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DYNAMICS IN SURVIVAL ANALYSIS: CONDITIONAL GAUSSIAN PROPERTY VERSUS CAMERON-MARTIN FORMULA

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ABSTRACT

This paper describes the stochastic process model for mortality rates of the population. The key question is the relationship between conditional and unconditional survival functions. The Cameron and Martin solution to the problem is compared to the solution based on the Conditional Gaussian Approach. The advantages of the Gaussian approach are discussed. The proof of the main formula for averaging uses the martingale specification of the random hazard rate.

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A.I. Yashin

1. Introduction

The well-known Cameron and Martin formula [1, 2, 3] gives a way of calculating the mathematical expectation of the exponent which is the functional of a Wiener process. More precisely, let $(\Omega, H, \mathbf{P})$ be the basic probability space, $\mathbf{H} = (H_u)_{u \geq 0}$ be the nondecreasing right-continuous family of σ-algebras, and H_0 is completed by the events of $\mathbf{P}$-probability zero from $H = H_\infty$. Denote by W_u n-dimensional $\mathbf{H}$-adapted Wiener process and $Q(u)$ a symmetric non-negative definite matrix whose elements $q_{ij}(u), i,j = 1,2,\ldots,n$ satisfy for some t the condition

$$\int_0^t \sum_{i,j=1}^n |q_{i,j}(u)| \, du < \infty. \tag{1}$$

The following result is known as a Cameron-Martin formula.

Theorem 1. *Let (1) be true. Then*

$$\mathbf{E}\exp[-\int_0^t (W_u, Q(u)W_u)du] = \exp[\frac{1}{2}\int_0^t Sp\,\Gamma(u)du]. \tag{2}$$

where $(W_u, Q(u)W_u)$ *is the scalar product equal to* $W_u^*, Q(u)W_u$, *and* $\Gamma(u)$ *is a symmetric nonpositive definite matrix, being a unique solution of the Ricatti matrix equation*

$$\frac{d\Gamma(u)}{du} = 2Q(u) - \Gamma^2(u); \tag{3}$$

$\Gamma(t) = 0$ *is a zero matrix.*

The proof of this formula given in [3] uses the property of likelihood ratio for diffusion type processes. The idea of using this approach comes from Novikov [4]. Using this idea Myers in [5] developed this approach and found the formula for averaging the exponent when, instead of a Wiener process, there is a process satisfying a linear stochastic differential equation driven by a Wiener process. His result may be formulated as follows.

Theorem 2. *Let* $Y(t)$ *be an* m*-dimensional diffusion process of the form*

$$dY(t) = a(t)Y(t)dt + b(t)dW_t,$$

with deterministic initial condition $Y(0)$. *Assume that matrix* $Q(u)$ *has the properties described above. Then the next formula is true:*

$$\mathbf{E}\exp[-\int_0^t Y^*(u)Q(u)Y(u)du] = \tag{4}$$

$$\exp[Y^*(0)\Gamma(0)Y(0) + Sp\int_0^t b(u)b^*(u)\Gamma(u)du]$$

where $\Gamma(u)$ *is the solution of matrix Ricatti equation*

$$\frac{d\Gamma(u)}{du} = Q(u) - (\Gamma(u) + \Gamma^*(u))a(u) - \frac{1}{2}(\Gamma(u) + \Gamma^*(u))b(u)b^*(u)(\Gamma(u) + \Gamma^*(u)), \tag{5}$$

with the terminal condition $\Gamma(t) = 0$.

These results have direct implementation to survival analysis: any exponent on the left-hand sides of (2) and (4) can be considered as a conditional survival function in some life cycle problem [5, 6, 7]. The stochastic process in the exponent is interpreted in terms of spontaneously changing factors that influence mortality or failure rate.

Such interpretation was used in some biomedical models. The quadratic dependence of risk from some risk factors was confirmed by the results of numerous physiological and medical studies [6]. The results are also applicable to the reliability analysis.

