^* f_{ij} \tilde{A}_i^{-1}$. $\underset{\sim}{A}_i$ is defined also with respect to the equilibrium populations $\underline{n}^*$: $A_i = \sum_k n_k^* f_{ik}$.

Necessary and sufficient conditions for local stability are that the eigenvalues of $\underset{\sim}{J}$ all have negative real parts. The eigenvalues of $\underset{\sim}{J}$ can be computed. From equation (34) it is clear that they depend on: the geography of the system as expressed in the barriers to movement and origin/destination characteristics (f_{ij}); and the geography of exogenous mobility patterns $(\underset{\sim}{R})$ (see lemma 1 of Appendix B). In any empirical application stability can be determined by computing equation (34).

If all eigenvalues of J have negative real parts then the equilibrium $\underline{n}^*$ is globally stable when it is unique (see above). In this case $\underline{n}^*$ will express the stable distribution of population in the various cities. This equilibrium will be approached directly (if all eigenvalues are real) or cyclically (if some eigenvalues are complex) as the system evolves. If one or more of the eigenvalues have positive real parts then the equilibrium point is a "saddle-point" in N-1 dimensional space. Any slight deviation from $\underline{n}^*$ will lead to further deviations as some cities move away from equilibrium drawing the rest of the system along behind (Hirsch and Smale 1974).

2.5 A Two City Example with Zero Population Growth

Consider two cities, and assume for simplicity that $r_1 = r_2 = 1$. Then

$$\partial n_1/dt = n_1(n_1\alpha_{11} + n_2\alpha_{21} - 1) \qquad (35)$$

$$\partial n_2/dt = n_2(n_2\alpha_{12} + n_2\alpha_{22} - 1) \qquad (36)$$

where $\alpha_{ij} = f_{ij}A_i^{-1}$.

Conditions for equilibrium, from (8), are:

$$-1 + n_1\alpha_{11} + n_2\alpha_{21} = 0 \quad \Rightarrow dn_1/dt = 0 \qquad (37.1)$$

$$-1 + n_1\alpha_{12} + n_2\alpha_{22} = 0 \quad \Rightarrow dn_2/dt = 0 \qquad (37.2)$$

or

$$-1 + \frac{f_{11}}{A_1} n_1 + \frac{f_{21}}{A_2} n_2 = 0 \qquad (38.1)$$

$$-1 + \frac{f_{12}}{A_1} n_1 + \frac{f_{22}}{A_2} n_2 = 0 \qquad (38.2)$$

Solving for a common denominator, these conditions become:

$$-(f_{11}n_1 + f_{12}n_2)(f_{21}n_1 + f_{22}n_2) + f_{11}n_1(f_{21}n_1 + f_{22}n_2) + f_{21}n_2(f_{11}n_1 + f_{12}n_2) = 0 \quad (39.1)$$

$$-(f_{11}n_1 + f_{12}n_2)(f_{21}n_1 + f_{22}n_2) + f_{12}n_1(f_{21}n_1 + f_{22}n_2) + f_{22}n_2(f_{11}n_1 + f_{12}n_2) = 0 \quad (39.2)$$

Cancelling out common terms, and dividing by n_1 in the first equation and n_2 in the second, we find that $dn_1/dt = 0$ if:

$$f_{21}(f_{11} - f_{12})n_1 + f_{12}(f_{21} - f_{22})n_2 = 0 \quad (40.1)$$

and $dn_2/dt = 0$ if:

$$f_{21}(f_{12} - f_{11})n_1 + f_{12}(f_{22} - f_{21})n_2 = 0 \quad (40.2)$$

Five cases of equilibrium can then be identified:

(i) $n_1 = n_2 = 0$

(ii) $n_1 = -\dfrac{f_{12}(f_{22} - f_{21})}{f_{21}(f_{12} - f_{11})} n_2 = \beta n_2 \quad (41)$

It may readily be checked that this is the eigenvector of HF associated with a unit eigenvalue.

(iii) $f_{22} = f_{21}$, and $f_{11} = f_{12}$

(iv) $n_1 = 0$ and $f_{22} = f_{21}$

(v) $n_2 = 0$ and $f_{12} = f_{11}$

Cases (iii), (iv), and (v) are special cases representing situations where the propensity for inter- and intra-urban interaction are identical in at least one city. In case (iii), where the cities are collapsed into a single location, any population pattern is stable. In cases (iv) and (v) one city is absent, the population is again reduced to one location and the other city is stable at any value. In all these cases the urban population is essentially collapsed onto the head of a pin, and stability (and thus dynamics) become trivial. Case (i) is also trivial.

The stable populations in case (ii) clearly depend on the strength of inter- *versus* intra-urban migration, and thus the dynamics of population change depend on the signs of $f_{22} - f_{21}$ and $f_{11} - f_{12}$. Four possibilities exist:

Case A Inter-urban migration dominates: $f_{22} < f_{21}$; $f_{11} < f_{12}$. Stability occurs on the ray $n_1 = \beta n_2$ ($\beta > 0$). In the positive quadrant ($n_i > 0$), the one of substantive interest, if $n_1 < \beta n_2$, $dn_1/dt > 0$, and $dn_2/dt < 0$. When $n_1 > \beta n_2$ then $dn_1/dt < 0$ and $dn_2/dt > 0$. This is shown in Figure 2a. Clearly the equilibrium ray is stable in this case: populations of the cities will converge to the point A over time. Clearly $n_1^* = \frac{N}{1 + \beta}$, $n_2^* = \frac{N\beta}{1 + \beta}$.

Case B Intra-urban migration dominates ($f_{12} < f_{11}$; $f_{21} < f_{22}$). In the positive quadrant, if $n_1 < \beta n_2$, $dn_1/dt < 0$, and $dn_2/dt > 0$. But if $n_1 > \beta n_2$ then $dn_1/dt > 0$ and $dn_2/dt < 0$. In short, the pattern is the converse of case A (Figure 2b). The equilibrium ray is unstable, and depending on initial conditions one city or the other will die out.

Case C Migration patterns are dominated by the pull of city 1 ($f_{11} > f_{12}$; $f_{21} > f_{22}$). In this case (and case D below) the cosine of β is negative. Thus there are no stable population combinations in the positive quadrant. For all feasible initial conditions $dn_1/dt > 0$ and $dn_2/dt < 0$. City 2 will thus always die out.

