# SLOW AND HEAVY VIABLE TRAJECTORIES OF CONTROLLED PROBLEMS
## PART 1. SMOOTH VIABILITY DOMAINS

Jean-Pierre Aubin

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INTERNATIONAL INSTITUTE FOR APPLIED SYSTEMS ANALYSIS
A-2361 Laxenburg, Austria

## PREFACE

In this paper, the first product of the Dynamics of Macrosystems group within SDS, Jean-Pierre Aubin first explains what is meant by slow and heavy trajectories of controlled problems. He then goes on to derive differential equations for these trajectories when the viability domain is smooth. Heavy trajectories seem to be present in the evolution of social and economic macrosystems, and thus their study may provide a clue to the dynamics of such systems.

ANDRZEJ WIERZBICKI
Chairman
System and Decision Sciences

ABSTRACT

We define slow and heavy viable trajectories of differential inclusions and controlled problems. Slow trajectories minimize at each time the norm of the velocity of the state (or the control) and heavy trajectories the norm of the acceleration of the state (or the velocity of the control). Macrosystems arising in social and economic sciences or biological sciences seem to exhibit heavy trajectories.

We make explicit the differential equations providing slow and heavy trajectories when the viability domain is smooth.

SLOW AND HEAVY VIABLE TRAJECTORIES
OF CONTROLLED PROBLEMS

PART 1. SMOOTH VIABILITY DOMAINS

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## 1. SLOW AND HEAVY VIABLE TRAJECTORIES OF DIFFERENTIAL INCLUSIONS

Let us consider a viability domain $K$ - a closed subset of a finite-dimensional state space $X$ - and a set-valued map $F$ from $K$ to $X$ satisfying

(1.1) $\quad F$ is upper semicontinuous with compact convex values .

The simplest viability problem is stated as follows.

For any $x_0 \in K$, find trajectories of the differential inclusion

(1.2) $\quad x'(t) \in F(x(t)) \quad , \quad x(0) = x_0 \quad ,$

which are viable in the sense that

(1.3) $\quad \forall t \geq 0 \quad , \quad x(t) \in K \quad .$

We define the *feedback map* $R$ by

(1.4) $\quad \forall x \in K \quad , \quad R(x) := F(x) \cap T_K(x)$

where

$$T_K(x) := \left\{ v \in X \,\middle|\, \liminf_{h \to 0+} \frac{d_K(x+hv)}{h} = 0 \right\}$$

is the *contingent cone* to K at x.

The main viability theorem states that a necessary and sufficient condition for the viability problem to get a solution is that

$$(1.5) \qquad \forall x \in K \quad , \quad R(x) \neq \emptyset$$

and then, that viable trajectories are solutions to the "feedback" differential inclusion

$$(1.6) \qquad x'(t) \in R(x(t)) \quad , \quad x(0) = x_0$$

(See Haddad [1981], Aubin and Cellina [1984]).

Let us denote by $m(R(x))$ the subset of elements of $R(x)$ with minimal norm. (It is nonempty when $R(x) \neq \emptyset$ and is made of a unique solution when $T_K(x)$ is convex, which happens whenever K is convex or smooth.)

Definition 1.1

We shall say that a trajectory of the differential inclusion

$$(1.7) \qquad x'(t) \in m(R(x(t))) \quad , \quad x(0) = x_0$$

▲

is a *slow (or lazy) viable trajectory.*

Such slow solutions may not exist, even when R is convex-valued. There are instances of viability problems having a unique trajectory, which is a slow one: this is the case when $-F$ is maximal monotone, because, in this case, there exists a unique solution to the differential inclusion

$$x'(t) \in F(x(t))$$

which is actually the solution to the differential equation

$$x'(t) = m(F(x(t)) = m(R(x(t)))$$

(see Brézis [1973]).

This is also the case when K is convex and $F(x) = G(x) - N_K(x)$, $N_K(x) := T_K(x)^-$ denoting the normal cone to K at x. Indeed, Cornet [1981] proved that *when G is continuous*, there are slow trajectories, solutions to the differential equation

$$x'(t) = m(F(x(t))) = m\left(\Pi_{T_K(x(t))}(G(x(t))\right) \quad . \quad \blacksquare$$

We shall come back later to the existence of a slow trajectory of the viability problem.

We now propose to single out another kind of (viable) trajectories, which seem to be present in the evolution of macrosystems arising in social and economic sciences (which motivated viability theory in the first place.) They are trajectories which no longer *minimize* at each time the norm of the velocity, but *the nor 1 of the acceleration*: we propose to call them *heavy trajectories*.