The way of proving the Cameron-Martin formula and its generalizations given in [1, 2, 3] does not use an interpretation and unfortunately does not provide any physical or demogaphical sense to the variables $\Gamma(u)$ that appear on the right-hand side of the formulas (2) and (4). Moreover, the form of the boundary conditions for equation (3) and (5) on the right-hand side complicate the computing of the Cameron-Martin formula when one needs to calculate it on-line for many time moments t. These difficulties grow when there are some additional on-line observations correlated with the influential factors.

Fortunately there is the straightforward method that allows avoidance of these complications. The approach uses the innovative transformations random intensities or compensators of a point process. Usage of this "martingale" techniques allows to get a more general formula for averaging exponents which might be a more complex functional of a random process from a wider class.

If the functional is of a quadratic form one can get another constructive way of averaging the exponent using the conditional Gaussian property. The goal of this paper is to illustrate this approach.

2. Results Formulation

We shall start from the following general statement.

Theorem 3. *Let* $Y(u)$ *be an arbitrary* $\mathbf{H}$*-adapted random process and* $\lambda(Y,u)$ *is some non-negative* $\mathbf{H}^y$*-adaptive function such that for some* $t \geq 0$

$$\mathbf{E} \int_0^t \lambda(Y,u)du < \infty \tag{6}$$

Then

$$\mathbf{E} \exp[-\int_0^t \lambda(Y,u)du] = \exp[-\int_0^t \mathbf{E}\,[\lambda(Y,u) \mid \mathbf{T} > u]du] \tag{7}$$

where $\mathbf{T}$ *is the stopping time associated with the process* $Y(u)$ *as follows:*

$$\mathbf{P}\,(\,\mathbf{T} > t \;\mid\; H_t^y) = \exp[-\int_0^t \lambda(Y,u)du\,] \tag{8}$$

and $H_t^y = \bigcap_{u>t} \sigma\{Y(v), v \leq u\,\}$ *is* σ*-algebra generated by the history of the process* $Y(u)$ *up to time* t , $\mathbf{H}^y = (H_t^y)_{t \geq 0}$.

The proof of this statement based on the idea of "innovation", widely used in martingale approach to filtration and stochastic control problems [3, 8, 9] is given in the Appendix.

Another form of this idea appeared and was explored in the demographical studies of population heterogeneity dynamics [7, 10, 11]. Differences among the individuals or units in these studies were described in terms of a random heterogeneity factor called "frailty". This factor is responsible for individuals' susceptibility to death and can change over time in accordance with the changes of some external variables, influencing the individuals' chances to die (or to have failure for some unit if one deals with the reliability studies).

When the influence of the external factors on the failure rate may be represented in terms of a function which is a quadratic form of the diffusion type Gaussian process, the result of Theorem 3 may be developed as follows:

Theorem 4. *Let the m-dimensional* H-*adapted process* $Y(u)$ *satisfy the linear stochastic differential equation*

$$dY(t) = [a_0(t) + a_1(t)Y(u)]dt + b(t)dW_t, \quad Y(0) = Y_0,$$

where Y_0 *is the Gaussian random variable with mean* m_0 *and variance* γ_0. *Denote by* $Q(u)$ *a symmetric non-negative definite matrix whose elements satisfy condition (1). Then the next formula is true*

$$\mathbf{E}\exp[-\int_0^t (Y^*(u)Q(u)Y(u))du] = \exp[-\int_0^t (m_u^* Q(u)m_u \tag{9}$$

$$+ Sp(Q(u)\gamma_u))du].$$

The processes m_u *and* γ_u *are the solutions of the following ordinary differential equations:*

$$\frac{dm_t}{dt} = a_0(t) + a_1(t)m_t - 2\gamma_t Q(t)m_t \tag{10}$$

$$\frac{d\gamma_t}{dt} = a_1(t)\gamma_t + \gamma_t a_1^*(t) + b(t)b^*(t) - 2\gamma_t Q(t)\gamma_t \qquad (11)$$

with the initial conditions m_0 *and* γ_0, *respectively.*

The proof of this theorem is based on the Gaussian property of the conditional distribution function $\mathbf{P}(Y(t) \le x \mid \mathbf{T} > t)$. This situation recalls the well-known generalization of the Kalman filter scheme [3, 12, 13, 14] (see Appendix).

