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THE DUAL FOREST ITERATION METHOD FOR THE STOCHASTIC TRANSPORTATION PROBLEM

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PREFACE

In this paper, the author develops a dual forest iteration method for the stochastic transportation problem.

The algorithm described in this paper iterates from one dual forest to another with the values of the dual objective function strictly increasing. At most two one-dimensional monotonic equations have to be solved at each step. In this sense it is simple compared with the primal forest iteration method, which may require the solution of more than two such equations at each step. The use of the algorithm is illustrated by application to a numerical example.

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ABSTRACT

This paper presents a dual forest iteration algorithm for solving the stochastic transportation problem. The algorithm iterates from one dual forest to another with the values of the dual objective function strictly increasing in the nondegenerate case. It therefore converges in a finite number of steps. At each step it is necessary to solve at most two one-dimensional monotone equations. If the computation is interrupted before completion, a primal feasible solution, and upper and lower bounds to the optimal value of the objective function can be obtained. A numerical example is also presented.

THE DUAL FOREST ITERATION METHOD FOR THE STOCHASTIC TRANSPORTATION PROBLEM

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1. INTRODUCTION

The forest iteration method for solving the stochastic transportation problem (STP) was first presented in [12]. The major advantage of this method, and one which distinguishes it from other approaches (see [1,4,5,6,7,8,10,11,15, 16,18,19,20,21,22,23]), is its finite convergence.

The forest iteration method iterates from one base forest triple to another with the values of the objective function strictly decreasing. At each step it is necessary to solve a small number of one-dimensional monotone equations.

It is interesting to consider whether it is possible to construct a polynomial-time algorithm capable of solving the STP. Looking at the deterministic case, we see that there is still no known polynomial-time primal simplex method for the transportation problem (TP), but that a polynomial-time dual simplex algorithm for the transportation problem has been given by Ikura and Nemhauser [9], and some of the special properties of dual transportation polyhedra have been explored by Balinski [2,3]. A first step in the direction of a polynomial-time algorithm for solving the STP would therefore seem to be to establish a finitely convergent dual forest iteration method for solving this problem.

This paper presents such a method for solving the STP. It iterates from one dual forest to another with the values of the dual objective function strictly increasing. At most two one-dimensional monotone equations must be solved at each step. In this sense it is simple compared with the primal forest iteration method, which may require the solution of more than two one-dimensional monotone equations at each step.

Since the graph of a basic optimal solution of an STP is a forest [12], it is clear that there are at least three possible iteration techniques: splitting, pivoting and connecting. However, they are quite different from their primal counterparts. The primal forest iteration method uses some primally infeasible points and a technique called *cutting* to pull these infeasible points back into the feasible region. In our dual algorithm we keep the whole process in the dual feasible region, and it is therefore also more direct.

Section 2 presents the dual form of the STP. The dual forest and the basic dual forest solution are defined in Section 3. We prove that the values of the dual objective function corresponding to all basic dual forest solutions associated with a particular dual forest are the same. In Section 4 we define the almost basic dual forest solution, a generalization of the basic dual forest solution which allows us to change the implicit prices in order to increase the value of the dual objective function. The iteration techniques used in three different cases are described in Sections 5, 6 and 7. Section 8 is concerned with the dual forest iteration algorithm and the associated finite convergence theorem. In Section 9, we illustrate the use of the algorithm by applying it to an example first considered in [12].

If the computation is interrupted, we have a lower bound to the optimal value of the objective function but no primal feasible solutions. However, we have an approximation of the certainty equivalent, which in turn yields a primal feasible approximation solution and an upper bound to the optimal value of the objective function. This is discussed in Section 10.

The concept of forests is not simply a convenient computational tool; it is inherent in the structure of the STP. We go into this more deeply in Section 11.

2. THE DUAL FORM OF THE STP

The standard formulation of the stochastic transportation problem with a dummy node is as follows [4,12,15,19,20,21,22]:

$$\min_{x,w} \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} + \sum_{j=1}^{n} \varphi_j(w_j)$$

$$\text{s.t.} \quad \sum_{j=1}^{n+1} x_{ij} = a_i\,, \qquad i = 1,\dots,m \tag{2.1}$$

$$\sum_{i=1}^{m} x_{ij} = w_j\,, \qquad j = 1,\dots,n+1$$

$$x_{ij} \ge 0\,, \qquad \forall i,j$$

where $n+1$ is the dummy node and

$$\varphi_j(w_j) = q_j^- \int_{\xi_j < w_j} (w_j - \xi_j) \mathrm{d}F_j(\xi_j) + q_j^+ \int_{\xi_j > w_j} (\xi_j - w_j) \mathrm{d}F_j(\xi_j) \quad . \tag{2.2}$$

The variables and parameters are defined in the following way:

a_i – total number of items available at i, $a_i > 0$

c_{ij} – cost of shipping one item from i to j, $c_{ij} \geq 0$

x_{ij} – number of items shipped from i to j

w_j – total number of items supplied to j (to be determined)

ξ_j – observed value of $\hat{\xi}_j$

$\hat{\xi}_j$ – random variable reflecting demand at j

F_j – marginal distribution function of $\hat{\xi}_j$ (known)

q_j^- – salvage cost per unit excess at j, $q_j^- \geq 0$

q_j^+ – penalty cost per unit shortage at j, $q_j^+ \geq 0$

Assume that F_j is continuous for all j. We know from [10,18] that φ_j is continuously differentable and convex and that

$$\varphi_j'(w_j) = -q_j^+ + q_j F_j(w_j) \quad , \tag{2.3}$$

where $q_j = q_j^+ + q_j^-$.

Since the feasible region is compact and the objective function is continuous, (2.1) always has a solution.

In the general case some arcs (routes between points) may not be available (see the numerical example given in Section 9). This may be overcome by considering such arcs to be available, but with very large cost coefficients.

From convex programming theory [13,14] and the duality theory of stochastic programming [17], the dual form of (2.1) is:

$$\begin{aligned} \max_{w,u,v} \quad & \sum_{i=1}^{m} a_i u_i + \sum_{j=1}^{n} w_j v_j + \sum_{j=1}^{n} \varphi_j(w_j) \\ \text{s.t.} \quad & u_i + v_j \leq c_{ij} \,, \qquad \forall i,j \\ & -v_j = \varphi_j'(w_j) \,, \quad j = 1,\ldots,n \\ & v_{n+1} = 0 \quad , \end{aligned} \tag{2.4}$$

where $c_{i,n+1} = 0$ for $i = 1,\ldots,m$. Our aim in this paper is to solve (2.4). Once we have done this, we also have the solution of (2.1).

If F_j is not continuous, we can use $\partial\varphi_j$ instead of φ_j', as in [12]. We shall not consider this case any further here.

3. BASIC DUAL FOREST SOLUTIONS

It is shown in [12] that (2.1) has an optimal forest solution (x,w) with multiplier (u,v) such that

$$\begin{aligned}
\sum_{j=1}^{n+1} x_{ij} &= a_i \,, && i = 1,\ldots,m \\
\sum_{i=1}^{m} x_{ij} &= w_j \,, && j = 1,\ldots,n+1 \\
x_{ij} &\geq 0 \,, && (i,j) \in f \\
x_{ij} &= 0 \,, && (i,j) \not\in f \\
u_i + v_j &= c_{ij} \,, && (i,j) \in f \\
u_i + v_j &\leq c_{ij} \,, && (i,j) \not\in f \\
-v_j &= \varphi_j'(w_j) \,, && j = 1,\ldots,n \\
v_{n+1} &= 0 \quad ,
\end{aligned} \tag{3.1}$$

where f is a forest. This is also a sufficient condition. In [12] we defined a forest as a union of trees in the transportation tableau, where the trees concerned are not connected to each other and the row index set of the forest is $\{1,\ldots,m\}$. The properties of the forest are described in some detail in [12]. The duality theory of stochastic programming [17] implies that the optimal forest solution of (2.1) is also an optimal solution of (2.4).

We divide (3.1) into three hierarchical levels:

1. Dual forest feasibility condition (DFF):

$$\begin{aligned}
u_i + v_j &= c_{ij} \,, && (i,j) \in f \\
u_i + v_j &\leq c_{ij} \,, && (i,j) \not\in f \\
-v_j &= \varphi_j'(w_j) \,, && j = 1,\ldots,n \\
v_{n+1} &= 0 \quad .
\end{aligned}$$

2. Primal forest feasibility condition and complementary condition (PFC):

$$\sum_{j=1}^{n+1} x_{ij} = a_i , \qquad i = 1,\dots,m$$

$$\sum_{i=1}^{m} x_{ij} = w_j , \qquad j = 1,\dots,n+1$$

$$x_{ij} = 0 , \qquad (i,j) \notin f \quad .$$

3. Primal non-negativity condition (PNN):

$$x_{ij} \geq 0 , \qquad (i,j) \in f \quad .$$

Definition 3.1. Suppose that we have a forest f on the $m \times (n+1)$ transportation tableau and a vector $(x,w,u,v) \in \mathbf{R}^{m\times(n+1)+(n+1)+m+(n+1)}$ such that the DFF and the PFC are satisfied. Then we call f a *dual forest* of (2.1) and (x,w,u,v) a *basic dual forest solution* of (2.1). ▪

Clearly, the optimal forest is a dual forest and the optimal forest solution (together with its multiplier) is a basic dual forest solution of (2.1).