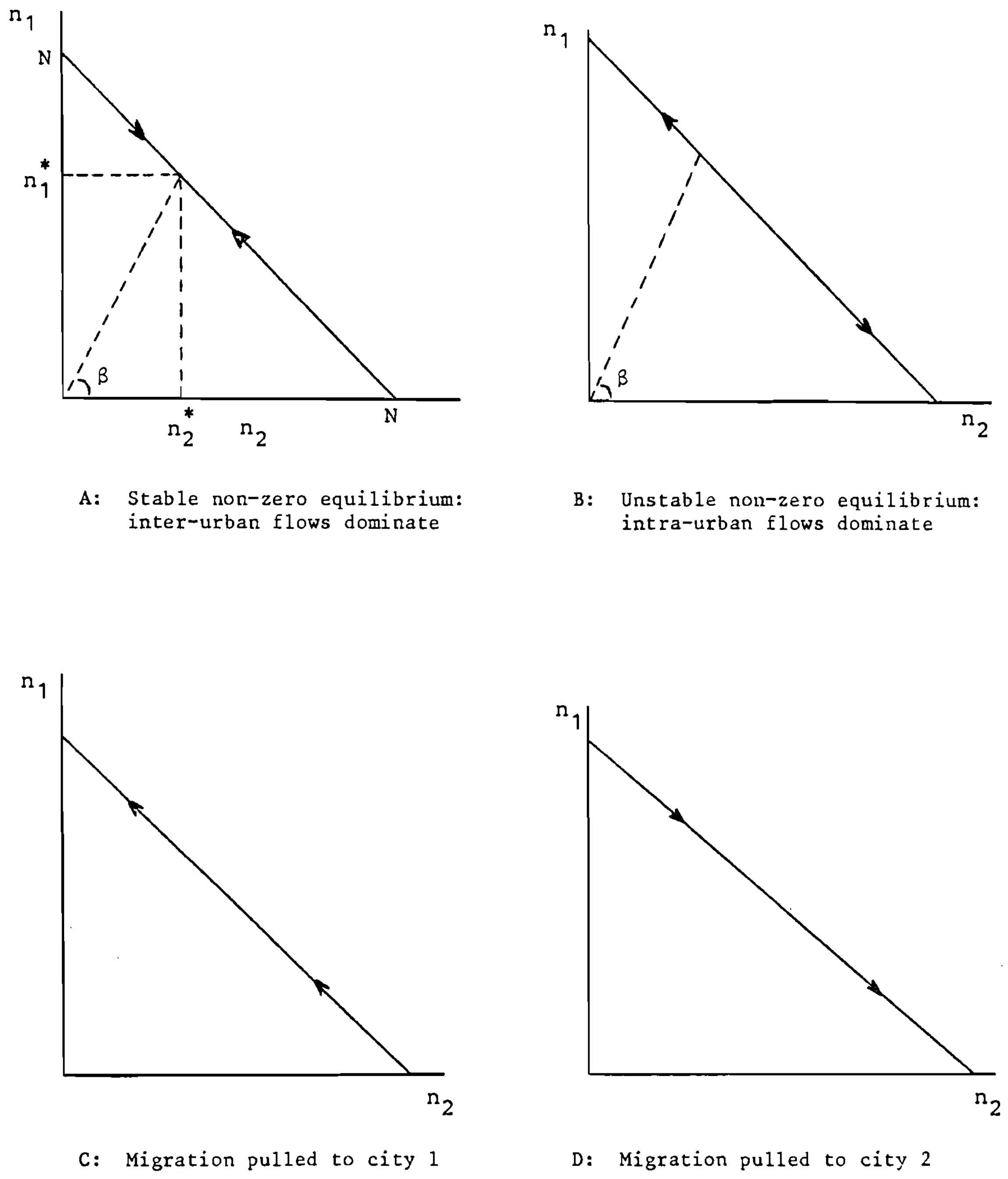


Figure 2. Population dynamics with two cities: a graphical depiction of alternative dynamics.

Case D Migration is dominated by the pull of city 2 ($f_{22} > f_{21}$; $f_{12} > f_{11}$). This is the converse of case C. If $n_1, n_2 > 0$ then $dn_2/dt > 0$ and $dn_1/dt < 0$. For full results see Figures 2c and 2d.

Cases C and D describe an extreme case of primacy; that one city cannot even retain an equal proportion of its own population. Case A suggests neither city can retain an equal proportion of its own population. The most realistic case is case B; where the equilibrium is unstable, and where dynamic trends "bifurcate" around the unstable equilibrium point depending on the initial size of city 1 compared to city 2. Once one city has declined to zero the other city's population represents an equilibrium. This is because the system has undergone a structural change. If one city no longer exists the other one has nowhere to send its migrants [m_{ij} for $i \neq j$ is zero from equation (1)] and thus the size of f_{ij} no longer matters.

2.6 Summary

For the ZPG model it has been shown that an equilibrium combination of populations always exists, in the sense that if the city size distribution matches this equilibrium there will be no change in city populations in the absence of some external shock. In short, in equilibrium total inmigration equals total outmigration for each city. This equilibrium distribution can be computed [equation (29)], and it depends on the mobility rates, and the geography of migration as expressed through f_{ij}. Further, we can expect this equilibrium distribution to be unique. The stability or instability of the population dynamics about this equilibrium can be determined by computing $\underset{\sim}{J}$ and its eigenvalues [equation (34)]. These stability properties also depend only on r_i and f_{ij}. Because of the uniqueness of equilibrium, the stability of $\underset{\sim}{J}$ also characterizes the global stability of the system, completing the qualitative analysis of inter-urban population dynamics with zero population growth. Finally a two-city example was presented to illustrate the analysis.

3. POPULATION DYNAMICS WITH NATURAL INCREASE

The model to be analyzed here is:

$$dn_{jt}/dt = n_{jt}\left[\sum_i r_i n_{it} f_{ij} A_{it}^{-1} - r_j + \beta_j\right] \tag{12}$$

3.1 Equilibrium States

In order to achieve a state of static equilibrium, it is clear that some rates of natural increase, β_j, must be negative. This case will not be dealt with here, since it does not represent typical real world characteristics of urban population change. Thus equilibrium must be conceived of dynamically, and two cases seem worthy of consideration.

TYPE A: Simple dynamic equilibrium. Here equilibrium is given by:

$$dn_{jt}/dt = kn_{jt} \qquad \forall j \tag{42}$$

This equilibrium can be regarded as equivalent to that achieved in linear multiregional demographic projection models (Rogers 1975); population growth is identical everywhere, and a stable vector of relative population sizes exists.

TYPE B: Weighted equilibrium with geographically varying growth rates

$$dn_{jt}/dt = k_j n_{jt} \tag{43}$$

In this case the growth rate in each city may be different, but in each it is constant over time.

3.2 Simple Dynamic Equilibria

The existence of such an equilibrium can be concluded if there exists a set of relative population sizes that solves the following problem:

$$kn_j^* = S_j(\underline{n}^*) \qquad \forall j \tag{44}$$

where

$$S_j(\underline{n}^*) = n_j^* \left[\sum_i r_i n_i^* f_{ij} A_{it}^{-1} - r_j + \beta_j\right]$$

or in matrix form:

$$k\underline{n}^* = S(\underline{n}^*) \tag{45}$$

This is a non-linear eigenvalue equation, where k is an eigenvalue and $\underline{n}^*$ an eigenvector of the vector function $S = [S_1,\ldots,S_j,\ldots,S_J]$.

The results assembled by Nikaido (1968) can be used to show that at least one simple dynamic equilibrium exists, with a positive identical growth rate for all cities, and a corresponding positive vector of relative population sizes (a city size distribution). (See theorem 1, Appendix C.) Indeed all solutions to (45) are positive, according to this theorem. This generalizes the results of Feeney (1973) and Ledent (1978). Solutions for the equilibrium growth ray(s) may be obtained by use of a non-linear eigenvalue program (cf. Andersson and Persson 198) and each solution must be treated as a candidate whose stability should be analyzed.

3.3 Geographically Variable Dynamic Equilibria

Theorem 2 of Appendix C shows that no relationship satisfying (43) can be characterized as an equilibrium ray. Therefore equilibria of type B do not exist.

3.4 Stability

Having shown that dynamic equilibria for this system are characterized by equation (42), and that such equilibria exist and can be computed, it remains to test such equilibria for stability. This is most easily checked for by using the logarithmic form. Dividing (12) by n_{jt}:

$$d \log n_{jt} / dt = \left[\sum_i r_i n_{it} f_{ij} A_{it}^{-1} - r_j + \beta_j \right] \tag{46}$$

But in equilibrium

$$d \log n_{jt} / dt = k \tag{47}$$

Thus defining

$$\hat{\hat{n}}_{jt} = \log n_{jt} - k \tag{48}$$

we wish to show that

$$d\, \hat{\hat{n}}_{jt} / dt < 0 \tag{49}$$

Taking a Taylor expansion around the equilibrium point and retaining linear terms:

$$d\, \hat{\hat{\underline{n}}}_t / dr = \underset{\sim}{J}_2\, \hat{\hat{\underline{n}}}_t \tag{50}$$

From theorem 3 of Appendix C:

$$\underset{\sim}{J}_2 = (\underset{\sim}{R}) \underset{\sim}{X} - \underset{\sim}{X}' (\underset{\sim}{R}) (\underset{\sim}{n}^*) \underset{\sim}{X} \tag{51}$$

Using the definition of $\underset{\sim}{X}$ as $(\underset{\sim}{A})^{-1} \underset{\sim}{F}$:

$$\underset{\sim}{J}_2 = [I - \underset{\sim}{F}' (\underset{\sim}{A})^{-1} (\underset{\sim}{n}^*)] (\underset{\sim}{R}) (\underset{\sim}{A})^{-1} \underset{\sim}{F} \tag{52}$$

The stability of equilibrium will depend on whether the eigen-values computed for $\underset{\sim}{J}_2$ have negative real parts.