Note that a similar approach to the averaging of the survival function was studied in [6] under the assumption that the conditional Gaussian property take place. The mortality rate in this paper was assumed to be influenced by the values of some randomly evolving physiological factors such as blood pressure or serum cholesterol level.

We will illustrate the results and ideas on several examples.

3. Examples

3.1. Failure Rate as a Function of a Random Variable

Let $(\Omega, H, \mathbf{P})$ be the basic probability space, $Y(\omega)$ and $T(\omega)$ be two random variables, such that $T(\omega) > 0$ with a probability one and has a continuous distribution function. $Y(\omega)$ and $T(\omega)$ will be interpreted as external environmental factor and termination (death) time, respectively.

Assume that the external factor influences failure rate by means of random variable $\mathbf{Z} = Y^2$. Let $\sigma(\mathbf{Z})$ be a σ-algebra in Ω generated by the random variable $\mathbf{Z}$. Denote by $F(t, \mathbf{Z}) = \mathbf{P}(\mathbf{T} \le t \mid \sigma(\mathbf{Z}))$ the $\sigma(\mathbf{Z})$-conditional distribution function of termination time $\mathbf{T}$. Assume that $F(t, \mathbf{Z})$ has the form

$$F(t,\mathbf{Z}) = 1 - e^{-\mathbf{Z}\int_0^t \lambda(u)du} \tag{12}$$

where $\lambda(t)$, $t \geq 0$, is deterministic function of t that may be interpreted as the age-specific mortality rate for an average (standard) individual [11].

Let $\bar{F}(t)$ denote the unconditional distribution function for $T(\omega)$,

$$\bar{F}(t) = \mathbf{P}(\mathbf{T} < t)$$

and $\bar{\lambda}(t)$ is determined by the equality

$$\bar{\lambda}(t) = \frac{\frac{d\bar{F}(t)}{dt}}{1-\bar{F}(t)}$$

This function was called "observed" mortality rate in [11] since it represents mortality approximated by empirical death rates which are evaluated without taking population heterogeneity into account. It can be easily shown [11] that

$$\bar{\lambda}(t) = \bar{\mathbf{Z}}(t)\,\lambda(t)$$

where

$$\bar{\mathbf{Z}}(t) = \mathbf{E}(\mathbf{Z} \mid \mathbf{T} > t)$$

is the conditional mathematical expectation of $\mathbf{Z}$ given the event $\{\mathbf{T} > t\}$.

The form of the $\bar{\lambda}(t)$ as a function of time is determined by the the conditional distribution of frailty $\mathbf{Z}$ and $\lambda(t)$. It turns out that if the frailty $\mathbf{Z}$ is generated by Gaussian random variable Y, the analytical form for $\bar{\mathbf{Z}}(t)$ might be easily found. Moreover, this conditional distribution of Y is Gaussian, as shown in the following theorem.

Proposition 1. *Let* $Z = Y^2$, *where* Y *is a Gaussian random variable with mean* a *and variance* σ^2. *Then the conditional distribution of* Y *given the event* $\{ T > t \}$ *is also Gaussian, with a mean* m_t *and variance* γ_t *that satisfy the equations*

$$\frac{dm_t}{dt} = -2\,\lambda(t)\, m_t\, \gamma_t, \quad m_0 = a \tag{13}$$

$$\frac{d\gamma_t}{dt} = -2\,\lambda(t)\, \gamma_t^2, \quad \gamma_0 = \sigma^2 \tag{14}$$

The result of this statement follows from Theorem 4. It can also be proved independently using Bayes' rule. According to this rule the conditional density of random variable Y may be represented in the form:

$$g(x,t) = \frac{h(x)\, P(T > t \mid Y = x)}{P(T > t)} \tag{15}$$

where (from the definitions of Z and T)

$$h(x) = \frac{1}{(2\pi\gamma_0)^{1/2}}\, e^{-\frac{(x-m_0)^2}{2\gamma_0}}$$

$$P(T > t \mid x) = e^{-x^2 \int_0^t \lambda(u)du}$$

and

$$g(x,t) = \frac{d}{dx} P(Y \leq x \mid T > t)$$

Substituting the formulas for $h(x)$ and $P(T > t)$ into the equation for $P(x \mid T > t)$ leads to