Theorem 3.1. *The values of the dual objective function corresponding to basic dual forest solutions associated with a given dual forest f are the same.* ▪

Proof. Assume that (x,w,u,v) is a basic dual forest solution associated with f. From (2.3), we know that

$$-v_j = \varphi_j'(w_j) = -q_j^+ + q_j F_j(w_j) \quad . \tag{3.2}$$

We also know that the value of (u,v) is determined by

$$u_i + v_j = c_{ij} , \qquad (i,j) \in f$$

together with a parameter α_t on each component tree t of f, i.e.,

$$v_j = v_j^0 + \alpha_t , \qquad j \in N_t \tag{3.3}$$

$$u_i = u_i^0 - \alpha_t , \qquad i \in M_t \quad ,$$

where M_t and N_t are the row index set and the column index set of t, respectively, and (u^0,v^0) is uniquely determined by f. We can use the fact that $v_{n+1} = 0$ to evaluate the parameter on the tree with the dummy node $n+1$. For other component trees, we know from (3.2) and (3.3) that w_j is a strictly decreasing function of α_t, since F_j is a nondecreasing function. However, the PFC implies that

$$\sum_{j \in N_t} w_j = \sum_{i \in M_t} a_i \tag{3.4}$$

for each component tree t. This determines α_t. Therefore, (u,v) is uniquely determined on f. Now suppose that $(\bar{x},\bar{w},u,v)$ is another basic dual forest solution associated with f. From (3.2) and the fact that F_j is nondecreasing, we have

$$-v_j = \varphi_j'(w_j + \zeta(\bar{w}_j - w_j))$$

for $0 \le \zeta \le 1$. The difference between the dual objective function values corresponding to $(\bar{x},\bar{w},u,v)$ and (x,w,u,v) is given by (2.4) as

$$\sum_{j=1}^{n} [(\bar{w}_j - w_j)v_j + \varphi_j(\bar{w}_j) - \varphi_j(w_j)]$$

$$= \sum_{j=1}^{n} [(\bar{w}_j - w_j)v_j + (\bar{w}_j - w_j)\int_0^1 \varphi_j'(w_j + \zeta(\bar{w}_j - w_j))\mathrm{d}\zeta] = 0 \quad ,$$

thus proving the theorem. ▪

We can therefore talk about the dual objective function value of a dual forest f, and shall denote it by $S(f)$.

4. ALMOST BASIC DUAL FOREST SOLUTIONS

Assume that we have a dual forest $\bar{f}$ with a basic dual forest solution $(\bar{x}, \bar{w}, \bar{u}, \bar{v})$. Assume also that it is not optimal, i.e., there exists a pair of indices (k,l) such that $\bar{x}_{kl} < 0$, $(k, l) \in f$. Is it possible to find another dual forest $\hat{f}$ with a basic dual forest solution $(\hat{x}, \hat{w}, \hat{u}, \hat{v})$ such that $S(\hat{f}) > S(\bar{f})$?

If this can be done then we can develop an algorithm which iterates from one dual forest to another with strictly increasing dual objective function values; this algorithm converges in finitely many steps since the number of dual forests is finite.

The following definition generalizes the above question:

Definition 4.1. Suppose that we have a forest f on the $m \times (n+1)$ transportation tableau and a vector $(x,w,u,v) \in \mathbf{R}^{m \times (n+1)+(n+1)+m+(n+1)}$ such that the PFC is satisfied and the DFF is partially satisfied, i.e.,

$$u_i + v_j = c_{ij}\,, \qquad (i,j) \in f\,,\ (i,j) \neq (k,l)$$

$$u_i + v_j < c_{ij}\,, \qquad \text{all other } (i,j)$$

$$-v_j = \varphi_j'(w_j)\,, \qquad j = 1,\dots,n$$

$$v_{n+1} = 0 \quad ,$$

where $(k,l) \in f$ and $x_{kl} < 0$. Then we call f an *almost dual forest* of (2.1), (x,w,u,v) an *almost basic dual forest solution* of (2.1), and (k,l) the *singular arc* of f. ▪

We see that a dual forest is also an almost dual forest. Notice that there are many almost basic dual forest solutions associated with f.

Suppose that $\bar{f}$ is an almost dual forest of (2.1) with an almost basic dual forest solution $(\bar{x}, \bar{w}, \bar{u}, \bar{v})$ and singular arc (k,l). Suppose that $(k,l) \in t$, where t is a component tree of $\bar{f}$. After deleting arc (k,l) from t, we get two trees t_1 and t_2. Let M, M_1 and M_2 be the row index sets of t, t_1 and t_2, and N, N_1 and N_2 be the column index sets of t, t_1 and t_2 (see Figure 1).

Let

$$A := \min \left\{ c_{ij} - \bar{u}_j - \bar{v}_j \mid i \in M_2,\ j \in \bar{N} \right\}$$

$$B := \min \left\{ c_{ij} - \bar{u}_j - \bar{v}_j \mid i \in \bar{M},\ j \in N_1 \right\}$$

$$C := \min \left\{ c_{ij} - \bar{u}_j - \bar{v}_j \mid i \in M_2,\ j \in N_1 \right\} \quad ,$$

where $\bar{M} = \{1,\dots,m\} \setminus M$, $\bar{N} = \{1,\dots,n+1\} \setminus N$.

If one of A, B, or C is zero, then we have a *degenerate* situation.

Suppose that

$$c_{pq} - \bar{u}_p - \bar{v}_q = A = 0 \quad ,$$

where $p \in M_2$, $q \in \bar{N}$. Then we simply let arc (p,q) enter the basis, i.e., we have a new almost dual forest $\tilde{f} = \bar{f} \cup \{(p,q)\}$ with an almost basic dual forest solution $(\bar{x}, \bar{w}, \bar{u}, \bar{v})$ and singular arc (k,l). We have thus carried out a *degenerate connecting operation* (see Figure 2). The number of trees in $\tilde{f}$ is one less than the number of trees in $\bar{f}$. This means that the sets M_2 and $\bar{N}$ are now different,