3.5 Endogenizing the Propensity to Migrate

Suppose that the fraction of population migrating is related in some positive manner to the accessibility of opportunities (as represented by other cities). The validity of this notion when choices are available has been rigorously derived in Sheppard (1980). One way of representing this is:

$$r_{it} = M \sum_j (f_{ij} n_{jt})^{\beta} \qquad (53)$$

where M is a very small number designed to keep r_{it} less than one. Substituting (53) into (44); $S_j(\underline{n}^*)$ splits into two terms, one of which is homogeneous of order 1, and one of which is homogeneous of order greater than one:

$$S_j(\underline{n}^*) = S_j^1(\underline{n}^*) + S_j^2(\underline{n}^*)$$

where

$$S_j^2(\underline{n}^*) = (\beta_j - r_j)n_{jt}^* \qquad (54)$$

Then if α is a scalar:

$$S_j(\alpha\underline{n}^*) = \alpha^{1+\beta} S_j^1(\underline{n}^*) + \alpha S_j^2(\underline{n}^*) \qquad (55)$$

This difference may be critical. The research of Okabe shows that simple population equilibria in his model only exist when the non-linear model is homogeneous of order one (Okabe 1979, theorem 6). Further, it seems that the parallels that Nikaido (1968) was able to draw between linear and non-linear eigenvalues may hinge on homogeneity of order less than or equal to one; he provides no results for homogeneity of higher orders. On this basis it may be reasonable to speculate that endogenizing r_j as a function related to accessibility of other cities may substantially reduce the probability of finding simple dynamic population equilibria at all.

3.6 Summary

The introduction of natural increase leads to the conclusion that simple dynamic population equilibria, with properties analogous to the multiregional stable growth projections of linear models exist. This would explain why simulations of a non-linear model by Ledent (1978) always led to such results. However it should be noted that several equilibrium paths can be expected, and there is no reason to believe that they will be necessarily stable. This all depends on analysis of the eigenvalues of $\underset{\sim}{J}_2$ [equation (51)] for each given equilibrium path calculated from solving the non-linear eigenvalue equation (45).

The presence of several equilibrium paths means that the population dynamics are governed by more than one regime, and it becomes difficult to make statements about global stability unless Lyapunov conditions can be derived (Gandalfo 1971). The sort of possibilities for urban population dynamics are illustrated in Figure 3, which for graphical purposes is presented as a two-city problem. Lines OA, OB and OC represent three dynamic equilibrium paths; three solutions to equation (45). Analysis of (51) for each case in this hypothetical example shows that OA and OC are stable (having eigenvalues with negative real parts), while OB is unstable. As a result, the population dynamics split into two regimes. To the right of OB populations tend away from OB, into the domain of attraction of OC (represented by the dashed line), leading to a stable pattern dominated by city 2. To the left of OB the converse occurs. Thus the outcome depends critically on what happens when the city size distribution is near OB. No matter how accurate forecast models may be, random external shocks, such as international migration, may push the process from one regime to the other, leading to dramatically different outcomes. The best way to counter such unexpected outcomes is to have as complete knowledge as possible about the various equilibria and their stability.

In a system of many cities, the picture can be much more complicated, and a pattern of population change that fluctuates widely and is difficult to predict can result. In such situations external shocks can play a far more vital role than is desirable (Allen 1976, 1982). A final point to note is that the number, location, and stability of equilibrium paths depends ultimately on the model of migration and on the relative location of cities, as expressed in $(\underset{\sim}{R})$ and $\underset{\sim}{F}$. This is as true in this case as in the case of zero population growth.

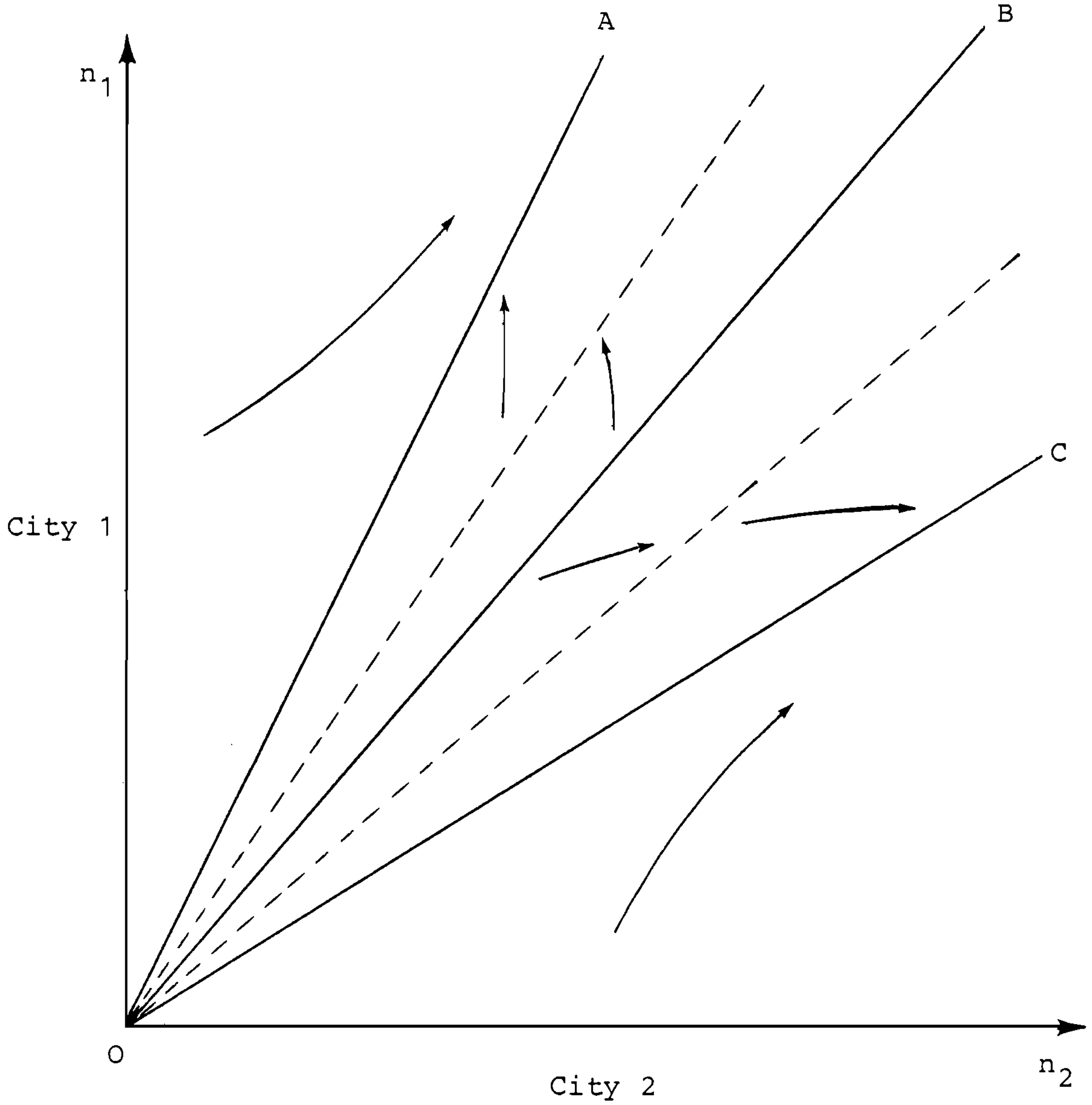


Figure 3. Multiple equilibrium paths for population changes with natural increase.

Making rates of mobility, r_j, endogenously depend on accessibility to other cities leads to a model which seems to exhibit properties suggesting that it cannot be treated as a non-linear model that is homogeneous of order one. It seems that a possible result will be a reduction in the number of equilibrium paths, perhaps to zero, and also a reduction in the likelihood that stable equilibria exist.