$$g(x,t) = f(t)\, e^{-\frac{[x(2\sigma^2\Lambda(t)+1) - a]^2}{2\sigma^2(2\sigma^2\Lambda(t)+1)}}$$

where

$$\Lambda(t) = \int_0^t \lambda(u)\mathrm{d}u$$

and $f(t)$ is some function that does not depend on x and acts as a normalizing factor. It is evident that this form of the conditional density $g(x,t)$ corresponds to a Gaussian distribution with $\frac{a}{2\sigma^2\Lambda(t)+1}$ and $\frac{\sigma^2}{2\sigma^2\Lambda(t)+1}$ as mean and variance, respectively. Substituting these values for m_t and γ_t, it is not difficult to check that they satisfy the equations given in the theorem.

Remark. Note that results of this theorem may be represented by the following averaging formula

$$\mathbf{E}\exp[-Y^2\int_0^t \lambda(u)du] = \exp[-\int_0^t (m_u^2 + \gamma_u)\lambda(u)du] \tag{16}$$

which has the form similar to the Cameron-Martin result.

3.2. Mortality in a Structurized Population

Assume that some population may be represented as a collection of several groups of individuals (men and women, ethnic groups, etc.). Introduce a random variable $\mathbf{Z}$ taking a finite number of possible values $(1,2,\ldots,K)$ with *a priori* probabilities $p_1,p_2,\ldots,p_K$. Let the age-specific mortality rate of the average individual depend on the value of the random variable $\mathbf{Z}$; - this will be associated with a particular social group. Assume that the survival probability of a person from group j with a history H_t^Y of environmental or physiological characteristics up to time t may be written as follows:

$$\mathbf{P}\,(\,\mathbf{T} > t \mid H_t^Y, \{\mathbf{Z} = j\}\,) = e^{-\int_0^t Y^2(u)\,\lambda(j,u)\,du}$$

where $Y(t)$ is the process described in formulation of the Theorem 4.

If the observer takes into account the differences between the people belonging to different social groups he should produce K different patterns of age-specific mortality rates $\bar{\lambda}(i,t)$, $i = \overline{1,K}$.

Proposition 2. *The mortality rates corresponding to the conditional survival probabilities*

$$\mathbf{P}\,(\,\mathbf{T} > t \mid \mathbf{Z} = i\,) = e^{-\int_0^t \bar{\lambda}(i,u)\,du} \quad , \quad i = \overline{1,K} \quad .$$

are given by the formulas

$$\bar{\lambda}(i,t) = \lambda(i,t)\,(\,m_t^2(i) + \gamma_t(i)\,), \quad i = \overline{1,K}$$

where K different estimations $m_t(i), \gamma_t(i)$ are the solutions of the following equations:

$$\frac{dm_t(i)}{dt} = a_0(t) + a_1(t)\,m_t(i) - 2\,m_t(i)\,\gamma_t(i)\,\lambda(i,t)\,, \quad m_0(i), \quad i = \overline{1,K}$$

$$\frac{d\,\gamma_t(i)}{dt} = 2\,a_1(t)\,\gamma_t(i) + b^2(\,t\,) - 2\,\lambda(i,t)\,\gamma_t^2(i)\,, \quad \gamma_0(i), \quad i = \overline{1,K} \quad .$$

If evolution of the environmental or physiological factors also depends on random variable $\mathbf{Z}$, there are K different processes influencing the mortality rates in each of the K population's group respectively.

$$dY_i(t) = a_0(i,t) + a_1(i,t)\,Y_i(t)\,dt + b\,(i,t)\,d\,W_{i,t}\,, \; Y_i(0) = Y_{i,0}$$

where the $Y_{i,0}$ are Gaussian random variables with means $m_0(i)$ and variances $\gamma_0(i)$, and the $W_{i,t}$ are independent **H**-adapted Wiener processes. The formula for $\bar{\lambda}(i,t)$ will be the same as before, but the equations for $m_t(i)$ and $\gamma_t(i)$ will contain different parameters $a_0(i,t), a_1(i,t), b(i,t)$:

$$\frac{dm_t(i)}{dt} = a_0(i,t) + a_1(i,t)\, m_t(i) - 2\, m_t(i)\, \gamma_t(i)\, \lambda(i,t)\,, \quad m_0(i), \quad i = \overline{1,K}$$

$$\frac{d\,\gamma_t(i)}{dt} = 2\, a_1(i,t)\, \gamma_t(i) + b^2(i,t) - 2\, \lambda(i,t)\, \gamma_t^2(i)\,, \quad \gamma_0(i), \quad i = \overline{1,K} \quad .$$