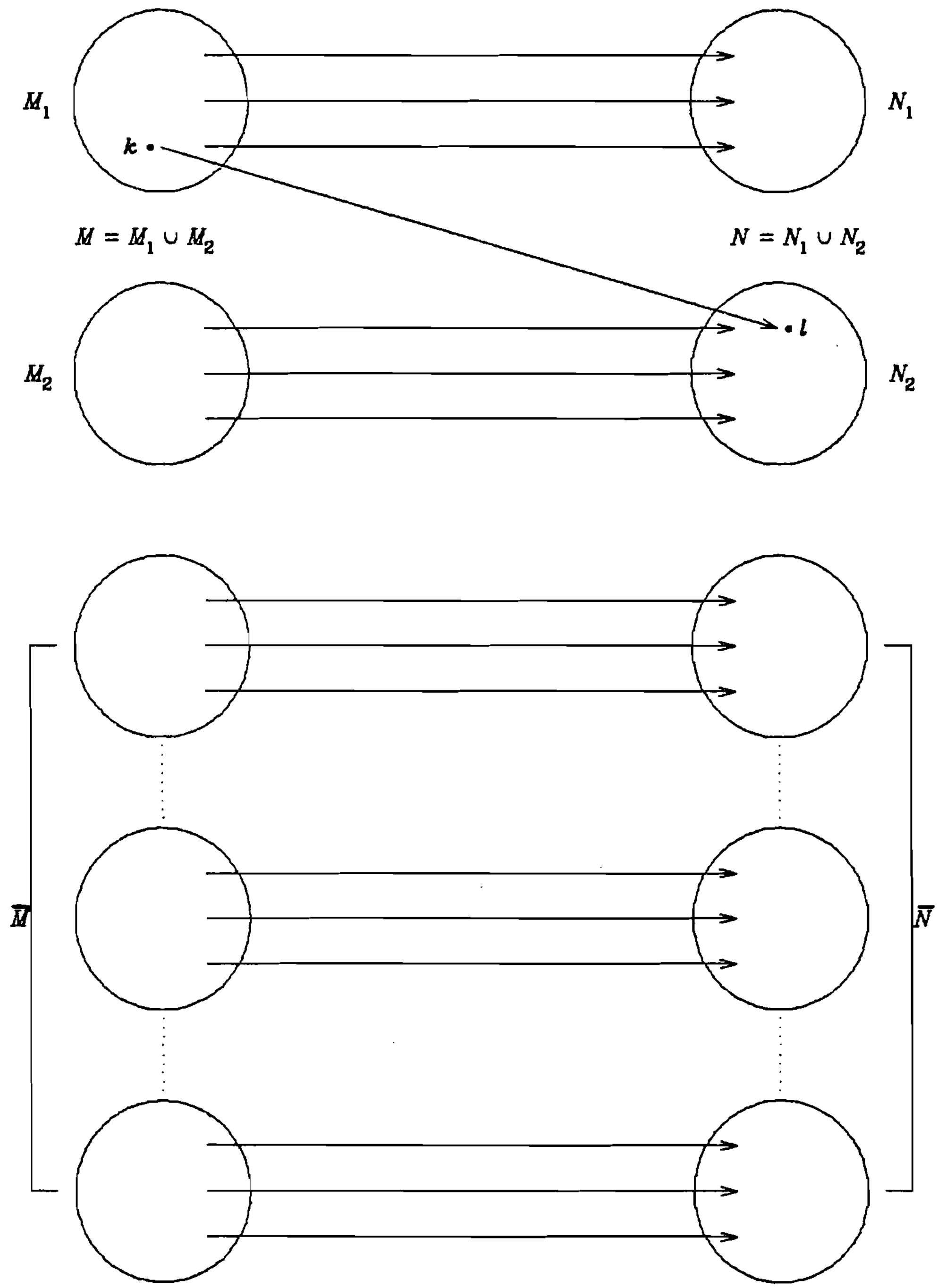

Figure 1. Almost basic dual forest solutions

so that the value of A may also have changed. If not, i.e., if A is still zero, we can have another degenerate connecting process. Since the number of trees in a forest is no more than n, we must have a positive A after at most $(n-1)$ such operations.

We obtain similar results if B is zero.

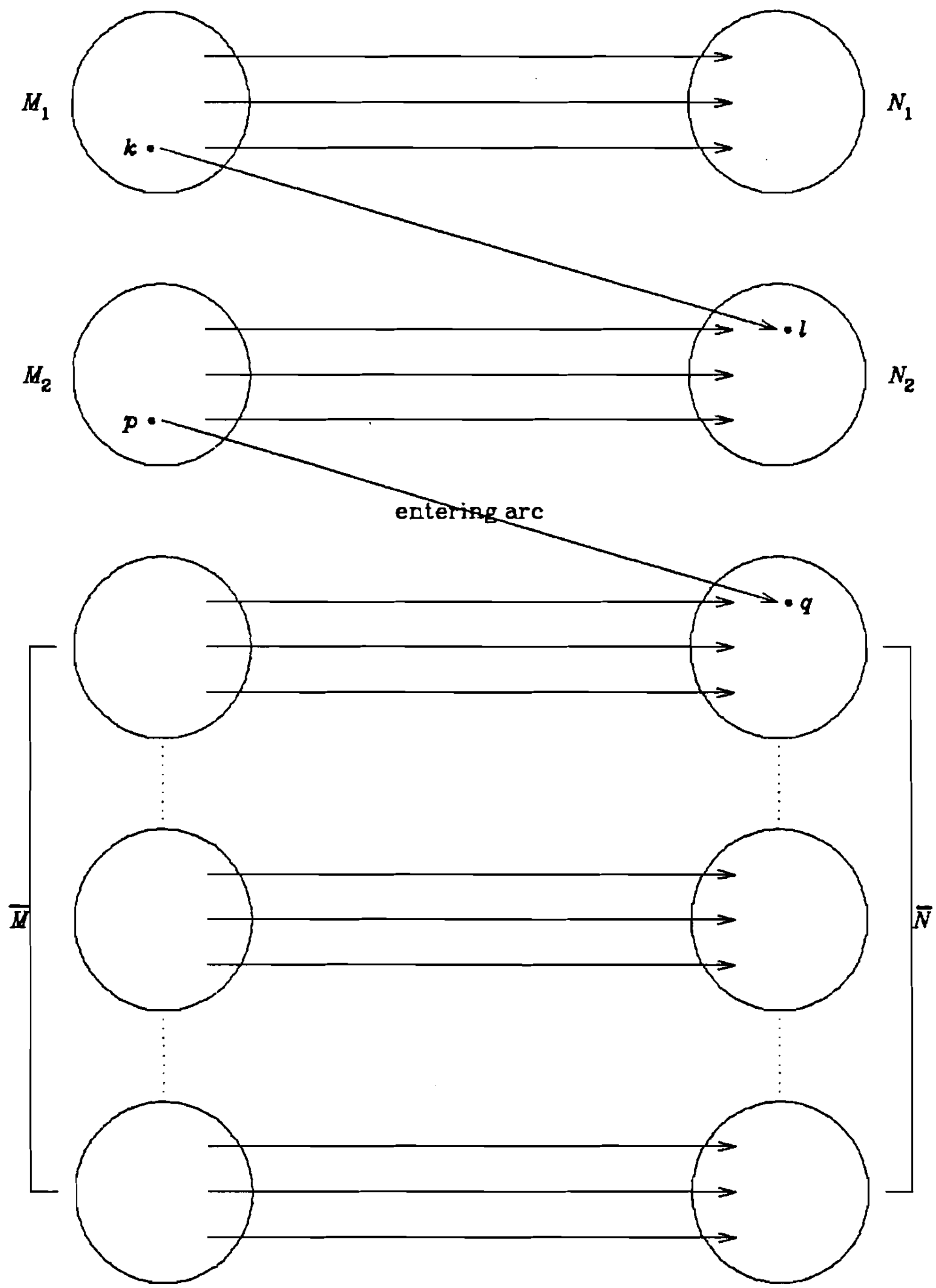

Figure 2. Connecting

However, the situation with $C = 0$ is more complicated, and results in *degenerate pivoting*. We avoid this by assumption:

Nondegeneracy assumption. We shall not encounter the case $C = 0$. ▪

Thus, in general, we can suppose that A, B, $C > 0$. We now discuss the methods used to remove the singular arc from the basis. There are three cases, which will be treated separately.

5. CASE I, $n+1 \in \mathbf{N}_1$

In this case, we decrease v_l by a positive quantity α, and modify the other components of $(\bar{x}, \bar{w}, \bar{u}, \bar{v})$ in such a way that it is still an almost basic dual forest solution associated with f and the value of the dual objective function is strictly increased.

It is clear that we must modify the components of $(\bar{u}, \bar{v})$ as follows:

$$u_i = \bar{u}_i + \alpha , \qquad i \in M_2 \tag{5.1}$$

$$v_j = \bar{v}_j - \alpha , \qquad j \in N_2 \ .$$

The components of $(\bar{x}, \bar{w})$ should also be altered using

$$-v_j = \varphi_j'(w_j) , \qquad j \in N_2$$

$$w_{n+1} = \bar{w}_{n+1} - \sum_{j \in N_2} (w_j - \bar{w}_j) \quad ,$$

and the PFC.

To retain dual feasibility, we must have

$$\alpha \le A, \quad \alpha \le C \quad . \tag{5.2}$$

It is easy to see that

$$x_{kl} = \sum_{j \in N_2} w_j - \sum_{i \in M_2} a_i \quad . \tag{5.3}$$

We know from (3.2), (5.1) and (5.3) that x_{kl} is a function of α and increases with increasing α.

The following lemma states that whenever x_{kl} is negative, an increase in α results in an increase in the value of the dual objective function.

Lemma 5.1. *Let G denote the value of the objective function in problem* (2.4) *and consider G as a function of α. Then*

$$G(\alpha) \ge G(0) - x_{kl}(\alpha)\alpha \quad . \quad \blacksquare$$

Proof.

From Definition 4.1 and the convexity of φ_j we have

$$\varphi_j(w_j) \ge \varphi_j(\bar{w}_j) - \bar{v}_j(w_j - \bar{w}_j)$$

for all $j \in N_2$. Now we have

$$\begin{aligned} & G(\alpha) - G(0) \\ &= \sum_{i \in M_2} (a_i u_i - a_i \bar{u}_i) + \sum_{j \in N_2} (w_j v_j - \bar{w}_j \bar{v}_j) + \sum_{j \in N_2} (\varphi_j(w_j) - \varphi_j(\bar{w}_j)) \\ &\geq \alpha \sum_{i \in M_2} a_i + \sum_{j \in N_2} ((w_j v_j - \bar{w}_j \bar{v}_j) - \bar{v}_j (w_j - \bar{w}_j)) \\ &= -x_{kl}(\alpha)\alpha \quad . \qquad \blacksquare \end{aligned}$$

Assume that $D = \min(A, C)$, and let

$$\begin{aligned} \tilde{u}_i &= \bar{u}_i + D , && i \in M_2 \\ \tilde{v}_j &= \bar{v}_j - D , && j \in N_2 \\ \tilde{w}_j &= F_j^{-1}((q_j^+ - \tilde{v}_j)/q_j) , && j \in N_2 \quad . \end{aligned} \tag{5.4}$$

We can then determine $\tilde{x}_{kl}$ using (5.3).