4. POPULATION DYNAMICS WITH LIMITS TO GROWTH

The model proposed is:

$$dn_{jt}/dt = n_{jt}\left[\sum_i r_i n_{it}(q_j - n_{jt}) f_{ij} A_{it}(n) - r_j + \gamma_j (q_j - n_{jt})\right] \tag{18}$$

where $A_{it}(n) = \sum_j n_{jt}(q_j - n_{jt}) f_{ij}$ and q_j represents the growth limit. As pointed out in section 1, this model is inconsistent as it stands since r_j is exogenous. Thus when $q_j = n_j$ for all j, all migration should be zero since $A_j(n)$ would be zero. But r_j is still positive, implying that people are leaving the cities. This inconsistency can be resolved by making r_j endogenous. The solution to be used here is:

$$r_{it} = \alpha_i \, A^{\beta}_{it}(n) \tag{56}$$

Substituting into (18):

$$dn_{jt}/dt = n_{jt}\,(q_j - n_{jt})\left[\sum_i \alpha_i n_{it} A^{\beta-1}_{it}(n) - \alpha_j A^{\beta}_{jt}(n) + \gamma_j (q_j - n_{jt})\right] \tag{57}$$

For this model a series of static equilibria exist. No dynamic equilibrium exists because the populations are bounded from above and below making unlimited growth or decline impossible. The static equilibria are listed in theorem 1 of

Appendix D. They consist of all possible combinations of $n_j^* = q_j$ or $n_j^* = 0$, plus any interior equilibrium points that might exist where some or all equilibrium populations are between zero and q_j: $0 < n_j^* < q_j$. For example in a two city system, the following equilibria are possible:

$$n_1^* = q_1 \quad , \quad n_2^* = q_2$$

$$n_1^* = q_1 \quad , \quad n_2^* = 0$$

$$n_1^* = 0 \quad , \quad n_2^* = q_2$$

$$n_1^* = 0 \quad , \quad n_2^* = 0$$

$$0 < n_i^* < q_i \quad \text{for either or both cities}$$

Of these five possibilities the fifth one may occur in more than one way: multiple interior equilibria are possible. This is because solutions for the fifth type are solutions to a non-linear eigenvalue equation (theorem 1, Appendix D), and several eigenvectors may exist for any eigenvalue equal to one.

For a system of H cities, there will be 2^H boundary equilibria, and an indeterminate number of interior equilibrium points. Fortunately, in the case where γ_j, the rate of natural increase, is non-zero for all cities, only one boundary equilibrium point is stable, thus the others may be ignored. This can be shown by analyzing the Jacobean matrix that determines local stability conditions about any equilibrium point (lemma 1, Appendix D).

To illustrate this, consider Figure 4. Here a two city case is illustrated. Population dynamics are confined to the rectangle OACD, due to the growth limits q_1 and q_2. Points C, A, B, and O represent the four boundary equilibria listed above, and D,E represent possible interior equilibria points. For further analysis it is useful to distinguish two cases.

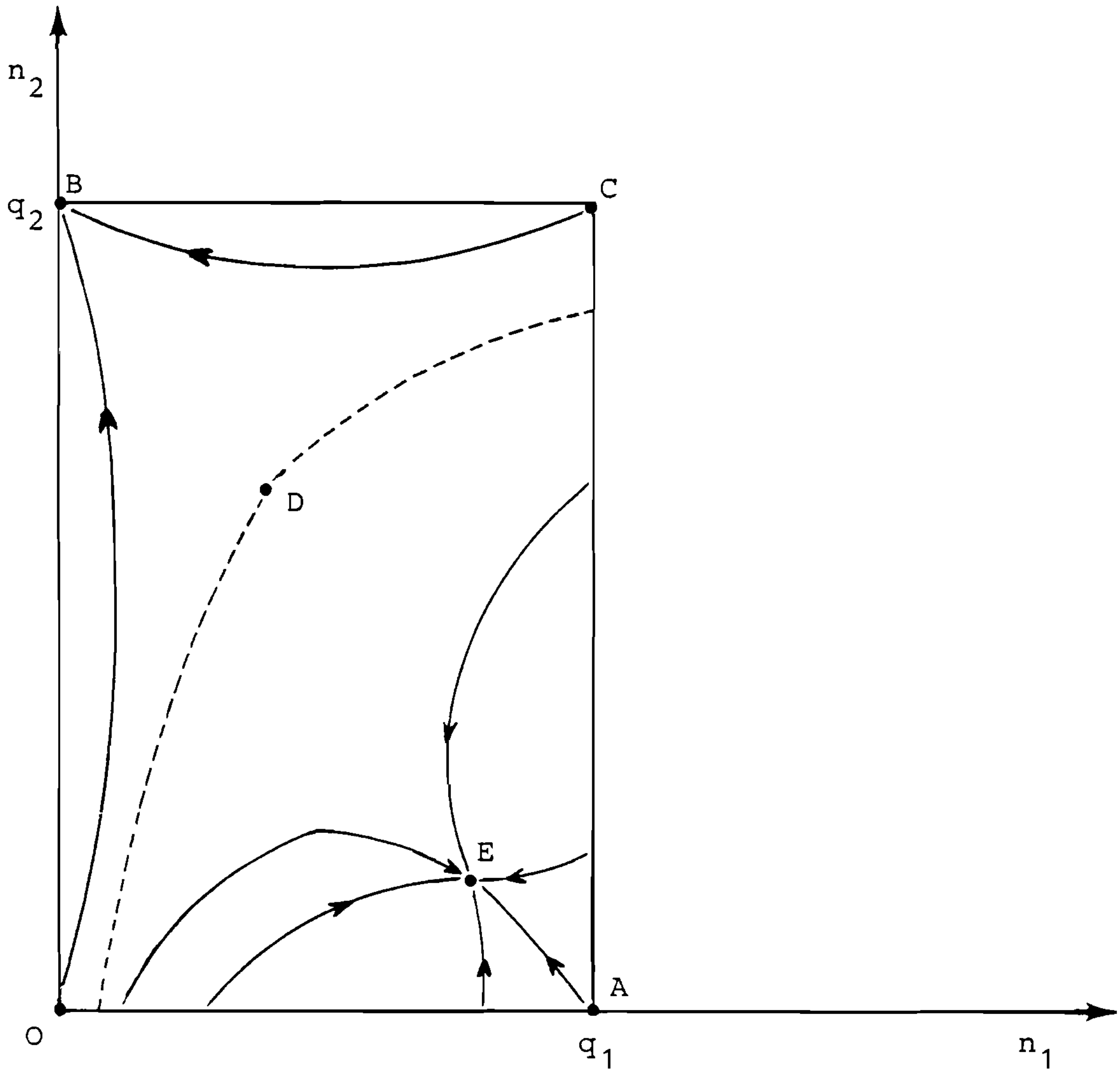


Figure 4. Population dynamics in a two-city system with limits to growth.

4.1 Natural Increase in All Cities

If γ_j is positive in all cities, then the only stable equilibrium is $n_j^* = q_j$ for all j. This is because this equilibrium is the only boundary equilibrium that is stable (corollary 1 and 2 of Appendix D), and because no interior equilibria can exist (corollary 3 of Appendix D). Thus if γ_j is positive everywhere the limiting city size distribution is given by the growth limits imposed. Population change may oscillate before equilibrium is achieved (see lemma 2, Appendix D), but the equilibrium is globally stable.

In light of this, the critical comments made in section 1 about imposing such limits are particularly important. The city size distribution is predefined, whereas in reality the benefits of city size should be deduced since they vary from place to place, depending on the geography of the urban system. Any attempt to define an optimal city size that ignores this context is fraught with problems (Richardson 1973), and in this sense the model of population dynamics with limits to growth can give little insight into how city size distributions are generated.

By a similar argument, if γ_j is negative everywhere then the city system will die out.

4.2 Natural Increase and Decrease Both Exist

If the set of cities is divided into cities j with natural increase ($\gamma_j > 0$), and the other cities k experiencing natural decrease ($\gamma_k < 0$) then the only stable boundary equilibrium is where $n_j^* = q_j$ for the former group, and $n_k^* = 0$ for the latter group (corollary 4 of Appendix D). However, in this case interior equilibria may also exist, so this boundary equilibrium may not be globally stable.

In the absence of such interior equilibria, the city size distribution is ultimately dependent solely on which cities experience natural increase, and which experience natural decrease. Thus the city size distribution would be defined

q_j and γ_j. However, because such interior equilibria cannot be ignored, the first step should be to determine whether, and how many, such equilibria exist by solving the non-llinear eigenvalue problem of equation (D.4) in Appendix D. Then the stability of these points may be determined using lemma 1 of Appendix D to construct the Jacobean matrix for each interior equilibrium point. This would complete the picture.