If the observer does not differentiate between people from different groups the observed age-specific mortality rate $\bar{\lambda}(t)$ will depend on the proportion $\pi_i(t)$, $i = \overline{1,K}$, of individuals in the different groups. These proportions coincide with the conditional probabilities of the events $\{ \mathbf{Z} = i \}$, $i = \overline{1,K}$, given $\{ \mathbf{T} > t \}$, and can be shown to satisfy the following equations

$$\pi_j(t) = \pi_j(0) - \int_0^t \pi_j(u) \, (\bar{\lambda}(j,u) - \sum_{i=1}^{i=K} \bar{\lambda}(i,u)\, \pi_i(u) \,)\, du \quad .$$

where $\pi_j(0) = p_j$. In this case $\bar{\lambda}(t)$ may be represented as follows:

$$\bar{\lambda}(t) = \sum_{i=1}^{i=K} \bar{\lambda}(i,u) \pi_i(t) \qquad (17)$$

3.3. Evaluation of Mortality Rate in Multistate Demography

Assume that $\mathbf{Z}_t$ is a finite state continuous time Markov process with vector initial probabilities $p_1, \cdots p_K$ and intensity matrix

$$\mathbf{R}(t) = ||r_{i,j}(t)||, \quad i,j = \overline{1,K} \quad t \geq 0.$$

with bounded elements for any $t \geq 0$. The process $\mathbf{Z}_t$ can be interpreted as a formal description of the individuals' transitions from one state to another in the multistate population model. Denote $H_t^z = \sigma\{Z_u, u \le t\}$. The following statement is the direct corollary of Theorem 3.

Proposition 3. *Let the process* Z_t *be associated with the death time* **T** *as follows:*

$$\mathbf{P}(\mathbf{T} > t \mid H_t^z) = \exp[-\int_0^t \lambda(Z_u, u)du].$$

Then the next formula is true:

$$\mathbf{E}\exp[-\int_0^t \lambda(Z_u, u)du] = \exp[-\int_0^t \sum_{i=1}^{i=K} \lambda(i,u)\pi_i(u)du]$$

where the $\pi_i(t)$ *are the solutions of the following system of the ordinary differential equations:*

$$\frac{d\pi_j(t)}{dt} = \sum_{i=1}^{i=K} \pi_i(t) r_{i,j}(t) - \pi_j(t)\,[\,\bar{\lambda}(j,t) - \sum_{i=1}^{i=K} \bar{\lambda}(i,t)\,\pi_i(t)\,],$$

with $\pi_j(0) = p_j$.

The variables $\pi_j(t)$, $j = \overline{1,K}$ can be interpreted as the proportions of the individuals in different groups at time t.

APPENDIX

3.4. Proof of Theorem 3

Let $\mathbf{H} = (H_t)_{t\geq 0}$ be a nondecreasing right-continuous family of σ-algebras in Ω and let H_0 be completed by sets of $\mathbf{P}$-zero measure from $H = H_\infty$.

Denote by $Y(t)$, $t \geq 0$, the continuous time $\mathbf{H}$-adapted process defined on $(\Omega, H, \mathbf{P})$ that describes the evolution of these factors. Denote by $\mathbf{H}^y$ the family of σ-algebras in Ω generated by the values of the random process $Y(u)$:

$$\mathbf{H}^y = (H_t^y)_{t\geq 0}, \quad H_t^y = \bigcap_{u>t} \sigma\{Y(v), \quad v \leq u\} \quad .$$

Assume that H_t^y-conditional distribution function of death time $\mathbf{T}$ may be represented by the formula

$$\mathbf{P}(\mathbf{T} \leq t \mid H_t^y) = 1 - e^{-\int_0^t \lambda(Y,u)\,du} \tag{A1}$$

where $\lambda(Y,u)$ was introduced before.

