VARIATIONAL PRINCIPLES AND CONSERVATION CONDITIONS IN VOLTERRA'S ECOLOGY AND IN URBAN RELATIVE DYNAMICS*

Dimitrios S. Dendrinos**
Michael Sonis***

November 1984
CP-84-49

Contribution to the Metropolitan Study: 14

*Paper presented at the Second World Congress of the Regional Science Association, Rotterdam, Netherlands - June 1984.

**Dimitrios S. Dendrinos
Professor of Urban Planning
The University of Kansas
Lawrence, Kansas 66045
USA

Funding from the National Science Foundation under Contract #SES-821-6620 is gratefully acknowledged.

***Michael Sonis
Associate Professor of Geography
Bar-Illan University
52-100 Ramat-Gan - ISRAEL

Part of this work was done while at University of Kansas as Visiting Foreign Distinguished Scholar, Sept. 15-Oct.15, 1983 under the Sponsorship of the Mid-America State Universities Association.

INTERNATIONAL INSTITUTE FOR APPLIED SYSTEMS ANALYSIS
A-2361 Laxenburg, Austria

CONTRIBUTIONS TO THE METROPOLITAN STUDY:

1. Anas, A. and L.S. Duann (1983) Dynamic Forecasting of Travel Demand. Collaborative Paper, CP-83-45. International Institute for Applied Systems Analysis (IIASA), A-2361 Laxenburg, Austria.

2. Casti, J. (1983) Emergent Novelty and the Modeling of Spatial Processes. Research Report, RR-83-27. IIASA, Laxenburg, Austria.

3. Lesse, P.F. (1983) The Statistical Dynamics of Socio-Economic Systems. Collaborative Paper, CP-83-51. IIASA, Laxenburg, Austria.

4. Haag, G. and W. Weidlich (1983) An Evaluable Theory of a Class of Migration Problems. Collaborative Paper, CP-83-58. IIASA, Laxenburg, Austria.

5. Nijkamp, P. and U. Schubert (1983) Structural Change in Urban Systems. Collaborative Paper, CP-83-57. IIASA, Laxenburg, Austria.

6. Leonardi, G. (1983) Transient and Asymptotic Behavior of a Random-Utility Based Stochastic Search Process in Continous Space and Time. Working Paper, WP-83-108. IIASA, Laxenburg, Austria.

7. Fujita, M. (1984) The Spatial Growth of Tokyo Metropolitan Area. Collaborative Paper, CP-84-03. IIASA, Laxenburg, Austria.

8. Andersson, A.E. and B. Johansson (1984) Knowledge Intensity and Product Cycles in Metropolitan Regions. Working Paper, WP-84-13. IIASA, Laxenburg, Austria.

9. Johansson, B. and P. Nijkamp (1984) Analysis of Episodes in Urban Event Histories. Working Paper, WP-84-75. IIASA, Laxenburg, Austria.

10. Wilson, A.G. (1984) Transport and the Evolution of Urban Spatial Structure. Collaborative Paper, CP-84-41. IIASA, Laxenburg, Austria.

11. Anas, A. (1984) The Combined Equilibrium of Travel Networks and Residential Location Markets. Collaborative Paper, CP-84-42. IIASA, Laxenburg, Austria.

12. Batten, D., P. Newton and J. Roy (1984) Nested Dynamics of Metropolitan Processes and Policies - Melbourne. Collaborative Paper, CP-84-47. IIASA, Laxenburg, Austria.

13. Mackett, R.L. (1984) Nested Dynamics of Metropolitan Processes and Policies - Leeds. Collaborative Paper, CP-84-48. IIASA, Laxenburg, Austria.

14. Dendrinos, D.S. and M. Sonis (1984) Variational Principles and Conservation Conditions in Volterra's Ecology and in Urban Relative Dynamics. Collaborative Paper, CP-84-49. IIASA, Laxenburg, Austria.

15. Batten, D. (1984) The Changing Economic Structure of Metropolitan Regions: A Preliminary Comparative Analysis. Collaborative Paper, CP-84-50. IIASA, Laxenburg, Austria.