For example, in the two city case of Figure 4, suppose that γ_1 is negative and γ_2 is positive, that interior equilibrium point D is unstable, whereas E is stable. Then the qualitative behavior of the two city system is depicted by the flow lines on the figure. It can be seen that, depending on the starting point, any one of two static city size distributions can evolve. In addition, if the system is at any point in time near the dotted line on this figure it may 'flip' from tendencies toward B, say, to changes leading to E, simply as a result of some small external shock. That several interior equilibrium points may exist even for a simple two city system is shown by Weidlich and Haag (1980). Thus for a many city system a large number of such equilibria could exist, making the population dynamics complex and the equilibrium outcome highly dependent on random fluctuations that cannot be foreseen in forecasting populations.

Ultimately, the number, position, and stability of equilibria will depend on the migration model and on the relative location of the cities. It is thse factors that determine f_{ij}. Note that limit cycles around an unstable equilibrium point will not be expected to occur. One of the boundary equilibria is always stable, implying that the Poincaré-Bendix theorem, which would deduce the dxistence of a limit cycle, cannot be applied (cf. Weidlich and Haag 1980). Notice also that the analysis is somewhat more complicated if γ_j is zero in some cities (corollary 5 of Appendix D). Here, for example, stability of at least one boundary equilibrium point is not guaranteed, and thus the possibility of limit cycles cannot be ruled out.

5. GENERALIZATIONS

The model examined in detail in this paper has represented the rate of migration between two regions as:

$$m_{ij}(t) = n_{it}n_{jt}f_{ij} / \sum_k n_{kt}f_{ik} \tag{58}$$

In the model with natural increase, the proof of the existence of simple dynamic equilibria depended crucially on this specification, because of the necessity to show that dn_j/dt is homogeneous of order one (Appendix C). However, it seems possible that this assumption may be relaxable to allow any functional form of the distribution component of the migration model, of the form:

$$m_{ij}(t) = n_{it}h_i(n_{jt}, f_{ij}(t)) / \sum_k h_i(n_{kt}, f_{ik}(t)) \tag{59}$$

Here h_i could be any function relating n_{jt} and f_{ij}. One example is a generalized gravity or intervening opportunity type of model:

$$h_i(n_{jt}, f_{ij}(t)) = n_{jt}^{\alpha}f_{ij} \tag{60}$$

The reason for believing that stable equilibria still exist for this more general model is because the normalization process guarantees that:

$$\sum_j m_{ij}(t) = n_{it} \tag{61}$$

implying that the sum of population change is homogeneous of degree one:

$$\sum_j d(\alpha n_{jt}) / dt = \alpha \sum_j d_{njt} / dt \tag{62}$$

A formal proof of this would require a relaxation of (a) in Appendix C in proving theorem 1 there. If this speculation is correct, it would suggest that any empirically useful model of migration behavior may be used for $h_i(n_{jt},f_{ij})$, and the existence of simple dynamic equilibria would still exist. Whether such equilibria are stable, however, is another question. This would depend on the form taken by the Jacobean matrix. Similarly, in the model with limits to growth, the stability of the boundary equilibria, and the existence and stability of interior equilibria, would depend on the function chosen for $h_i(n_{jt},f_{ij}(t))$.

Introducing an exponent exceeding one on n_{it} in equation (59), however, would seem to introduce the same problems as exist if homogeneity of order greater than one is assumed in theorem 1 of Appendix C.

In this case,

$$\sum_j d(\alpha n_{jt})/dt > \alpha \sum_j dn_{jt}/dt \tag{63}$$

and even the existence of simple dynamic equilibria for the model with natural increase becomes questionable.

A second generalization of the investigation here is to examine other problems of spatio-temporal change than migration. Examples are the dynamics of commodity flows, information difussion, and individual spatial behavior. Any model of the following form:

$$ds_{jt}/dt = \sum_i dI_{ij}(t)/dt + g_i s_{jt} \tag{64}$$

$$I_{ij}(t) = e_i s_{it} s_{jt} f_{ij} / \sum_k s_{kt} f_{ik} \tag{65}$$

has the dynamic properties shown in this paper. Here $I_{ij}(t)$ is a spatial flow from i to j at time t, and s_{jt} represents a spatial stock at location j, time t. g_j and e_i are parameters.

Finally, there is no reason to restrict the indices i and j to refer only to location. For instance they could represent any states in a multi-state demographic, economic, or sociological model for which equations (64) and (65) hold. Indeed Volterra (1939; see also Rugh 1981) originally developed a quadratic model of multi-state dynamics for the simple reason that incorporating second order polynomial relationships could provide a better approximation to some general non-linear dynamic model than would a linear model. In this sense, the use of a model of the type analyzed in this paper can be regarded as a natural generalization of linear multi-state models with constant transition rates.

6. CONCLUSIONS

This paper has examined the effects of incorporating simple non-linearities in a model of migration rates on the dynamics of population change in the absence of age distributions. In particular, given a set of cities, the dynamics and predictability of a city size distribution is the subject of investigation.

In the case of zero population growth with constant mobility rates, the existence of one static equilibrium city size distribution is guaranteed. However, stability is another issue. Indeed in the two city case it was shown that the more plausible scenario of migration behavior (intra-urban migration exceeds inter-urban migration) that the equilibrium is unstable. This tends to support the conjecture (Sheppard 1982) that the concept of a stable city size distribution is called into question once non-linearities in migration behavior are allowed for.

When natural increase is introduced with constant mobility rates, it seems that a number of equilibrium city size distributions exist, in contradistinction to the expectation that generally only one will exist in the previous case. On the one hand, this fact alone should increase the probability that at least one such distribution is stable. Thus non-linearities here would certainly not preclude a stable city size distribution. However, there is also the likelihood that several city size distributions represent stable equilibria, which makes it harder to predict the outcome. Furthermore, once endogenously determined mobility rates are introduced, then it seems quite likely that few or no such equilibria exist. This issue certainly requires further investigation.

Introducing limits to growth implies again that at least one stable, static, city size distribution exists. However, this one is basically determined by the exogenously introduced growth limits, so it reveals nothing about now city size distributions are generated as a result of inter-urban and rural-urban migrations. Other static equilibria may also exist, if some cities exhibit negative rates of natural population change. These will not be fixed by the exogenously imposed limits, but once again the possibility of multiple stable city size distributions is real.

From the point of view of city size distributions, then, a single stable distribution can no longer be guaranteed, but is also not precluded, once non-linearities in migration are allowed. As to the shape of such distributions however, nothing useful can be deduced from this analysis, except a confirmation of the notion that no single shape can be expected (Sheppard 1982). The shape will depend on migration behavior and on the relative location of cities, as expressed in the variables r_i (mobility rate) and f_{ij} (the rate of migration from city i to city j, when both cities are of unit size).

From the point of view of developing population forecasts, the possibility of multiple stable equilibria once non-linearities are allowed in transition rates may have significant implications. As suggested in the paper, the outcome of population change when multiple stable equilibria exist can lead to different long run behavir as a result of relatively small and uncontrollable fluctuations. If so, then the most accurate prediction of migration, birth and death rates would not be enough to generate a general population forecast that is accurate, or even approximately accurate. This would suggest a different strategy for population forecasting. Instead of concentrating on a single long run population distribution, it would be necessary to attempt to generate the full range of possible stable distributions in a way suggested by the analyses here. The differences between these forecasts (which can be large) can then be evaluated, and an attempt made to evaluate the probability that each outcome will occur, given the current population distribution and a model of non-stationary transition rates. Perhaps the central message here is that the likely long-run outcome will depend on the initial population distribution as well as the model of transitions, in contradistinction to the linear case where only the latter information is required.

APPENDIX A: Choice of a Functional Form for Migration

Okabe (1979) has published a paper on urban population dynamics where the following functional form was used (see also Wikdar and Karmeshu, 1982):

$$dn_{it}/dt = \beta_i n_{it} + \sum_j \left(M^*_{jl}(t) - M^*_{ij}(t) \right) \tag{A.1}$$

where

$$M^*_{ij}(t) = G_i n_{it}^{\alpha} n_{jt}^{\gamma} d_{ij}^{-k} \tag{A.2}$$

Model (A.1) is consistent in the sense that total populations are accounted for:

$$\sum_i dn_{it} / dt = \sum_i \beta_i n_{it} \tag{A.3}$$

as can be seen by summing equation (A.1) over i. However, as Ledent (1978) notes, there is no reason why the sum of out-migrants should be less than or equal to the total population in i. As a result, more people may move from a city than actually live there, and populations for individual cities would be negative. Both of these results are clearly undesirable. In addition, model (A.1) assumes that the entire population is mobile. No distinction is made between stayers and migrants that stay within a city; the sum of the two is supposed to be captured by M^*_{ii}. In continuous time, however, such a presumption is clearly unrealistic. It seems more reasonable to introduce a rate of mobility, r_i, stating what fraction of the population in city i are migrants at any one time, and to separate migrants from non-migrants.