The first difficulty which arises is that a solution to a differential inclusion is only absolutely continuous, and therefore, not necessarily twice differentiable. To use the second derivative in the distribution sense does not help because the concept of *heavy* trajectory requires the existence of the acceleration at each time, or, at worst, at almost each time.

However, another straightforward strategy consists in differentiating the feedback differential inclusion (1.6) to reveal a law governing the acceleration of a viable trajectory, of the form

$$x''(t) \in DR(x(t), x'(t))(x'(t))$$

where $DR(x,v)$ is the *contingent derivative* of the set-valued map R at a point $(x,v)$ of its graph. It is the set-valued map from X to X defined by

(1.8) $\qquad w \in DR(x,v)(u) \iff (u,w) \in T_{Graph(R)}(x,v)$

(see Aubin [1981] and Aubin and Ekeland [1984]).

So, if $m(DR(x,v)(v))$ denotes the subset of elements with minimal norm of the derivative $DR(x,v)$ of R at $(x,v) \in Graph(R)$ in the direction v, we can propose the following formal definition.

Definition 1.2

We shall say that a trajectory of the differential inclusion

$$(1.9) \qquad \begin{cases} \text{i)} & x''(t) \in m(DR(x(t),u'(t))(R(x(t))) \\ & (x(0),x'(0)) = (x_0,v_0), \text{ given in Graph } (R) \end{cases}$$

is a *heavy* viable trajectory. ▲

The problem of the existence of *heavy* viable trajectories is by no means trivial, for many reasons. In the first place, we have in general very little information about the feedback map R. Even when K is convex (and thus, R is convex valued), the graph of $x \to T_K(x)$ is not necessarily closed, even locally compact. Furthermore, the computation of the contingent derivative of R may be difficult, not to mention that the minimal requirements of either convexity or smoothness are missing. In the second place, a recent paper by Cornet and Haddad [1983] shows that the viability problem for second-order differential inclusions is quite different - and more difficult - than the first-order case.

Despite these dark omens, we will solve this problem in the case when K is a smooth manifold and when the set-valued map F is parametrizable.

But before that we state this problem in a slightly more general framework, suitable for the definition of *slow* and *heavy* viable trajectories of *control problems*.

## 2. SLOW AND HEAVY VIABLE TRAJECTORIES OF A CONTROLLED SYSTEM

Let us consider a viability domain $K$ - a closed subset of a finite dimensional state space $X$-, a finite dimensional control space $U$, an upper semicontinuous map $F$ from $K$ to $U$, with compact convex values, and a continuous function $f$ from graph $(F)$ to $X$, affine with respect to the controls.

We define the viability problem for controlled systems as follows. For any $x_0 \in K$, find trajectories of

$$(2.1) \qquad \begin{cases} \text{i)} \; x'(t) = f(x(t),u(t)) \text{ where } u(t) \in F(x(t)) \\ \text{ii)} \; x(0) = x_0 \end{cases}$$

which are viable in the sense that

$$(2.2) \qquad \forall t \geq 0 \quad , \quad x(t) \in K$$

By taking $U = X$, $f(x,u) = u$, we obtain the particular case of a viability problem for a differential inclusion. (Observe also that (2.1) is a differential inclusion $x'(t) \in G(x(t))$ where $G(x) = f(x,F(x)))$.

We define the feedback map $R$ by

$$(2.3) \qquad \forall x \in K, \; R(x) := \{u \in F(x) \,|\, f(x,u) \in T_K(x)\} \quad .$$

<u>Definition 2.1</u>

We shall say that a trajectory of the controlled system

$$(2.4) \qquad \begin{cases} \text{i)} \; x'(t) = f(x(t),u(t)) \text{ where } u(t) \in m(R(x(t))) \\ \text{ii)} \; x(0) = x_0 \end{cases}$$

is a *slow* (or lazy) *viable trajectory*.

We shall say that a trajectory of the system of differential inclusions

$$(2.5)\quad \begin{cases} \text{i)}\ x'(t) = f(x(t),u(t)) \text{ where } u(t) \in R(x(t)) \\ \text{ii)}\ u'(t) \in m(DR(x(t),u(t)))(f(x(t),u(t))) \\ \text{iii)}\ (x(0),u(0)) = (x_0,u_0), \text{ given in Graph } (R) \end{cases}$$

is *heavy viable* trajectory.

▲

So, in the framework of control problems, slow trajectories are associated with controls with minimal norm and heavy trajectories are associated with controls evolving with minimal velocity.