Using the terminology of the martingale theory [3, 15] and the recent compensator representation results [16] one can say that the process

$$A(t) = \int_0^{t\Delta T} \lambda(Y,u)\, du$$

is an H^{xy}-predictable compensator of the life cycle process

$$X_t = 1(T \le t), \quad t \ge 0 \quad ,$$

where $H^{xy} = (H_t^{xy})_{t\ge 0}$, $H_t^{xy} = H_t^x \vee H_t^y$, $H_t^x = \sigma\{X_u, u \le t\}$. This means that the process

$$M_t = 1(T \le t) - A(t), \quad t \ge 0$$

is an H^{xy}-adapted martingale. If the termination time T is viewed as the time of death, the process $\lambda(Y,u)$, $0 \le u \le t$, may be regarded as the age-specific mortality rate for an individual with history $Y_0^t = \{Y(u),\ 0 \le u \le t\}$.

Let $H^x = (H_t^x)_{t\ge 0}$. Denote by $\bar{A}(t)$ the H^x-predictable compensator of the life cycle process X_t. According to the definition of the compensator and the compensator representation results [3, 17] one can write

$$\bar{A}(t) = -\int_0^{t\Delta T} \frac{dP(T \ge u)}{P(T \ge u)} = \int_0^{t\Delta T} \bar{\lambda}(u)du$$

The formula for $\bar{\lambda}(u)$ is the result of the following statement:

Lemma 1. *Let $Y(t)$ and T are related as it is described by the formula (17). Then*

$$\bar{\lambda}(t) = E[\lambda(Y,t) \mid T \ge t]$$

Proof. Note that the process

$$\overline{M}_t = \mathbf{E}\,(\,M_t \mid H^z_t),\quad t \geq 0$$

is $\mathbf{H}^z$- adapted martingale that can be represented in the form

$$\overline{M}_t = \mathbf{I}\,(\,\mathbf{T} \leq t\,) - \int_0^{t \wedge \mathbf{T}} \mathbf{E}\,[\,\lambda(Y,u) \mid H^z_u]du + N_t$$

where

$$N_t = \mathbf{E}\,[\int_0^{t \wedge \mathbf{T}} \lambda(Y,u)du \mid H^z_t] - \int_0^{t \wedge \mathbf{T}} \mathbf{E}\,[\,\lambda(Y,u) \mid H^z_u]du$$

The process N_t seems to be $\mathbf{H}^z$-predictable martingale. To prove that, it is enough to check the martingale property

$$\mathbf{E}(N_t \mid H^z_v) = N_v$$

that easily follows from the equality

$$\mathbf{E}\,[\int_{v \wedge \mathbf{T}}^{t \wedge \mathbf{T}} \lambda(Y,u)du \mid H^z_v] - \mathbf{E}\,[\int_{v \wedge \mathbf{T}}^{t \wedge \mathbf{T}} \mathbf{E}\,[\,\lambda(Y,u) \mid H^z_u]du \mid H^z_v]$$

and the process

$$\mathbf{I}\,(\,\mathbf{T} \leq t\,) - \int_0^{t \wedge \mathbf{T}} \mathbf{E}\,[\,\lambda(Y,u) \mid H^z_u]du$$

is $\mathbf{H}^z$-adapted martingale. Note further that σ-algebra H^z_u has the atom $\{\mathbf{T} > u\}$ [18] and consequently

$$\int_0^{t \wedge \mathbf{T}} \mathbf{E}\,[\,\lambda(Y,u) \mid H^z_u]du = \int_0^{t \wedge \mathbf{T}} \mathbf{E}\,[\lambda(Y,u) \mid \mathbf{T} > u\,]du$$