Ledent (1978) introduces this modification; and indeed constrains the values of r_i so that the sum of stayers of outmigrants equals the total population for each city. The resulting model is the one used in this paper (Ledent, 1978, p.8):

$$m_{ij}(t) = f_{ij}\, n_{it}\, n_{jt} / \sum_k n_{kt}\, f_{ik} \qquad \text{(A.4)}$$

However, Ledent chooses not to model migrations directly using this form, but rather to use equation (A.1) in combination with a second constraint equation ensuring that migrants equal the total population. The strategy is then to model the dynamics by iterating between the constraint equation and the equation for population dynamics (Ledent, 1978, p.14). This is clearly rather cumbersome.

The second approach to this problem is used by DePalma (1982):

$$dn_{it}/dt = \beta_{it}\, n_{it} + \sum_j \left[n_{jt} w_{ji} - n_{it} w_{ij} \right] \qquad \text{(A.5)}$$

subject to the constraint that $\sum_j w_{ij} = 1$. See also Weidrich and Haag (1980) and Papageorgiou (1982) for similar formulations. In this case the constraint on w_{ij} ensures the consistency missing in Okabe's model. Indeed an appropriate choice of constant G_i in (A.2) would convert (A.1) into the form of (A.5). Equation (A.5) results from the classic

Kolmogorov equations or probability theory.

However, once w_{ji} is time variant (due to a dependence on population) version (A.5) is also difficult to handle analytically if the accounting identity on migrants is to be retained. This is because the dynamics of the constraint equations must also be modeled.

In this paper, the version given by (A.4) will be directly analyzed (cf. also DePalma and Lefevre, 1982). It has the advantage that the constraint equation is incorporated directly into the migration function, ensuring that the accounting relations are continually satisfied. Despite the relatively complex nature of (A.4) as compared to (A.2), it turns out that this does not greatly hinder the search for analytical results.

APPENDIX B: Equilibrium and Stability for the AGP Model

Consider the matrix $\underset{\sim}{M}$ of equation (29):

$$\underset{\sim}{M} = (\underset{\sim}{H})(\underset{\sim}{R})^{-1}(\underset{\sim}{F}). \tag{B.1}$$

THEOREM 1. *$\underset{\sim}{M}$ has at least one eigenvalue equal to one.*

Proof. If

$$(\underset{\sim}{M} - \underset{\sim}{I})\underline{n}^{*} = 0 \tag{29}$$

then $\underset{\sim}{M}$ must have an eigenvalue of one; or equivalently $\underset{\sim}{M} - \underset{\sim}{I}$ must be singular. Define $\underset{\sim}{F}^{-1} = \underset{\sim}{G}$. Then:

$$(\underset{\sim}{M} - \underset{\sim}{I}) = [(\underset{\sim}{H})(\underset{\sim}{R})^{-1} - \underset{\sim}{G}]\,\underset{\sim}{F} \tag{B.2}$$

Therefore

$$\mathrm{Det}(\underset{\sim}{M} - \underset{\sim}{I}) = \mathrm{Det}[(\underset{\sim}{H})(\underset{\sim}{R})^{-1} - \underset{\sim}{G}].\mathrm{Det}\ \underset{\sim}{F} \tag{B.3}$$

Thus if $[(\underset{\sim}{H})(\underset{\sim}{R})^{-1} - \underset{\sim}{G}]$ is singular, so is $(\underset{\sim}{M} - \underset{\sim}{I})$, since in each case the determinant will be zero. Define $\underset{\sim}{Q} = [(\underset{\sim}{H})(\underset{\sim}{R})^{-1} - \underset{\sim}{G}]$.

Then $\underset{\sim}{Q}$ has elements:

$$q_{ii} = \sum_{k \neq i} r_k q_{ki} / r_i \qquad \text{(B.4)}$$

$$q_{ij} = g_{ij} \qquad j \neq i \qquad \text{(B.5)}$$

Now create the vector $\underline{\mu}$, with elements μ_i defined as:

$$\mu_i = \sum_{k \neq j} q_{ki}\, r_i \qquad \text{(B.6)}$$

Thus $\underline{\mu}$ is the sum of all rows of Q except for the j-th row, each row i being first weighted by $-r_i$. Substituting (B.4) and (B.5) into (B.6):

$$\mu_i = - \sum_{k \neq i} r_k g_{ki} + \sum_{m \neq i,j} r_m q_{mi} = -r_j g_{ji} \qquad (i \neq j) \qquad \text{(B.7)}$$

$$\mu_j = \sum_{k \neq j} r_k\, g_{kj} \qquad \text{(B.8)}$$

But from the definition of $\underset{\sim}{Q}$ [equations (B.4) and (B.5)], if $\underline{\mu}$ is divided by r_j then $\underline{\mu}$ becomes simply the j-th row of $\underset{\sim}{Q}$. Thus the j-th row of $\underset{\sim}{Q}$ is a linear combination of the other rows. It then follows that $\underset{\sim}{Q}$ has a zero determinant and is thus singular.

THEOREM 2. *If $\underset{\sim}{M}$ is indecomposable and primitive, then the equilibrium vector $\underline{n}^*$ is the right hand eigenvector associated with the largest eigenvalue of $\underset{\sim}{M}$. This largest eigenvalue is equal to one, and no other eigenvalue is as large. Then the internal equilibrium vector is unique.* (This generalizes a result of Ledent (1978, p.34)).

Proof. $\underset{\sim}{M}$ is non-negative (equation (B.1), and by assumption indecomposable and primitive. For such a matrix, from the Perron-Frobenius theorems, the eigenvector associated with the largest eigenvalue is strictly positive. But we know by Brounier fixed point theorem that a nonnegative equilibrium vector exists, and we know from equation (29) that it is an eigenvector of $\underset{\sim}{M}$. Finally, only one eigenvector of $\underset{\sim}{M}$ may be non-negative due to the orthogonality of eigenvectors in a primitive matrix. Therefore the equilibrium vector is given by the eigenvector associated with the largest eigenvalue of $\underset{\sim}{M}$, and that eigenvalue equals one. If $\underset{\sim}{M}$ is primitive, no other eigenvalue is as large. Therefore only one eigenvalue of $\underset{\sim}{M}$ equals one, ane the equilibrium associated with that eigenvalue is unique.

THEOREM 3. *The stability of the equilibrium point in the ZPG model depends on the eigenvalues of:*

$$\underset{\sim}{J}_1 = (\underset{\sim}{R})\underset{\sim}{X}(\underset{\sim}{n}^*) - (\underset{\sim}{n}^*)\underset{\sim}{X}'(R)(\underset{\sim}{n}^*)\underset{\sim}{X} - (\underset{\sim}{Y}) - (\underset{\sim}{R}) \qquad (B.9)$$

$\underset{\sim}{J}_1$ is the Jacobian of the ZPG model.

In the above theorem;

$$\underset{\sim}{X} = (\underset{\sim}{A})^{-1}\underset{\sim}{F}$$

$(\underset{\sim}{n}^*)$ is a diagonal matrix with the elements of $\underline{n}^*$ on the main diagonal, and $(\underset{\sim}{Y})$ is a diagonal matrix with the row sum of the i-th row of $\underset{\sim}{X}$ in the i-th diagonal entry.