FOREWORD

The Metropolitan Development Project was initiated in 1983 as a collaborative study. In 1984 efforts have been concentrated on creating a methodological basis for a more focused research phase starting in 1985. One of the priorities is to analyze the spatial dynamics of interacting populations.

This papaer contains an application of Volterra's ecological model to the issue of interurban population movements. In the paper it is argued that the Volterra paradigm is useful also for modelling of human populations. The issue of fast urban growth and decline is analyzed within this framework.

Åke E. Andersson
Leader
Regional Issues Project

November 1984

ABSTRACT

From an analytical viewpoint, Volterra's variational principles and their associated integrands in single and multiple species interaction under absolute growth conditions in the field of mathematical ecology are reconsidered and simplified. They are then compared with the conservation conditions found appropriate to hold in a class of dynamic problems of relative growth in urban analysis. The comparison assists in interpreting the integrands of geographical (spatial) associations as a "stationary effort fitness function" associated with a cumulative entropy measure of the relative urban dynamic spatial distributions.

From a substantive viewpoint, the paper shows the theoretical conditions, which would result in all spatial activity to be concentrated into a single point, so that inferences can be made regarding the conditions under which the activity will disperse. It also demonstrates that assuming a particular problem formulation, in this case a relative dynamic framework in an inactive environment, will result in obtaining spatial competitive exculsion. This is demonstrated in a parsimonious manner.

Introduction

Mathematical ecology formalizations of urban and regional spatial associations have been steadily making inroads into geographical analysis during the past five years, Dendrinos [2], Curry [1], Dendrinos and Mullally [3], Sonis [10], and others. This recent work supports the argument that well established ecological interactions can provide new and rich insights into the dynamic interdependencies of a broad class of geographical systems.

Central to this work is the role, derivation and interpretation of the variational principles and corresponding "fitness" functions which give rise (or underlie) the particular dynamics governing the evolution of such systems. Since the early developmental stages of the field of mathematical ecology the quest for these governing functions was viewed as an essential element. Although the interest in such a question has subsided since the early work by Volterra on this topic, found in Scudo and Ziegler [8], its import has not deminished.

Volterra was able to derive differential equations of absolute growth population dynamics of ecological associations as solution of variational principles similar to those of classical mechanics. Moreover, Volterra gave three different forms of these principles: the principle of "least action" for one-species population growth; the principle of "stationary action" and principle of "least vital action" for multispecies ecological dynamics. He drew from Maupertuis' notion of "quantity of action" and its use by Descartes (with the principle of momentum) and by Leibnitz (with the principle of kinetic energy) on the integrals of dynamic equations. Volterra's main result, however, emerged through the use of Hamilton's principle of stationary action reducing the biological associations (the dynamic equations) to the mechanics (kinetics) of a branch of problems in the calculus of variations.

Analytically, Volterra showed how the trajectories in the space of states of ecological associations (which describe how the system evolves over time) can be found as functions (extremals) which give the stationary value for a certain integral; i.e., the extremals X are the solution of the variational problem:

$$\int_0^T I\ (X,\dot{X},t)\ dt = \text{Const} \qquad (0.1)$$

when X is a state variable, T is a time horizon and dot stands for time derivative. A special choice of the integrand I allowed him to derive the equations of motion in ecological dynamics for a single species, logistic growth, ecology. In this case the integral (0.1) obtains a minimum. Further, he was able to derive more general formulations of governing integrands regarding the dynamics of a general class of conservative, multiple species ecological associations under absolute growth. Moreover, under certain conditions to be discussed later, Volterra derived the principle of "least vital action." All derivations are described in Part I of this paper.

Volterra defined conservative ecological associations pretty much like in classical mechanics: the total interaction among species is balanced in a manner that results in the value of the total interaction to equal zero. Ecologists did not extend Volterra's work in the following decades. They were disatisfied by the peculiar conditions associated with the existence and stability properties of Volterra's conservative systems. This, coupled with their almost exclusive interest in absolute growth dynamics, did not provide mathematical ecologists the opportunity to search for other kinds of conservative systems or relative growth dynamics where the existence properties of the equilibrium are not as unreasonable or restrictive as the original Volterra formulations.