If $\tilde{x}_{kl} \leq 0$, then the value of the dual objective function is strictly increasing. We have

$$\tilde{w}_{n+1} = \sum_{i \in M} a_i - \sum_{j \in N^*} \tilde{w}_j \quad , \tag{5.5}$$

where $N^* = N \setminus \{n+1\}$.

When $D = A$, we carry out a connecting operation. This is as described in Section 4 except that it is *nondegenerate* in the sense that we have modified (w, u, v) (see also Figure 2).

When $D = C$, we carry out a *pivoting* operation, which is again nondegenerate. Suppose that

$$c_{rs} - \bar{u}_r - \bar{v}_s = C = 0 \quad ,$$

where $r \in M_2$, $s \in N_1$. Then we let arc (r,s) enter the basis and arc (k,l) leave the basis, i.e., we have a new dual forest $\tilde{f} = \{\bar{f} \cup \{(r,s)\}\} \setminus \{(k,l)\}$ with $\tilde{x}_{kl} = 0$ (see Figure 3).

In both cases, the other components of $\tilde{x}$ can be determined using the PFC. In fact, only the components of x in t_2 are changed in the connecting process; in the pivoting process we also modify components in the cycle consisting of (r,s) and members of t.

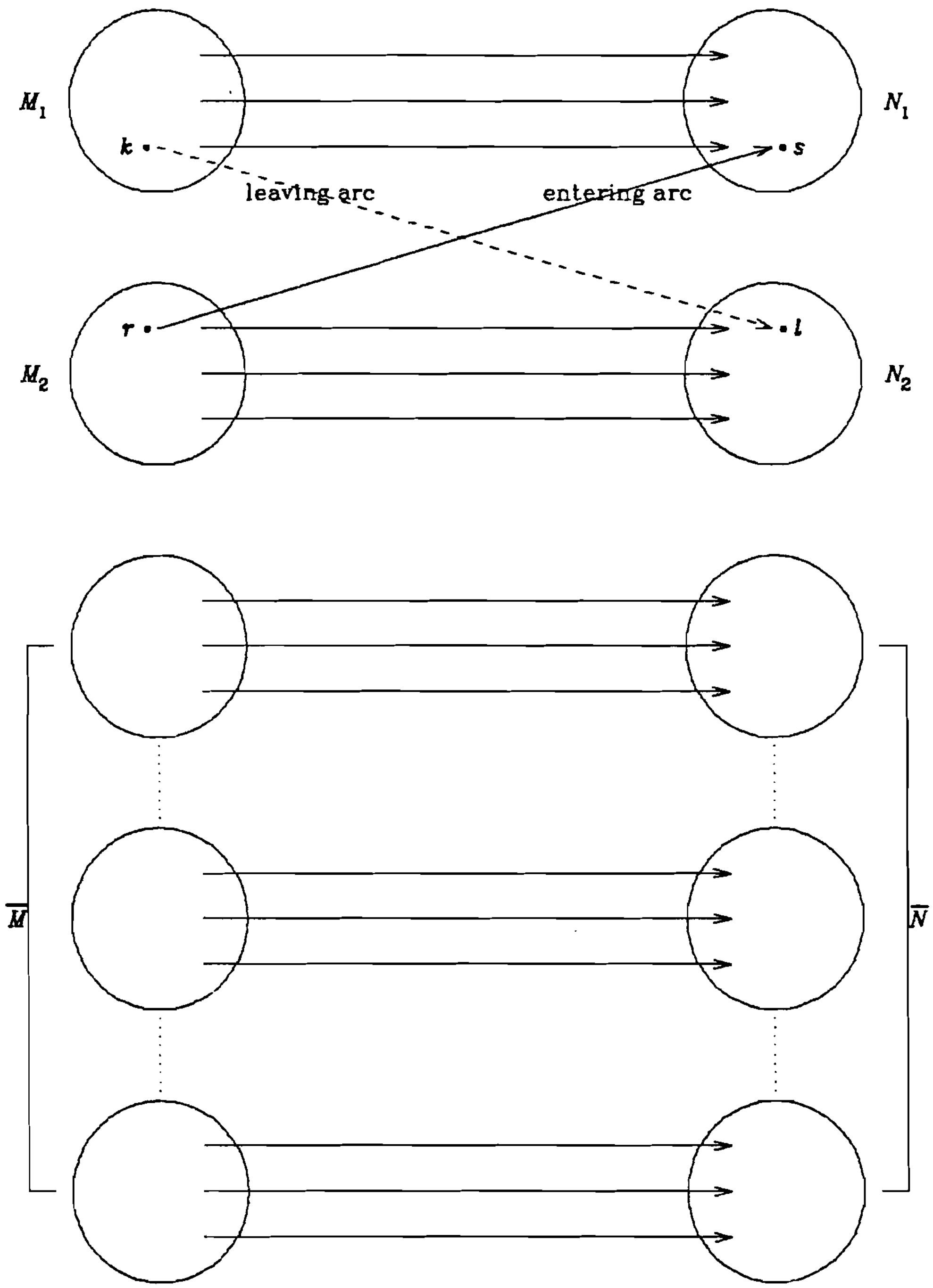

Figure 3. Pivoting

If $\tilde{x}_{kl} > 0$, then we have increased the value too much. We should rather stop at $\tilde{x}_{kl} = 0$ to ensure that the value of the dual objective function is strictly increasing. Then (3.2), (5.1), (5.3) and

$$\tilde{x}_{kl} = x_{kl} = 0 \tag{5.6}$$

form a one-dimensional monotone equation in α. Suppose the solution is $\tilde{\alpha}$. Replacing D in (5.4) by $\tilde{\alpha}$ yields the value of $(\tilde{x},\tilde{w},\tilde{u},\tilde{v})$. Let $\tilde{f} = \bar{f} \setminus \{(k,l)\}$. We now have a *splitting* process with arc (k,l) leaving the basis (see Figure 4). According to Lemma 5.1, we have a strict increase in the value of the dual objective function.

In a pivoting or splitting process, we remove the negative flow arc (k,l) and increase the value of the dual objective function. Therefore, $\tilde{f}$ and $(\tilde{x},\tilde{w},\tilde{u},\tilde{v})$ are, respectively, the new dual forest $\hat{f}$ and the new basic dual forest solution $(\hat{x},\hat{w},\hat{u},\hat{v})$ which we sought at the beginning of Section 4.

If only connecting occurs, we may repeat the procedure. Since the number of the trees in $\tilde{f}$ is one less than in $\bar{f}$ when we have a connecting process, we should have either a pivoting or a splitting process after at most $(n-1)$ connecting processes. We then have a dual forest with an increased dual objective function value.

Notice that we need to solve a one-dimensional monotone equation only when splitting occurs.

6. CASE II, $n+1 \in \mathbf{N}_2$

This is similar to Case I, so we shall not go into so much detail here.

In this case, we decrease u_k by a positive quantity β, and modify the other components of $(\bar{x}, \bar{w}, \bar{u}, \bar{v})$ in such a way that it is still an almost basic dual forest solution associated with f and the value of the dual objective function is strictly increased.