The non-decreasing process on the right-hand side of this equality is $\mathbf{H}^z$-adapted and continuous and, consequently, it is $\mathbf{H}^z$-predictable. The uniqueness of $\mathbf{H}^z$-predictable compensator implies the formula

$$\bar{A}(t) = \int_0^{t \wedge \mathbf{T}} \mathbf{E}\,[\,\lambda(Y,u) \mid \mathbf{T} > u\,]du$$

and consequently

$$\bar{\lambda}(t) = \mathbf{E}[\ \lambda(Y,t)\ |\ \mathbf{T} > t\ ]$$

In particular cases when $\lambda(Y,u) = Y^*(u)Q(u)Y(u)$ where $Q(u)$ is the matrix with property (1), formula for $\bar{\lambda}(t)$ will be

$$\bar{\lambda}(t) = m_t^* Q(t) m_t + Sp\,(Q(t)\gamma_t),$$

where $m_t = \mathbf{E}[Y(t)|\ \mathbf{T} > t\ ]$ and $\gamma_t = \mathbf{E}[(Y(t) - m_t)(Y(t) - m_t)^*\ |\ \mathbf{T} > t\ ]$.

3.5. Proof of Theorem 4

Introduce the conditional characteristic function $f_t(\alpha)$ defined as follows:

$$f_t(\alpha) = \mathbf{E}\,(\ e^{i\alpha^* Y(t)}|\ \mathbf{T} > t\ ),\quad t \geq 0.$$

According to Bayes' rule, this can be approximated by

$$f_t(\alpha) = \mathbf{E}'\,(\ e^{i\alpha^* Y(t)}\ \varphi(t)\ )$$

where

$$\varphi(t) = e^{-\int_0^t (Y^*(u)Q(u)Y(u) - \overline{Y^*(u)Q(u)Y(u)})du}$$

and $\mathbf{E}'$ denotes the mathematical expectation with respect to marginal probability measure corresponding to the trajectories of the Wiener process W_u, $0 \leq u \leq t$, and

$$\overline{Y^*(u)Q(u)Y(u)} = \mathbf{E}(\ Y^*(u)Q(u)Y(u)\ |\ \mathbf{T} > u\ ).$$

Using Ito's differential rule one can represent the product $e^{i\alpha^* Y(t)}\varphi(t)$ as follows:

$$e^{i\alpha^* Y(t)}\varphi(t) = e^{i\alpha^* Y(0)} + \int_0^t i\alpha^* \, e^{i\alpha^* Y(u)}\, \varphi(u)\, (a_0(u) + a_1(u)\, Y(u))du$$
$$+ \int_0^t i\alpha^* \, e^{i\alpha^* Y(u)}\varphi(u) b(u) dW_u - \frac{1}{2}\int_0^t e^{i\alpha^* Y(u)}\varphi(u)\alpha^* b(u) b^*(u)\alpha du +$$
$$+ \int_0^t e^{i\alpha^* Y(u)}\, \varphi(u)[Y^*(u)Q(u)Y(u) - \overline{Y^*(u)Q(u)Y(u)}]du$$

Taking the mathematical expectation $\mathbf{E}'$ of both sides of this equality leads to

$$f_t(\alpha) = f_0(\alpha) + i\alpha^* \int_0^t a_0(u)\, f_u(\alpha)\, du + i\alpha \int_0^t a_1(u)\, \mathbf{E}'\, [\, e^{i\alpha^* Y(u)}\varphi(u)\, Y(u)\,]\, du$$
$$+ \frac{1}{2}\int_0^t \alpha^* b(u) b^*(u)\alpha\, f_u(\alpha)\, du - \int_0^t \mathbf{E}'\, [\, e^{i\alpha^* Y(u)}\, \varphi(u)\, Y^*(u)Q(u)Y(u)\,]\, du$$
$$+ \int_0^t f_u(\alpha)\, \overline{Y^*(u)Q(u)Y(u)}\, du$$

Notice that $f_0(\alpha)$ has the form:

$$f_0(\alpha) = e^{i\alpha^* m_0 - \frac{1}{2}\alpha^* \gamma_0 \alpha}$$

This particular form and the equation for $f_t(\alpha)$ generate the idea that one should search for an $f_t(\alpha)$ in the same form:

$$f_t(\alpha) = e^{i\alpha^* m_t - \frac{1}{2}\alpha^* \gamma_t \alpha} \tag{A.2}$$

where m_t and γ_t satisfy some ordinary differential equations

$$\frac{dm_t}{dt} = g(t), \quad m_0 \tag{A.3}$$

$$\frac{d\gamma_t}{dt} = G(t), \quad \gamma_0 \quad . \tag{A.4}$$

(We assume that the equations for m_t and γ_t have unique solutions.)