Although absolute growth conservative dynamic systems may be of little importance for dynamic spatial analysis in regional science, relative growth conservative dynamic systems are, Dendrinos (with Mullally) [4]. At least, they might be more important than absolute growth in urban/regional analysis, than they are in the field of animal and plant ecology. For example, in the area of aggregate urban dynamics, since total national growth may have very little to do with any particular urban area or region in a national economy (particularly when a very large number of metropolitan areas or regions are involved in absence of primacy), it is elasticities of growth that matter. Cities and regions, competing with one another for economic activity, attract or repulse growth depending on relative advantages they enjoy in the national space. Relative growth has been argued to be of importance in a variety of other geographical contexts, including intra-urban dynamics and the processes underlying innovation diffusion. In these contexts, the role of the environment as it may affect aggregate urban dynamics can be analytically studied and the purpose of this paper is precisely to do so. More specifically, issues are addressed regarding multi-urban interaction stability and conditions under which the existence of potentials can be shown.

Under absolute growth and for certain conservative ecological associations Volterra was able to derive a governing integrand. This paper shows that such an integrand can be derived for particular classes of spatial (urban conservative) systems, different than the Volterra ones, associated with relative growth. This is done in Part II of this paper. Relative growth dynamics is shown to be the solution of a governing integrand which measures the entropy of dynamic distribution and the interaction within the zero aggregate growth relative spatial distribution. The stationarity of the integral of such an integrand results in stationarity of a cumulative entropy measure of relative spatial distributions. This is of particular interest to geographical analysis since it provides insights into the fitness functions present in spatially interacting urban systems. We link these fitness functions to the notion of entropy, which we study (for the first time in dynamic spatial analysis) not only over space, but also over time. It is shown that over time entropy is maximized, but over space it is at a minimum in the (asymptotically stable) steady state, which represents urban competitive exclusion (i.e., total agglomeration of population into one locale) for dynamic spatial conservative systems. Interpretations and implications are discussed at the end of Part II, where the subject of uniqueness of such integrand is addressed. Further, suggestions on specific hypotheses for empirical testing are presented together with conclusions.

Part I. Volterra's Integrands.

In this Part the work of Volterra on absolute growth is summarized and, in certain instances, simplified and extended to acquire geographical and economic meaning. In Section A the derivation of the logistic growth path from a variational principle is supplied for the single species case. Section B deals with multiple species interactions and the definition of Volterra's conservative systems (under absolute growth), and their demographic energy notions. In Section C the variational principle generating Volterra's conservative "stationary action" dynamic multiple species interaction is derived. Finally, in Section D the principle of "least vital action" by Volterra in multiple species interaction is presented. The reasons to include a summary of Volterra's work in Part I is not only to bring this significant work in a capsule to the attention of regional scientists and urban/regional geographers who have not been exposed to it until now, but also to shed additional light on Volterra's work by elaborating on and/or simplifying certain derivations.

A. The principle of "least action" for one species logistic growth.

This section presents the first attempt to link entropy with dynamics of spatial systems, defining thus cumulative spatial entropy. Volterra in his classical work on "the calculus of variations and the logistic curve" found in Scudo and Ziegler [8] (p. 11-17) was the first to derive the Verhulst-Pearl equation of logistic (absolute) growth

$$\dot{x} = x(a - bx) \tag{1.1}$$

from the minimization of an integral $E = \int_0^T I(X,\dot{X},t)\,dt$, where:

$$X(t) = \int_0^t x(t)\,dt\ , \quad \text{or} \quad \frac{dX}{dt} = \dot{X} = x\ . \tag{1.2}$$

Volterra interprets X as the total (cumulative) quantity of life. The Euler condition for optimization of E is:

$$\frac{\partial I}{\partial X} - \frac{d}{dt}\left(\frac{\partial I}{\partial \dot{X}}\right) = 0 \tag{1.3}$$