We have

$$u_i = \bar{u}_i - \beta \,, \qquad i \in M_1 \tag{6.1}$$

$$v_j = \bar{v}_j + \beta \,, \qquad j \in N_1 \ .$$

To retain dual feasibility, we require

$$\beta \le B \,, \quad \beta \le C \ . \tag{6.2}$$

We have

$$x_{kl} = \sum_{i \in M_1} a_i - \sum_{j \in N_1} w_j \tag{6.3}$$

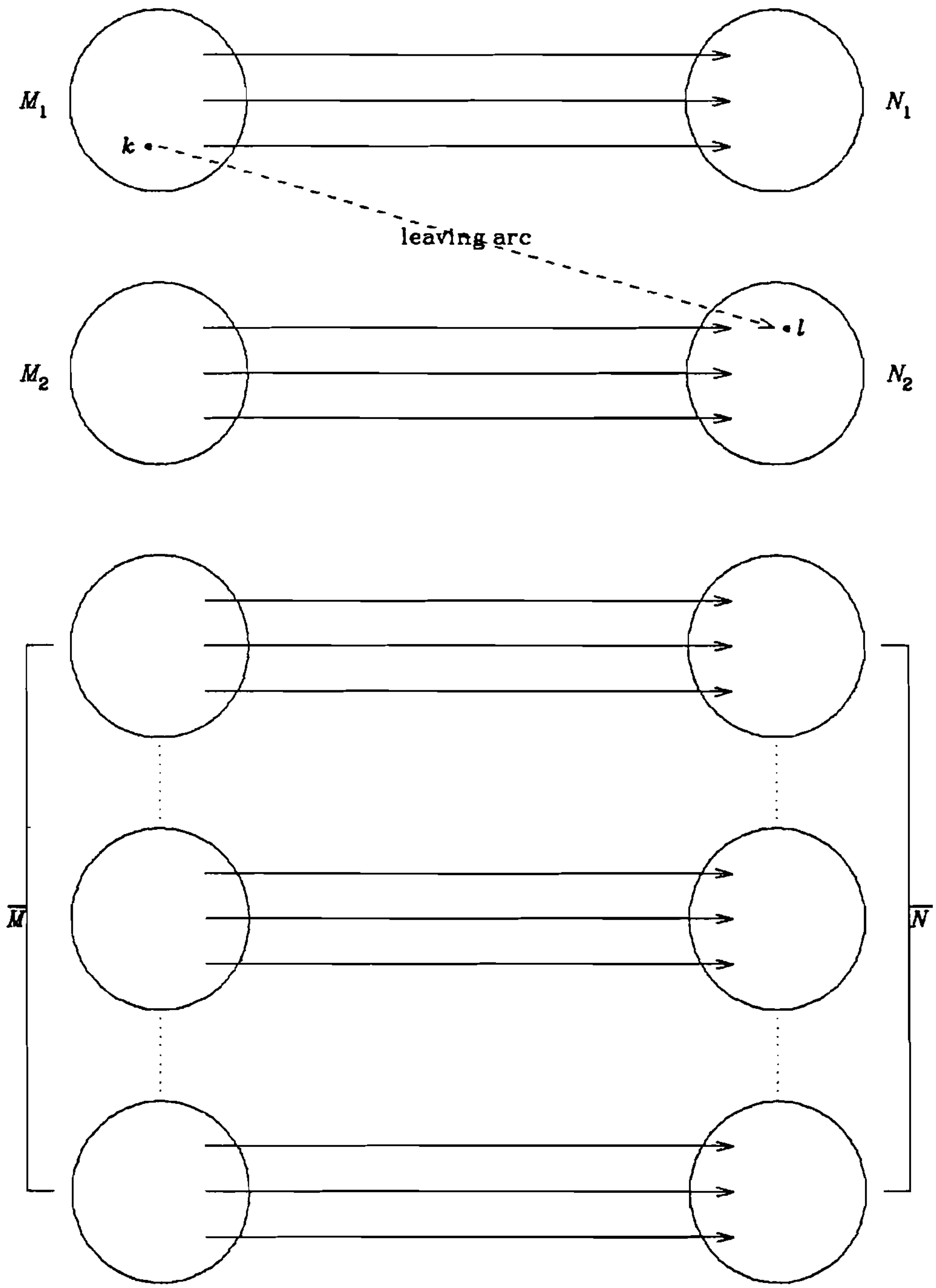

Figure 4. Splitting

and know that x_{kl} is a function of β and increases with increasing β.

Lemma 6.1. *Let G denote the value of the objective function in problem* (2.4) *and consider G as a function of β. Then*

$$G(\beta) \geq G(0) - x_{kl}(\beta)\beta \quad . \quad \blacksquare$$

Assume that $E = \min(B, C)$. Let

$$\tilde{u}_i = \bar{u}_i - E, \qquad i \in M_1$$

$$\tilde{v}_j = \bar{v}_j + E, \qquad j \in N_1 \qquad (6.4)$$

$$\tilde{w}_j = F_j^{-1}((q_j^+ - \tilde{v}_j)/q_j), \qquad j \in N_1$$

and determine $\tilde{x}_{kl}$ using (6.3).

If $\tilde{x}_{kl} \le 0$, then we get a strict increase in the value of the dual objective function. When $E = B$, we carry out a connecting operation; when $E = C$, we carry out a pivoting operation.

If $\tilde{x}_{kl} > 0$, then we have a splitting process.

In this case (3.2), (6.1), (6.3) and

$$\tilde{x}_{kl} = x_{kl} = 0 \qquad (6.5)$$

form a one-dimensional monotone equation in β. Suppose the solution is $\tilde{\beta}$. Replacing E in (6.4) by $\tilde{\beta}$ yields the value of $(\tilde{x}, \tilde{w}, \tilde{u}, \tilde{v})$. The additional comments made in Section 5 also hold in this case.

7. CASE III, $n+1 \in \bar{\mathbf{N}}$

In this case, we add arc $(k, n+1)$ to $\bar{f}$: $f' = \bar{f} \cup \{(k, n+1)\}$.

This is then the same as Case I except that we have two singular arcs: (k, l) and $(k, n+1)$ (see Figure 5). However, all the remarks made in Section 5 still hold here since $(k, n+1)$ is only a dummy arc, with a flow which does not affect the value of the dual objective function. The flow balance is given by

$$\tilde{x}_{k,n+1} = \bar{x}_{kl} - \tilde{x}_{kl} \quad .$$

Thus, $\tilde{x}_{k,n+1}$ is negative after the process outlined in Section 5. We then have an almost dual forest with a singular arc $(k, n+1)$, i.e., Case II. Applying the approach described in Section 6, we obtain a dual forest with an increased objective function value.

Therefore, we have to solve at most two one-dimensional monotone equations in Case III.

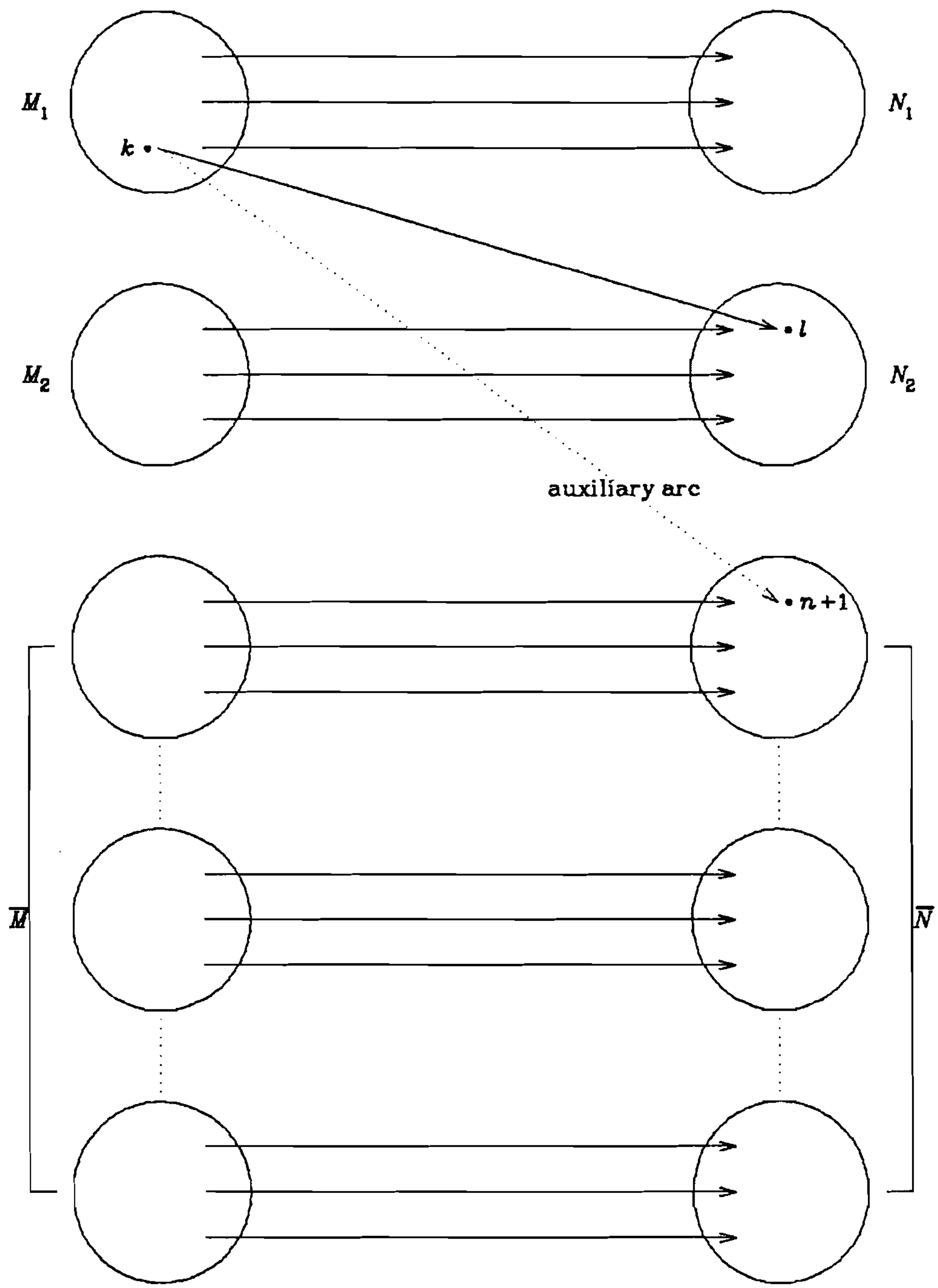

Figure 5. Case III

8. THE DUAL FOREST ITERATION METHOD

We can now outline the dual forest iteration algorithm.