Proof. Define:

$$x_{ij} = f_{ij} . A_i^{-1} \qquad (B.10)$$

Then

$$F_j(\underline{n}^*) = n_j^* \left[\sum_i r_i\, n_i^*\, x_{ij} - r_j \right] \qquad (B.11)$$

$$\partial x_{ij}/\partial n_k = -f_{ij} f_{ik} A_i^{-2} = -x_{ij} x_{ik} \qquad (B.12)$$

Now, if $k \neq j$:

$$\partial F_j/\partial n_k = n_j^* r_k x_{kj} + n_j^* \sum_i r_i n_i^* \partial x_{ij}/\partial n_k$$

$$= n_j^* \left[r_k x_{kj} - \sum_i r_i n_i^* x_{ij} x_{ik} \right] \qquad (B.13)$$

But

$$\partial F_j/\partial n_j = \sum_i r_i n_i^* x_{ij} + n_j^* \left[r_j x_{jj} - \sum_i r_i n_i^* x_{ij}^2 \right] - r_j \qquad (B.14)$$

Substituting (B.13) and (B.14) into the definition of the Jacobian matrix gives rise to equation (B.9). QED.

LEMMA 1. *The stability of equilibrium depends on the geography of migration,* $\underset{\sim}{F}$*, and the mobility rates* $(\underset{\sim}{R})$*.*

Proof. Expanding (B.9) to incorporate the definition of $\underset{\sim}{X}$:

$$\underset{\sim}{J}_1 = [I-(\underset{\sim}{n}^*)\underset{\sim}{F}'(\underset{\sim}{A})^{-1}](\underset{\sim}{R})(\underset{\sim}{n}^*)\underset{\sim}{A}^{-1}\underset{\sim}{F} - (\underset{\sim}{Y}) - (\underset{\sim}{R}) \qquad (B.15)$$

Recalling the definitions of $(\underset{\sim}{Y})$ and of $\underline{n}^*$ completes the proof.

APPENDIX C: Equilibrium and Stability for the Model with Natural Increase

Preliminaries. Consider the function $S_j(\underline{n})$ of equation (44):

$$S_j(\underline{n})_t = n_j \sum_i r_i n_i f_{ij} A_{it}^{-1} - r_j + \beta_j \qquad \text{(C.1)}$$

It can be shown that:

(a) $S(\underline{n}) \geq 0$ if $\underline{n} \geq 0$. See the equations (19)-(21).

(b) $S(\underline{n})$ is a continuous mapping from R_+^n R_+^n.

(c) $S(\underline{n})$ is homogeneous of the first order:

$$S(\alpha.\underline{n}) = \alpha.S(\underline{n})$$

where α is a scalar constant.

(d) $S(\underline{n})$ is monotonic, in a weak sense (Nikaido, 1968, p.150): i.e., if there are two population vectors $\underline{n}$, $\underline{m}$; whose $\underline{n} \geq \underline{m}$, but with $n_i = m_i = n$ for some i, then $S_i(\underline{n}) \not\geq S_i(m)$. This is because:

$$S_i(\underline{n}) \geq S_i(\underline{m}).$$

i.e.,

$$n\left[\sum_{j\neq i} r_j n_j f_{ji} A_j^{-1} + r_k\left[nf_{jj}/A_j - 1\right] + \beta_j\right] \geq n\left[\sum_{j\neq i} r_j m_j f_{ji} A_j^{-1} + r_k\left[nf_{jj}/A_j - 1\right] + \beta_j\right]$$

Suppose $n_j \geq m_j$ for all $j \neq i$. Then, within the square brackets, the second and third terms on each side of the inequality are equal, but the first term on the left hand side is greater than that on the right hand side.

(e) $S(\underline{n})$ can be said to be indecomposable according to the definition of Nikaido (1968, p.156). This requires that, for the case outlined in (d) above,

$$S_i(\underline{n}) > S_i(\underline{m})$$

for at least some elements i from the set of elements where $n_i = m_i$. This was proven for all i in this set in the analysis of the inequality of point (d).

THEOREM 1. *There exists at least one solution $k > 0$, $\underline{n}^* > 0$ to the equilibrium equation (45): At least one stable population dynamic equilibrium exists.*

Proof. Due to properties (a) and (b), at least one solution to (45) exists (Nikaido, 1968, theorem 10.1). Due to properties (a), (c), (d), and (e), all solutions $k, \underline{n}^*$ to equation (45) yield positive values for the non-linear eigenvalue and eigenvector (Nikaido, 1968, theorem 10.4). QED.

THEOREM 2. *No state of geographically varying growth rates can represent a dynamic equilibrium growth ray.*

Proof. From equation (43), and equation (44):

$$L\dot{\underline{n}} = \underline{k} \qquad (C.2)$$

$$L\dot{\underline{n}} = \left[F'(A)^{-1}(R)\underline{n}_t - \underline{r} + \underline{\beta}\right] \qquad (C.3)$$

where $L\underline{n}^{-1}$ is the vector [d log n_1/ dt ... d log n_J/dt]. Equating (C.2) and (C.3) it is obvious that this equation can only hold for an instant of time, since (C.3) depends on the time-varying vector $\underline{n}_t$, whereas (C.2) is time invariant. But for a relationship to represent dynamic equilibrium it must persist in time, in the absence of external shocks. QED.

THEOREM 3. *The stability of stable population dynamic equilibrium in the model of population change with natural increase depends on the eigenvalues of:*

$$\underset{\sim}{J}_2 = (\underset{\sim}{R}).\underset{\sim}{X} - \underset{\sim}{X}'(\underset{\sim}{R})(\underset{\sim}{n}^*)\underset{\sim}{X} \qquad (C.4)$$

Proof. The Jacobian of the system (46) is the matrix of partial derivatives $\partial \hat{F}_i(\underline{n}^*)/\partial n_j$, where

$$\hat{F}_j(\underline{n}^*) = \sum_i r_i\, n_i\, f_{ij}\, A^{-1} - r_j + \beta_j \qquad (C.5)$$

Now

$$\partial F_j(\underline{n}^*)/\partial n_k = r_k\, x_{kj} - \sum_i r_i\, n_i\, x_{ij}\, x_{ik} \qquad (C.6)$$

where $x_{ij} = f_{ij} A_i^{-1}$. Thus in matrix form:

$$J_2 = (\underset{\sim}{R})\underset{\sim}{X} - \underset{\sim}{X}'(\underset{\sim}{R})(\underset{\sim}{n}^*)\underset{\sim}{X}$$

APPENDIX D: Equilibrium and Stability for the Model with Limits to Growth

THEOREM 1. *Four types of equilibrium solutions exist:*

a) $n^*_j = q_j$ *for all j*

b) $n^*_j = 0$ *for all j*

c) n^*_j *equals zero for some cities, and equals* q_j *for all other cities*

d) An interior equilibrium exists such that, for some cities:

$$0 < n^*_j < q_j$$

Of these four, types a) to c) always exist, but d) may or may not exist.

Proof. A vector of populations $\underline{n}^*$ represents a static equilibrium solution to this model if:

$$dn_j^*/dt = n_j^* \quad q_j - n_j^* \; \sum_i \alpha_i n_i^* f_{ij} A_i^{\beta-1}(n) - \alpha_j A_j^{\beta}(n) + \gamma_j \; q_j - n_j^* \quad = 0 \qquad \text{(D.1)}$$

a) If $n_j^* = q_j$ for all j, $A_i(n)$ equals zero and it follows that equals zero

b) If $n_j^* = 0$ for all j, (D.1) equals zero

c) If $n_j^* = 0$ or q_j, for all j, $A_i(n)$ equals zero and (D.1) equals zero.

d) Equation (D.1) is zero if the expression in square brackets is zero:

$$q_j - n_j^* \; \sum_i \alpha_i n_i^* f_{ij} A_i^{\beta-1}(n) - \alpha_j A_j^{\beta}(n) + \gamma_j \; q_j - n_j^* \quad = 0 \qquad \text{(D.2)}$$

Rearranging (D.2):

$$\gamma_j^{-1} \quad q_j - n_j^* \; \sum_i \alpha_i n_i^* f_{ij} A_i^{\beta-1}(n) - \alpha_j A_j^{\beta}(n) + q_j \quad = n_j^* \quad \forall_j \qquad \text{(D.3)}$$

It can be seen that (D.3) may be interpreted as a non-linear eigenvalue equation:

$$T(\underline{n}^*) = n^* \qquad \text{(D.4)}$$

where $T(\underline{n}^*)$ is the left hand side of the equations (D.3). If a solution to this eigenvalue equation exists with an eigenvalue of one, and an associated eignevector such that $0<n_j^*<q_j$ for some j, then the fourth type of equilibrium will exist.