The integrand I, chosen by Volterra, is:

$$I = m_1 \frac{dX}{dt} \ln \frac{dX}{dt} + m_2 (a - b\frac{dX}{dt}) \ln (a - b\frac{dX}{dt}) + kX \tag{1.4}$$

where m_1, m_2, k are appropriate constants. In I the element of cumulative entropy is introduced. Then the components of the Euler condition are:

$$\frac{\partial I}{\partial X} = k \tag{1.5}$$

$$\frac{\partial I}{\partial \dot{X}} = m_1 (\ln \frac{dX}{dt} + 1) - m_2\, b\, \left(\ln (a - b\frac{dX}{dt}) + 1\right) \tag{1.6}$$

and:

$$\frac{d}{dt}\left(\frac{\partial I}{\partial \dot{X}}\right) = m_1 \frac{1}{\frac{dX}{dt}} \cdot \frac{d^2X}{dt^2} + m_2 \frac{b^2}{a-b\frac{dX}{dt}} \quad \frac{d^2X}{dt^2} \,. \tag{1.7}$$

If the three constants satisfy the conditions(1):

$$m_1 = bm_2; \quad k = am_1, \quad m_1 \neq 0, \tag{1.8}$$

then:

$$m_1\left(a - \left[\frac{1}{\frac{dX}{dt}} + \frac{b}{a-b\frac{dX}{dt}} \right] \frac{d^2X}{dt^2} \right) = 0 \tag{1.9}$$

which is identical to the original Verhulst-Pearl equation.

---

(1)There is a printing error on page 16 of Scudo and Ziegler [8] regarding $k = am_1$.

Volterra computed the second variation of the integral E with respect to X and found it positive, indicating E attains a minimum. In his original paper Volterra interprets this as "reducing the movement of population to a principle of minimum," Scudo and Ziegler [8], p. 15. In compliance with the spirit of later work by Volterra one must assume that he meant "minimum effort for adaptation," although Volterra does not explicitly state this. Note that this expression E does not possess a Hamiltonian, thus it is not a potential.

B. Conservation of demographic energy in multiple species interaction.

Volterra's general formulation of the (non-logistic) growth multiple species interaction absolute growth is:

$$\dot{x}_i = (a_i + \frac{1}{b_i} \sum_j a_{ji} x_j) x_i = \alpha_i x_i , \quad i,j = 1,2,\ldots I \tag{2.1}$$

where a_i is the "coefficient of self-growth", $\frac{1}{b_i}$ is an "equivalence factor" and α_i is a "demographic coefficient" (Volterra's terms [8], p. 239). Coefficients a_{ij} depict particular species associations depending on their sign. Also, Volterra calls a_i the "gross" growth rate, b_i an "average weight" and α_i the "net" growth rate.

The key notion of Volterra's elaboration is the "value of the whole association" V, or the "actual demographic energy":

$$V = \sum_i b_i x_i . \tag{2.2}$$

The differential of V is equal to:

$$dV = \sum_i b_i a_i x_i dt + \sum_i \sum_j a_{ji} x_i x_j dt \tag{2.3}$$

on the basis of which it is possible to introduce the "demographic potential energy" of an ecological association:

$$P = - \sum_i a_i \, b_i \, X_i \; , \quad X_i = \int_0^t x_i(t) \, dt \; . \tag{2.4}$$

If the value of the second term in the r.h.s. of condition (2.3) is always zero, i.e.:

$$\sum_i \sum_j a_{ji} \, x_i \, x_j = 0 \tag{2.5}$$

then the individual interactions will not affect the total ecological association. Requirement (2.5), according to Volterra, is the definition of a "conservative ecological association."

The system of differential equations (2.1) implies that:

$$\sum_i b_i \dot{x}_i = \sum_i a_i \, b_i \, x_i + \sum_i \sum_j a_{ji} \, x_i \, x_j \tag{2.6}$$

and, therefore, the definition (2.5) of a conservative ecological association a la Volterra is equivalent to the following condition:

$$\sum_i b_i \, (\dot{x}_i - a_i \, x_i) = 0 \tag{2.7}$$

Integration of (2.6) gives us:

$$\sum_i b_i \, (x_i - a_i X_i) = \text{Const} \tag{2.8}$$

or, due to (2.2), 2.4):

$$V + P = \text{Const.} \tag{2.9}$$

This is Volterra's "principle of conservation of demographic energy" which is the sum of actual demographic energy V and the potential demographic energy P ([8], p. 242).