Algorithm 8.1.

1. Start from a dual forest f^0 with a basic dual forest solution (x^0, w^0, u^0, v^0).
2. Having obtained a dual forest f^k with a basic dual forest solution (x^k, w^k, u^k, v^k), check its optimality. If the PNN is satisfied, then it is optimal and we stop. Otherwise, go to Step 3.
3. Using the iteration techniques discussed in Sections 4–7, find a new dual forest f^{k+1} with a basic dual forest solution $(x^{k+1}, w^{k+1}, u^{k+1}, v^{k+1})$. $k+1 \to k$. Go to Step 2. ▪

Theorem 8.1. *Under the nondegeneracy assumption given in Section* 4, *Algorithm* 8.1 *converges in finitely many steps.* ▪

Proof. Since the value of the dual objective function is strictly increasing at each step and the number of dual forests is finite, the theorem is proved. ▪

We shall now give one way of initiating a run of this algorithm. Note that if we have $q_j^+ < \min_i \{c_{ij}\}$ for some j, then we do not send any goods to j because the shipping cost is not less than the penalty cost. We can delete all such demands. Now let $u_i^0 = 0$ for all i and $v_j^0 = \min_i \{c_{ij}\}$ for all j. Since $-q_j^- \le 0 \le v_j^0 \le q_j^+$, we can solve w_j^0 using (3.2), where $j = 1,\ldots,n$. We obtain w_{n+1}^0 by subtracting the sum of the other w_j's from the sum of the a_i's. The value of x^0 can be found from the PFC. Clearly, (x^0, w^0, u^0, v^0) is a basic dual forest solution.

In Cases I and II it is only necessary to solve a one-dimensional monotone equation if we have splitting at that iteration. In Case III it may be necessary to solve two one-dimensional monotone equations.

9. A NUMERICAL EXAMPLE

We illustrate the use of the dual forest iteration algorithm by applying it to an example first considered in [12].

In this example $m = 4$, $n = 5$, and cells (2,1), (3,1) and (3,3) are not available. We take column 0 rather than column 6 as our dummy node column. The other data are given in Table 1.

Table 1

i	c_{ij}					a_i
	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$	
1	18	21	18	16	10	10
2		15	16	14	9	19
3		10		9	6	25
4	17	16	17	15	10	15
D_j	22	20	12	10	13	

The j-th random demand is uniformly distributed on $[0, D_j]$ and $q_j^- = 6D_j$, $q_j^+ = 0$. Therefore

$$\varphi_j(w_j) = \begin{cases} 3D_j^2 - 6D_j w_j & \text{if } w_j < 0 \\ 3(D_j - w_j)^2 & \text{if } w_j \in [0, D_j] \\ 0 & \text{if } w_j > D_j \end{cases} \tag{9.1}$$

and

$$\varphi_j'(w_j) = \begin{cases} -6D_j & \text{if } w_j < 0 \\ 6(w_j - D_j) & \text{if } w_j \in [0, D_j] \\ 0 & \text{if } w_j > D_j \end{cases} \tag{9.2}$$

for $j = 1,2,3,4,5$.

0. Initial Step

To obtain an initial dual forest and an initial starting basic dual forest solution, we let

$$u_i = 0 , \qquad \forall i$$

$$v_j = \min \{c_{ij} \mid i = 1,\ldots,m\} , \qquad \forall j$$

The corresponding values of $(u_i + v_j - c_{ij})$ are given in Table 2.

We have $f = \{(1,0),(2,0),(2,3),(3,0),(3,2),(3,4),(3,5),(4,0),(4,1)\}$. From (3.1) and (9.2), we deduce that

$$w_j = D_j - v_j / 6 , \qquad j = 1,2,3,4,5 . \tag{9.3}$$

Solving the PFC, we obtain the values of x given in Table 3 (see also Figure 6).

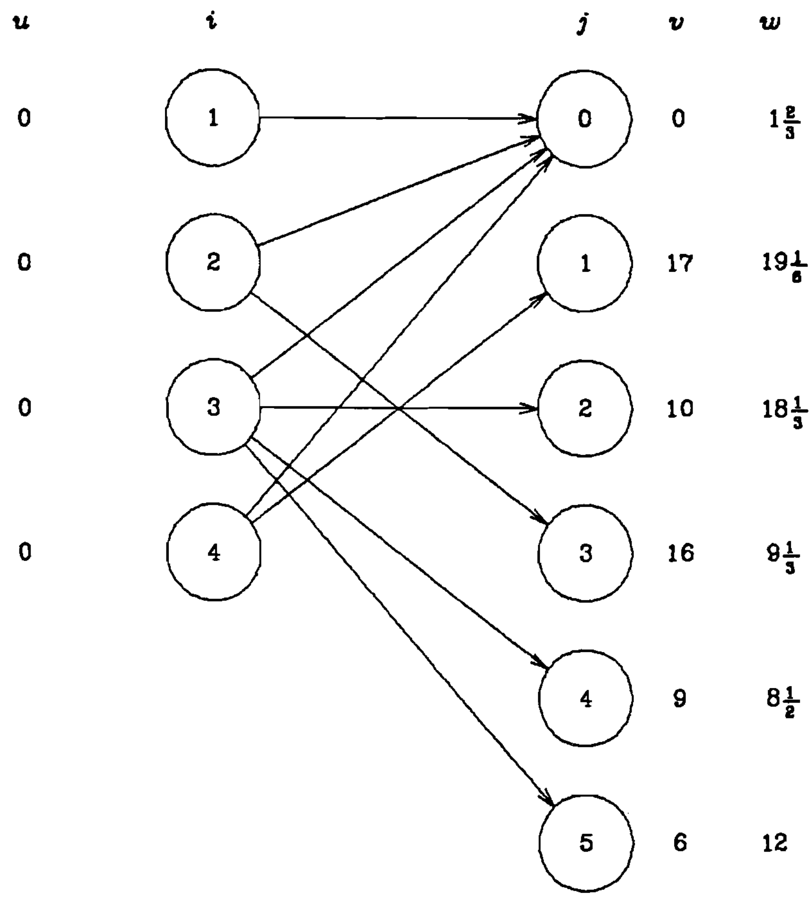

Figure 6. Initial dual forest

Table 2

i	$u_i + v_j - c_{ij}$						u_i
	$j = 0$	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$	
1	0	−1	−11	−2	−7	−4	0
2	0		−5	0	−5	−3	0
3	0		0		0	0	0
4	0	0	−6	−1	−6	−4	0
v_j	0	17	10	16	9	6	

Table 3

i	x_{ij}						a_i
	$j = 0$	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$	
1	10						10
2	$9\frac{2}{3}$			$9\frac{1}{3}$			19
3	$-13\frac{5}{6}$		$18\frac{1}{3}$		$8\frac{1}{2}$	12	25
4	$-4\frac{1}{6}$	$19\frac{1}{6}$					15
w_j	$1\frac{2}{3}$	$19\frac{1}{6}$	$18\frac{1}{3}$	$9\frac{1}{3}$	$8\frac{1}{2}$	12	

We see that (x,w,u,v) is a basic dual forest solution. The value of the dual objective function is 870.5.

1. Step 1: Pivoting

From Table 3, we see that either (3,0) or (4,0) can be taken as the singular arc. We choose arc (3,0).

We now have $(k,l) = (3,0)$, $M_1 = \{3\}$, $M_2 = \{1,2,4\}$, $N_1 = \{2,4,5\}$, $N_2 = \{0,1,3\}$. From Table 2, we know that $c_{25} - u_2 - v_5 = C = 3$.

We then decrease $\{u_3\}$ from $\{0\}$ to $\{-3\}$, and increase $\{v_2, v_4, v_5\}$ from $\{10,9,6\}$ to $\{13,12,9\}$. This decreases $\{w_2, w_4, w_5\}$ to $\{17\frac{5}{6}, 8, 11\frac{1}{2}\}$. From (6.3), we now have $x_{30} = -12\frac{1}{3}$.