The vector function $g(t)$ and matrix $G(t)$ can be found from the equation for $f_t(\alpha)$. In order to do this note that the following equalities hold:

$$f'_{\alpha t} = \mathbf{E}\,(\,i e^{i\alpha^{*}Y(t)}\,\varphi(t)\;Y(t))$$

$$f''_{\alpha\alpha t} = -\mathbf{E}\,(\,e^{i\alpha^{*}Y(t)}\,\varphi(t)\;Y(t)Y(t)^{*}(t))$$

where $f'_{\alpha t}$ and $f''_{\alpha\alpha t}$ denote the vector of the first order derivatives and the matrix of the second order derivatives respectively, of the function $f_t(\alpha)$ with respect to α.

Applying these formulas to the equation for $f_t(\alpha)$ we obtain (omitting the dependence of $f_t(\alpha)$ on α for simplicity):

$$f_t = f_0 + i\alpha^{*}\int_0^t a_0(u)\,f_u\,du + \alpha^{*}\int_0^t f'_{\alpha u}a_1(u)\,du$$
$$-\frac{1}{2}\int_0^t f_u\,\alpha^{*}b(u)b^{*}(u)\alpha\,du + \int_0^t Sp\,(Q(u)f''_{\alpha\alpha u})\,du + \int_0^t [m_u^{*}Q(u)m_u + Sp\,(Q(u)\gamma_u)]f_u\,du$$

Derivatives $f'_{\alpha t}$ and $f''_{\alpha\alpha t}$ may be calculated from equation (A.2):

$$f'_{\alpha t} = f_t(im_t - \frac{1}{2}\alpha^{*}\gamma_t - \frac{1}{2}\gamma_t\alpha)$$

$$f''_{\alpha\alpha t} = f_t(im_t - \frac{1}{2}\alpha^{*}\gamma_t - \frac{1}{2}\gamma_t\alpha)(im_t - \frac{1}{2}\alpha^{*}\gamma_t - \frac{1}{2}\gamma_t\alpha)^{*} - f_t\gamma_t.$$

Substituting these derivatives into the equation for $f_t(\alpha)$, differentiating with respect to t and using equations (A.3) and (A.4) for m_t and γ_t we obtain:

$$f_t\,[\,i\alpha^{*}g(t) - \frac{1}{2}\alpha^{*}G(t)\alpha\,] = i\alpha^{*}a_0(t)\,f_t + \alpha^{*}f_t(\,im_t - \frac{1}{2}\alpha^{*}\gamma_t - \frac{1}{2}\gamma_t\alpha)a_1(t)$$
$$-\frac{1}{2}f_t\alpha^{*}b(t)b^{*}(t) + f_t Sp\{Q(t)[(im_t - \frac{1}{2}\alpha^{*}\gamma_t - \frac{1}{2}\gamma_t\alpha)(im_t - \frac{1}{2}\alpha^{*}\gamma_t - \frac{1}{2}\gamma_t\alpha)^{*} - \gamma_t]\}$$
$$+ f_t[m_t^{*}Q(t)m_t + Sp\,(Q(t)\gamma_t)]$$

Taking the real and imaginary parts of this equality yields

$$g(t) = a_0(t)+a_1(t)m_t - 2\gamma_t Q(t)m_t \tag{A.5}$$

$$G(t) = a_1(t)\gamma_t + \gamma_t a_1^*(t) + b^2(t) - 2\gamma_t Q(t)\gamma_t \tag{A.6}$$

which lead to the equations for m_t and γ_t described in the theorem.

Notice that the form of the $f_t(\alpha)$ noted above corresponds to the Gaussian law for conditional distribution of the $Y(t)$ given the event $\{T > t\}$.

It is left to show now that equation (A.4) with $G(t)$ given by (A.6) has a unique solution. One can easily do this implementing the approach developed in [3], chapter 12.

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