The inverse of the feedback map R associates with any control u the subset $R^{-1}(u)$ of the states of the system controllable by u. We introduce the *state cells* C(u), which are the subsets (possibly empty) of $R^{-1}(u)$ defined by

$$C(u) := \{x \in R^{-1}(u) \mid f(x,u) \in DR^{-1}(u,x)(0)\} \quad .$$

We can regard a state cell C(u) as a subset of "internal" states to $R^{-1}(u)$. Starting with a state $x_0$ in $C(u_0)$ in the direction $f(x_0,u_0)$, a heavy trajectory "keeps" the constant control $u_0$ as long as the state x(t) remains in the state cell $C(u_0)$, because in this case the system of differential inclusions (25) can be written

$$(2.6)\quad \begin{cases} \text{i)}\ x'(t) = f(x(t),u_0) \text{ where } u_0 \in R(x(t)) \\ \text{ii)}\ 0 \in m(DR(x(t),u_0))(f(x(t),u_0)) \quad . \end{cases}$$

The control will start to evolve when the state of the system leaves the state cell $C(u_0)$, according to (2.5).

In the case of usual differential inclusions (1.1) (where U = X and f(x,u) = u), the cells are defined $C(v) := \{x \mid v \in DR^{-1}(v,x)(0)\}$. In such a cell $C(v_0)$, a heavy viable trajectory can be written $x_0 + tv_0$ as long as $x_0 + tv_0 \in C(v_0)$.

Therefore, state cells display areas of the viability domain where "quantitative growth" holds true.

## 3. DIFFERENTIAL EQUATIONS YIELDING SLOW AND HEAVY VIABLE TRAJECTORIES

In this section, we shall assume that the viability domain is smooth:

$$\text{(3.1)} \qquad K := \{x \in X \mid g(x) = 0\}$$

where

$$\text{(3.2)} \qquad \begin{cases} g \text{ is a } C^2\text{- function from } X \text{ to a finite-dimensional} \\ \text{space } Y \text{ and } g'(x) \in L(X,Y) \text{ is surjective for all } x \in K \end{cases}$$

We assume also that the control problem satisfies

$$\text{(3.3)} \qquad \begin{cases} \text{i)} & \forall x \in K \quad , \qquad F(x) = U \\ \text{ii)} & f(x,u) = A(x)u+b(x) \end{cases}$$

where

$$\text{(3.4)} \qquad \begin{cases} \text{i)} & x \in K \to A(x) \in L(U,X) \text{ is continuous and bounded} \\ \text{ii)} & x \in K \to b(x) \in X \text{ is continuous.} \end{cases}$$

We observe that in this case

$$\text{(3.5)} \qquad T_K(x) := \operatorname{Ker} g'(x)$$

and the feedback map R is defined by

$$\text{(3.6)} \qquad \begin{cases} R(x) = \{u \in U \mid g'(x)(A(x)u+b(x)) = 0\} \\ \qquad\quad = -(g'(u)A(x))^{-1}g'(x)b(x) \quad . \end{cases}$$

We shall assume that

$$\text{(3.7)} \qquad \forall x \in K,\ g'(x)A(x) \in L(U,Y) \text{ is surjective}$$

and we set

(3.8) $P(x) := A(x)^*g'(x)^*(g'(x)A(x)A(x)^*g'(x)^*)^{-1}$ ..

For any $x \in K$, $P(x) \in L(Y,U)$ is the *orthogonal right-inverse* of the surjective map $g'(x)A(x)$.

Remark

When the dynamics of the controlled system are not imposed by the model, but may be chosen at will, the problem arises whether we can determine them in terms of the viability constraints and we can find "minimal constructions".

Since assumption (3.7) requires that the maps $g'(x)A(x) \in L(U,Y)$ must be surjective, the dimension of the control space U must be at least equal to the dimension of the resource space Y. We may then choose

$$U = Y \quad .$$

Since we have assumed that the maps $g'(x)$ are surjective, we can take for map $A(x)$ a right inverse of $g'(x)$, and, in particular, the orthogonal right inverse of $g'(x)$ defined by

$$g'(x)^+ := g'(x)^*(g'(x)g'(x)^*)^{-1} \quad .$$

Then, in this case, $R(x) = -g'(x)b(x)$ and the feedback inclusion reduces to the differential equation

$$x'(t) = (1-A(x(t))g'(x(t))b(x(t)) \quad .$$

Observe that $(1-A(x)g'(x))$ is the projector onto $T_K(x) = \text{Ker } g'(x)$ whose kernel is the image of $A(x)$. When $A(x) = g'(x)^+$, $1-g'(x)^+g'(x)$ is the orthogonal projector onto $T_K(x)$.