We therefore have a pivoting process: arc (3,0) leaves the basis and arc (2,5) enters the basis (see Figure 7). The new values of (x,w,u,v) and $(u_i + v_j - c_{ij})$ are given in Tables 4 and 5.

The corresponding value of the dual objective function is 909.75.

2. Step 2: Pivoting

We choose arc (4,0) as the singular arc (see Table 5).

We now have $(k,l) = (4,0)$, $M_1 = \{4\}$, $M_2 = \{1,2,3\}$, $N_1 = \{1\}$, $N_2 = \{0,2,3,4,5\}$, and $c_{11} - u_1 - v_1 = C = 1$.

We then decrease $\{u_1\}$ from $\{0\}$ to $\{-1\}$, and increase $\{v_1\}$ from $\{17\}$ to $\{18\}$. This decreases $\{w_1\}$ to $\{19\}$. From (6.3), we now have $x_{40} = -4$.

We therefore have a pivoting process: arc (4,0) leaves the basis and arc (1,1) enters the basis (see Figure 8). The new values of (x,w,u,v) and $(u_i + v_j - c_{ij})$ are given in Tables 6 and 7.

The corresponding value of the dual objective function is $913\frac{5}{6}$.

3. Step 3: Pivoting

Following the same procedure as before, we choose $(k,l) = (2,0)$. We now have $M_1 = \{2,3\}$, $M_2 = \{1,4\}$, $N_1 = \{2,3,4,5\}$, $N_2 = \{0,1\}$, and $c_{15} - u_1 - v_5 = C = 1$.

We then decrease $\{u_2, u_3\}$ from $\{0,-3\}$ to $\{-1,-4\}$, and increase $\{v_2, v_3, v_4, v_5\}$ from $\{13,16,12,9\}$ to $\{14,17,13,10\}$. This decreases $\{w_2, w_3, w_4, w_5\}$ to $\{17\frac{2}{3}, 9\frac{1}{6}, 7\frac{5}{6}, 11\frac{1}{3}\}$. From (6.3), we now have $x_{20} = -2$.

We therefore have a pivoting process: arc (2,0) leaves the basis and arc (1,5) enters the basis (see Figure 9). The new values of (x,w,u,v) and $(u_i + v_j - c_{ij})$ are given in Tables 8 and 9.

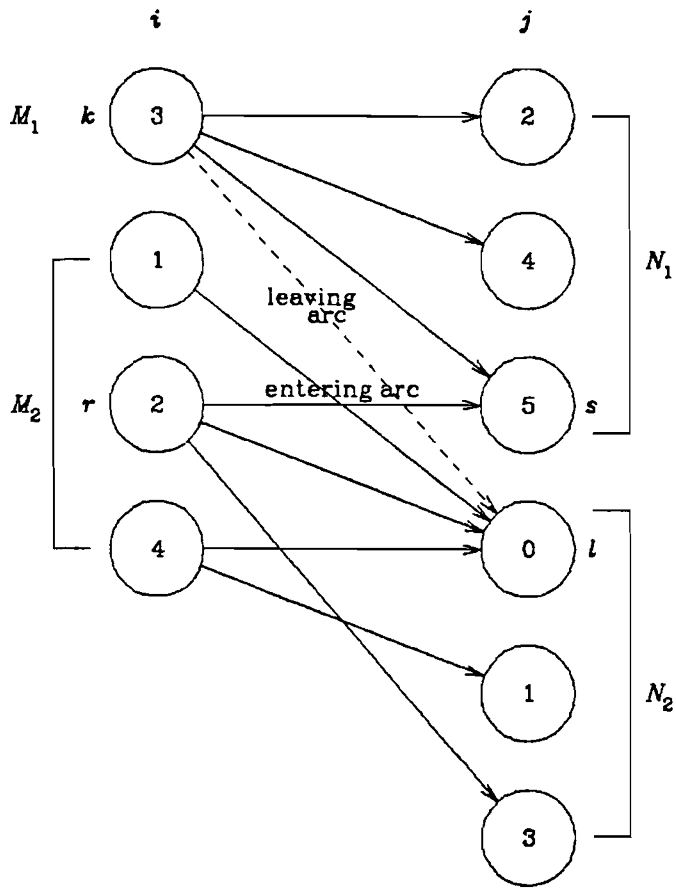

Figure 7. Step 1: Pivoting

Table 4

i	$u_i + v_j - c_{ij}$						u_i
	$j=0$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	
1	0	−1	−8	−2	−4	−1	0
2	0		−2	0	−2	0	0
3	−3		0		0	0	−3
4	0	0	−3	−1	−3	−1	0
v_j	0	17	13	16	12	9	

Table 5

i	x_{ij}						a_i
	$j=0$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	
1	10						10
2	$-2\frac{2}{3}$			$9\frac{1}{3}$		$12\frac{1}{3}$	19
3			$17\frac{5}{6}$		8	$-\frac{5}{6}$	25
4	$-4\frac{1}{6}$	$19\frac{1}{6}$					15
w_j	$3\frac{1}{6}$	$19\frac{1}{6}$	$17\frac{5}{6}$	$9\frac{1}{3}$	8	$11\frac{1}{2}$	

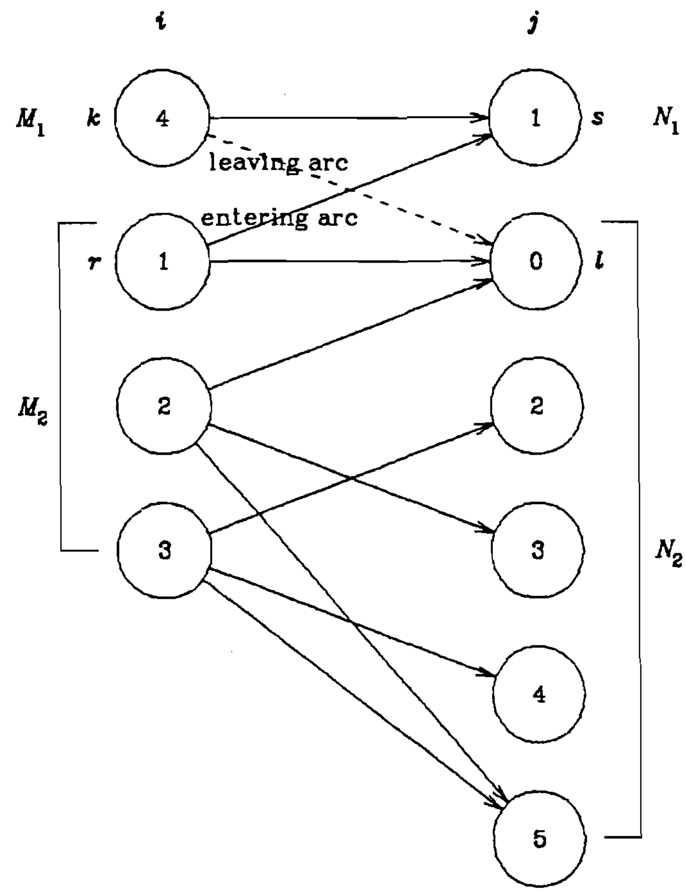

Figure 8. Step 2: Pivoting

Table 6

i	$u_i + v_j - c_{ij}$						u_i
	$j = 0$	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$	
1	0	0	−8	−2	−4	−1	0
2	0		−2	0	−2	0	0
3	−3		0		0	0	−3
4	−1	0	−4	−2	−4	−2	−1
v_j	0	18	13	16	12	9	

Table 7

i	x_{ij}						a_i
	$j = 0$	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$	
1	6	4					10
2	$-2\frac{2}{3}$			$9\frac{1}{3}$		$12\frac{1}{3}$	19
3			$17\frac{5}{6}$		8	$-\frac{5}{6}$	25
4		15					15
w_j	$3\frac{1}{3}$	19	$17\frac{5}{6}$	$9\frac{1}{3}$	8	$11\frac{1}{2}$	