LEMMA 1. *The elements of the Jacobian matrix of partial derivatives are:*

$$J_{jj} = \left(q_j - 2n_j^*\right)\left[(\beta-1)\, n_j^* \left(q_j - n_j^*\right)\sum_i \alpha_i n_i^* f_{ij}^2 A_i^{\beta-2}(n) + \sum_i \alpha_i n_i^* f_{ij} A_i^{\beta-1}(n)\right.$$

$$\left. + \beta\alpha_j n_j^* f_{jj} A_j^{\beta-1}(n)\right] + \alpha_j^2\left(q_j - n_j^*\right) n_j^* f_{jj} A_j^{\beta-1}(n)$$

$$- \alpha_j A_j^{\beta}(n) + \alpha_j\left(q_j - 2n_j^*\right) \qquad \text{(D.5)}$$

$$J_{kj} = n_k^*\left(q_k - n_k^*\right)\left[(\beta-1)\sum_i \alpha_i n_i^* f_{ik} A_i^{\beta-2}(n)\left(q_j - n_j^*\right) f_{ij} + \alpha_j f_{jk} A_j^{\beta-1}(n)\right]$$

$$- \left(q_j - 2n_j^*\right)\beta\alpha_k f_{kj} A_k^{\beta-1}(n) \qquad \text{(D.6)}$$

where J_{kj}, the k,j-th element of the Jacobian matrix, is $\partial\tilde{F}_k(\underline{n}^*)/\partial n_j$.

Proof. Taking a Taylor expansion around an equilibrium point, $\underline{n}^*$, and retaining just linear terms:

$$d\hat{n}_{jt}/dt = \sum_k \left(\partial\tilde{F}(n)/\partial n_k\right)\hat{n}_{kt} \qquad \text{(D.7)}$$

where $\partial\hat{F}_j(\underline{n}^*)/\partial n_k$ are the elements of the Jacobian matrix, and $\hat{n}_{kt}$ is $n_{jt} - n_j^*$, and:

$$\tilde{F}_j(\underline{n}^*) = n_j^*\left[\left(q_j - n_j^*\right)\sum_i \alpha_i n_i^* f_{ij} A_i^{\beta-1}(n) - \alpha_j A_j^{\beta}(n) + \gamma_j\left(q_j - n_j^*\right)\right] \qquad \text{(D.8)}$$

Therefore

$$\partial\tilde{F}_j(\underline{n}^*)/\partial n_j = (\beta-1)n_j^*\left(q_j-n_j^*\right)\sum_i \alpha_i n_i^* f_{ij} A_i^{\beta-2}(n)\left(q_j-2n_j^*\right)f_{ij}$$

$$+ \left(q_j-2n_j^*\right)\sum_i \alpha_i n_i^* f_{ij} A_i^{\beta-1}(n)$$

$$+ n_j^*\alpha_j f_{jj} A_j^{\beta-1}(n)\left[\beta\left(q_j-2n_j^*\right) + \alpha_j\left(q_j-n_j^*\right)\right]$$

$$- \alpha_j A_j^{\beta}(n) - \gamma_j \qquad \text{(D.9)}$$

and

$$\partial\tilde{F}_k(\underline{n}^*)/\partial n_j = (\beta-1)n_k^*\left(q_k-n_k^*\right)\sum_i \alpha_i n_i^* f_{ik} A_i^{\beta-2}(n)\left(q_j-n_j^*\right)f_{ij}$$

$$+ n_k^*\left(q_k-n_k^*\right)\alpha_j f_{jk} A_j^{\beta-1}(n)$$

$$- \beta\alpha_k A_k^{\beta-1}(n)\left(q_j-2n_j^*\right)f_{kj} \qquad \text{(D.10)}$$

Q.E.D.

COROLLARY 1. *A necessary and sufficient condition for the equilibrium point* $n_j^* = q_j$ *for all* j *to be locally stable is that* $\gamma_j > 0$ *for all* j.

Proof. If $n_j^* = q_j$ for all j, $A_i(n^*)$ equals zero. Substituting into (D.5) and (D.6):

$$J_{jj} = \gamma_j \; q_j - 2^*_j \;\; = -q_j \; \gamma_j \tag{D.11}$$

$$J_{jk} = 0 \tag{D.12}$$

Thus, if $\gamma_j > 0$, the Jacobian is a diagonal matrix with negative elements on the main diagonal. Therefore the eigenvalues are all negative and the equilibrium point is stable.

COROLLARY 2. *If $\gamma_j > 0$, for all j, then any equilibrium point of type (b) or (c) from Theorem 1 is unstable.*

Proof. If n^*_j equals 0 for some j, and perhaps also equal to q_k for some other cities k, then $A_i(n^*)$ is zero. Thus for those cities where $n^*_j = 0$:

$$J_{jj} = q_j \gamma_j \tag{D.13}$$

whereas elements J_{jk} for $j \neq k$ are still zero. Thus if γ_j is positive, some diagonal elements and therefore some eigenvalues of the Jacobian, are positive. Therefore these equilibria are unstable.

COROLLARY 3. *If $\gamma_j > 0$ for all j, then no equilibrium point exists where $0 < n^*_j < q_j$ for some j, i.e., equilibrium of type (d) in theorem 1 is not possible.*

Proof. In equilibrium total inmigration equals total outmigration:

$$\sum_{i,j} M_{ij} = \sum_{i,j} M_{ji}$$

But if $0 < n_j^* < q_j$ for some j;

$$d\ N_t/dt = \sum_j \gamma_j\ n_j^*\ q_j - n_j^* \quad > 0$$

where N_t is the total population in the system. Thus the population is increasing, and stable equilibrium is impossible.

COROLLARY 4. *If γ_k is negative for cities $\{k{:}k \in K\}$, where K is a subset of the complete set of cities $\{J\}$; $\{K\} \in \{J\}$, then only one of the equilibria of types (a), (b), and (c) of theorem 1 are stable. The one which is stable is that one where $n_k^* = 0$ for all $k \in K$; and $n_j^* = q_j$ for all other cities.*

Proof. If n_j^* is zero, then J_{jj} is negative only when γ_j is negative (equation (D.13)). If n_j^* is equal to q_j, J_{jj} is negative only when γ_j is positive (equation (D.11)). J_{jk} is always zero for $j \neq k$ and n_j^* equal to q_j or zero.

Q.E.D.

COROLLARY 5. *If the cities of the system are split into three disjoint sets $\{J\}$, $\{K\}$ and $\{L\}$ such that*

$$\gamma_j > 0 \qquad \forall_{j \in J}$$

$$\gamma_k < 0 \qquad \forall_{K \in K}$$

$$\gamma_l = 0 \qquad \forall_{l \in L}$$

then the equilibria given by: $n_j^* = q_j$ for all $j \in J$, $n_k^* = 0$ for all $j \in J$ and n_l^* equal to zero or q_l for all $l \in L$ may possibly be stable. All other equilibria are unstable.

Proof. By corollary 4, all other equilibria are unstable. For the equilibria defined in corollary 5, J_{jj} and J_{kk} are negative whereas J_{11} is zero. Also the off-diagonal elements in the Jacobian are zero. Thus the Jacobian has some negative and some zero eigenvalues. When zero eigenvalues are present then the values eigenvalues alone are not sufficient to determine stability. All that can be said is that necessary conditions for stability exist (Hirsch and Smale, 1974, p.187).

LEMMA 1. *If one city (j) has a population that approximates its limiting population, some while other cities have not yet approached this limit, then the population in city j will decline;* $dn_{jt}/dt < 0$.

Proof. Consider

$$dn_{jt}/dt = n_{jt}\left[\sum_i \alpha_i n_{it}\left(q_j - n_{jt}\right) f_{ij} A_i^{\beta-1}(n) - \alpha_j A_j^{\beta}(n) + \gamma_j\left(q_j - n_{jt}\right)\right] \qquad \text{(D.1)}$$

obtained by substituting (56) into (18). Suppose that:

$n_j \simeq q_j$

$n_k << q_k$ for some $k \neq j$.

Then in (D.1), the first and last terms in the square brackets are zero, whereas the second term is approximately $\alpha_j \left[\sum_{k \neq j} \left(q_k - n_k\right) f_{jk}\right]^{\beta}$. Thus $dn_{jt}/dt < 0$, since $q_k - n_k$ is non-zero for some k.

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