■

*Theorem 3.1*

We posit assumptions (3.2), (3.4) and (3.7).

a) For any $x_0 \in K$, slow viable trajectories do exist; they are the solutions to the differential equation

$$(3.9)\quad \begin{cases} x'(t) = (1-A(x(t))P(x(t))g'(x(t))b(x(t)) \\ x(0) = x_0 \end{cases}$$

b) For any $x_0 \in K$ and $u_0 \in -(g'(x_0)A(x_0))^{-1}g'(x_0)b(x_0)$, heavy viable trajectories do exist; they are the solutions to the system of differential equations

i) $x'(t) = A(x(t))u(t) + b(x(t))$

ii) $u'(t) = -P(x(t))$

$$(3.10)\quad [g'(x(t))(A'(x(t))u(t)+b'(x(t)) \cdot (A(x(t))u(t)+b(x(t)))$$
$$+ g''(x(t))(A(x)(t))u(t)+b(x(t)),A(x(t)u(t)+b(x(t)))]$$

iii) $(x(0),u(0)) = (x_0,u_0)$

▲

Proof

1. Proof of part a).

Since $R(x) = \{u|g'(x)A(x)u = -g'(x)b(x)\}$, then $m(R(x))$ is the unique solution that minimizes the norm $\|u\|$ of $u$ under a linear equality constraint. Since $P(x)$ denotes the orthogonal right inverse of $g'(x)A(x)$, then $m(R(x)) = -P(x)g'(x)b(x)$ and the slow variable trajectories are the solutions to $x'(t) = A(x(t))m(R(x(t)) + b(x(t))$, which can be written in the form (3.9).

2. We need to compute the contingent derivative of the feedback map R.

*Lemma 3.1*

We posit assumptions (3.2) and (3.7). Then

$$(3.11)\quad \begin{cases} \forall v \in \operatorname{Ker} g'(x), \quad DR(x,u)(v) = \\ -A(x)^{-1}g'(x)^{-1}(g''(x)(A(x)u+b(x),v)+g'(x)(A'(x)u+b'(x))\cdot v). \end{cases}$$

Proof

In this simple case, the graph of F can be written in the form

(3.12) $\quad \mathrm{Graph}(R) = \{(x,u) \mid h(x,u) = 0\}$

where we set

(3.13) $\quad h(x,u) = (g(x), g'(x)(A(x)u+b(x))$ .

This function h is $C^1$ and we check easily that

$$h'(x,u)(v,w) = (g'(x)v, g''(x)(A(x)u+b(x),v)$$

$$+ g'(x)(A'(x)u+b'(x))\cdot v + g'(x)A(x)w) \quad .$$

Since both $g'(x)$ and $g'(x)A(x)$ are surjective by assumption (3.7), then $h'(x,u)$ is surjective. Therefore, the contingent cone to Graph (R) at (x,u) - actually, its tangent space - is the set of pairs (v,w) such that $h'(x,u)(v,w) = 0$, or, equivalently, the set of pairs (v,w) such that $w \in DR(x,u)(v)$.

■

3. Now, we can compute explicitly $m(DR(x,u))(v)$, which is the element minimizing the norm $\|w\|$ under the linear equality constraint

$$g'(x)A(x)w = -g'(x)(A'(x)u+b'(x)\cdot v - g''(x)(A(x)u+b(x),v) \ .$$

Since $g'(x)A(x)$ is surjective, we deduce that

$$m(DR(x,u))(v) = -P(x)[g'(x)(A'(x)u+b'(x)\cdot v)+g''(x)(A(x)u+b(x),v)]$$

which is a continuous map. Therefore, heavy trajectories are given by the system of differential equations (3.10).

Example. Slow and heavy viable trajectories on the sphere

In this case, $g(x) := \langle Gx,x\rangle - 1$ where G is a symmetric positive definite operator from X to X*. Therefore,

$$m(R(x)) = A(x)^*Gx \frac{\langle Gx,b(x)\rangle}{\|A(x)^*Gx\|^2}$$

and equation (3.9) becomes

$$x'(t) = b(x(t)) - A(x(t))A(x(t))^*G(x(t)) \frac{\langle Gx(t), b(x(t)) \rangle}{\| A(x(t))^*Gx(t) \|^2} .$$

In the same way, we obtain

$$m(DR(x,u)(v)) = - \frac{A(x)^*Gx}{\| A(x)^*Gx \|^2} \Big( \langle G^*(A(x)u+b(x), v \rangle - \langle Gx, (A'(x)u+b'(x))v \rangle \Big) .$$

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