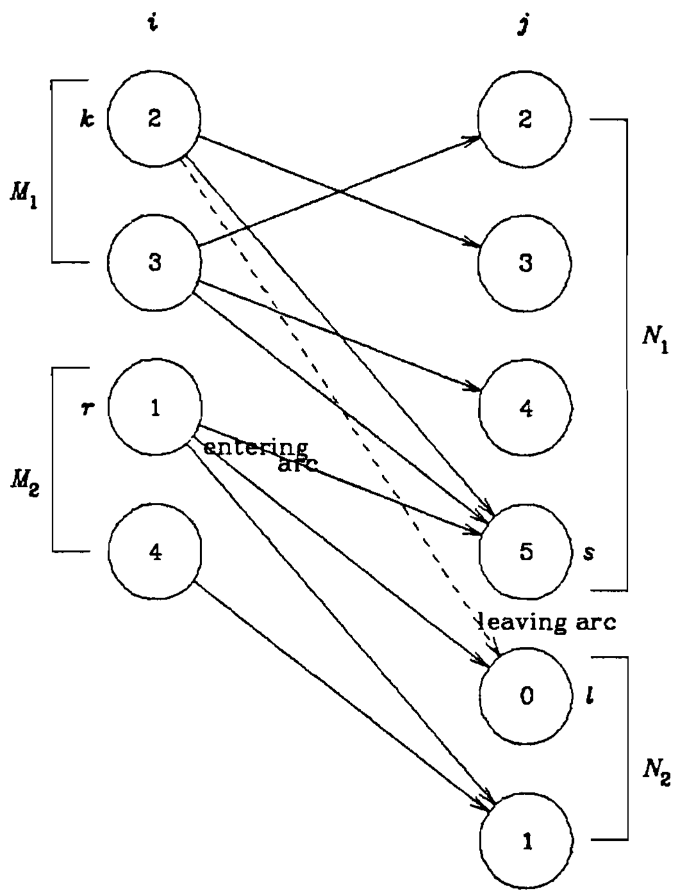

Figure 9. Step 3: Pivoting

Table 8

i	$u_i + v_j - c_{ij}$						u_i
	$j=0$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	
1	0	0	−7	−1	−3	0	0
2	−1		−2	0	−2	0	−1
3	−4		0		0	0	−4
4	−1	0	−3	−1	−3	−1	−1
v_j	0	18	14	17	13	10	

Table 9

i	x_{ij}						a_i
	$j=0$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	
1	4	4				2	10
2				$9\frac{1}{6}$		$9\frac{5}{6}$	19
3			$17\frac{2}{3}$		$7\frac{5}{6}$	$-\frac{1}{2}$	25
4		15					15
w_j	4	19	$17\frac{2}{3}$	$9\frac{1}{6}$	$7\frac{5}{6}$	$11\frac{1}{3}$	

The corresponding value of the dual objective function is $916\frac{1}{6}$.

4. Step 4: Splitting

We set $(k,l) = (3,5)$. We now have $M_1 = \{3\}$, $M_2 = \{1,2,4\}$, $N_1 = \{2,4\}$, $N_2 = \{0,1,3,5\}$, and $c_{22} - u_2 - v_2 = C = 2$.

We then decrease $\{u_3\}$ from $\{-4\}$ to $\{-6\}$, and increase $\{v_2, v_4\}$ from $\{14,13\}$ to $\{16,15\}$. This decreases $\{w_2, w_4\}$ to $\{17\frac{1}{3}, 7\frac{1}{2}\}$. From (6.3), we have $x_{35} = \frac{1}{6}$.

This shows that the value of x_{35} has been increased too much. We have a splitting process: arc (3,5) leaves the basis (see Figure 10). Relations (9.3), (6.1), (6.3) and (6.4) now yield a one-dimensional equation in β:

$$25 = a_3 = w_2 + w_4$$

$$= D_2 - \frac{\bar{v}_2 + \beta}{6} + D_4 - \frac{\bar{v}_4 + \beta}{6} = 25.5 - \frac{\beta}{3} \quad .$$

We have $\beta = 1.5$. The new values of (x, w, u, v) and $(u_i + v_j - c_{ij})$ are given in Tables 10 and 11.

The values of x given in Table 11 are non-negative, i.e., this is the optimal solution. The value of the optimal (dual) objective function value is $916\frac{13}{24}$, i.e., the same result as in [12].

In this example we had to solve only one one-dimensional monotone equation. In [12], we solved two such equations.

10. INTERRUPTING THE COMPUTATION

Suppose the computation is interrupted before completion. We then have a lower bound to the optimal value of the objective function but no primal feasible solution of x. However, we have an approximation of w, i.e., an approximation of the certainty equivalent [11,18]. Fixing w and solving the deterministic TP leads to a primal feasible solution and an upper bound to the optimal value of the objective function.

For example, suppose that the computation in Section 9 is interrupted after the first step. We have 909.75 as a lower bound to the optimal value of the objective function. Fixing w and solving the TP, we obtain the primal feasible solution given in Table 12.

The value of the objective function is 918.25, which is an upper bound to the optimal value of the objective function.

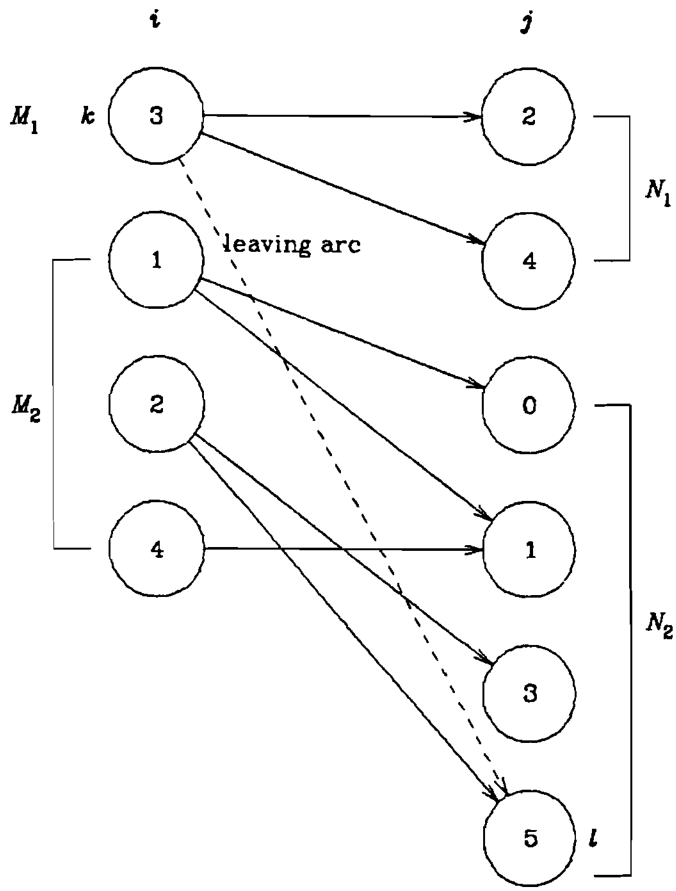

Figure 10. Step 4: Splitting

Table 10

i	$u_i + v_j - c_{ij}$						u_i
	$j = 0$	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$	
1	0	0	−5.5	−1	−1.5	0	0
2	−1		−1.5	0	−0.5	0	−1
3	−5.5		0		0	−1.5	−5.5
4	−1	0	−1.5	−1	−1.5	−1	−1
v_j	0	18	15.5	17	14.5	10	

Table 11

i	x_{ij}						a_i
	$j = 0$	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$	
1	$4\frac{1}{2}$	4				$1\frac{1}{2}$	10
2				$9\frac{1}{6}$		$9\frac{5}{6}$	19
3			$17\frac{5}{12}$		$7\frac{7}{12}$		25
4		15					15
w_j	$4\frac{1}{2}$	19	$17\frac{5}{12}$	$9\frac{1}{6}$	$7\frac{7}{12}$	$11\frac{1}{3}$	

Table 12

i	x_{ij}						a_i
	$j=0$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	
1	$3\frac{1}{6}$	$4\frac{1}{6}$				$2\frac{2}{3}$	10
2			$\frac{5}{6}$	$9\frac{1}{3}$		$8\frac{5}{6}$	19
3			17		8		25
4		15					15
w_j	$3\frac{1}{6}$	$19\frac{1}{6}$	$17\frac{5}{6}$	$9\frac{1}{3}$	8	$11\frac{1}{2}$	

11. REMARKS

We have seen that the concept of forests is a convenient methodological tool. However, this is not all: it is inherent in the structure of the problem. A transportation problem will always have an optimal tree. In turn, this spanning tree will always have u and v such that $u_i + v_j = c_{ij}$, even if there are some zero flows. In the case of the STP, however, there is generally no such spanning tree since this would require that $-v_j$ be a subgradient of φ_j at w_j. For example, we see from Table 10 that all the values of $(u_i + v_j - c_{ij})$ outside the optimal forest are strictly negative, which means that we cannot make an optimal spanning tree.

Another interesting point is that Ikura and Nemhauser also use the term "forest" in their discussion of a polynomial-time dual simplex algorithm for the TP. This may not be accidental.

If w is fixed then the problem becomes a TP. In this case there is no splitting and the proposed method becomes the dual simplex method for the TP. In this sense, the method may be seen as a stochastic extension of the dual simplex method for the TP. This is different from the primal forest iteration method for the STP.

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