In order for the association to be conservative, Volterra proves ([8] p. 165, employing unnecessarily complicated proofs), that the following two antisymmetry conditions must be met:

$$a_{ij} = -a_{ji}; \quad a_{ii} = 0 \quad . \tag{2.10}$$

A simpler proof goes as follows: the expression (2.5) can be written as:

$$\sum_i \sum_j a_{ji} x_i x_j = \sum_i a_{ii} x_i^2 + \sum_{\substack{i \, j \\ i>j}} \sum (a_{ij} + a_{ji}) x_i x_j = 0 \tag{2.11}$$

which identifies a polynomial of second degree in the independent variables $x_1, x_2, \ldots, x_I$. The condition requires that all coefficients be zero. This implies directly (2.10). Thus, conditions (2.10) are equivalent to the definition (2.5) of a conservative ecological association and to (2.7); they are necessary conditions for an equilibrium to exist, but not sufficient. At best, the equilibrium is neutrally stable, something which occurs when all eigenvalues have zero real parts; otherwise the equilibrium is unstable. This is a direct result from the fact that the sum of the real parts of the eigenvalues equals the sum of the diagonal elements of matrix $[a_{ij}]$, which is zero (since $a_{ii} = 0$, for all $i = 1,2,\ldots, I$).

The non-zero equilibrium states of Volterra's conservative ecological association immediately give us the <u>first integral</u> of the system of differential equations (2.1). (It will be recalled that a first integral of a system of differential equations is a function which has a constant value along each solution of the system of differential equations.) Following, is the derivation of the first integral, which has an interesting form from an economic theory standpoint.

The non-zero equilibrium state $(x_1^*, x_2^*, \ldots, x_I^*)$ of Volterra's conservative ecological association must evidently satisfy the "fundamental system" (Volterra's term):

$$a_i b_i + \sum_j a_{ji} x_j^* = 0 \;, \quad i,j = 1,2,\ldots,I \;. \tag{2.12}$$

The non-zero equilibrium state of this conservative association (if it exists), due to conditions (2.5), or (2.7), requires that:

$$\sum_i a_i b_i x_i^* = 0 \qquad (2.13)$$

Conditions (2.1), (2.7), (2.10), (2.12), (2.13) combined imply that:

$$\sum_i b_i x_i^* \dot{x}_i/x_i = \sum_i x_i^* (a_i b_i + \sum_j a_{ji} x_j) =$$

$$= \sum_i a_i b_i x_i^* + \sum_i \sum_j a_{ji} x_j x_i^* = - \sum_j x_j (\sum_i a_{ij} x_i^*) =$$

$$= \sum_j a_j b_j x_j = \sum_j b_j \dot{x}_j = \sum_i b_i \dot{x}_i .$$

Thus,

$$\sum_i (b_i \dot{x}_i - b_i x_i^* \dot{x}_i/x_i) = 0 , \qquad (2.14)$$

and, therefore,

$$\sum_i (b_i x_i - b_i x_i^* \ln x_i) = \text{Const}; \qquad (2.15)$$

this implies further that at all time periods:

$$\exp V / \prod_i \left(x_i^{x_i^*}\right)^{b_i} = \text{Const} , \qquad (2.16)$$

where V is the value of the whole association (2.2). Thus, the function

$$\exp V / \prod_i \left(x_i^{x_i^*}\right)^{b_i} \qquad (2.17)$$

is the first integral for the system (2.1). Condition (2.16) can be also written as:

$$\prod_i x_i^{b_i x_i^*} = \frac{\exp V}{\text{Const}} . \qquad (2.18)$$

This expression for the first integral carries some economic interpretation from either a utility or production function standpoint. It corresponds to a Cobb-Douglas type utility/production function, where $\underline{x}$ can be viewed as a vector of input factors in production, $\underline{x}^*$ their equilibrium values, and $\underline{b}$ as the vector of their prices. Quantity expV from condition (2.2) is then the total value added. The returns to scale are equal to V since (2.2) holds. The stationary principle, to be elaborated in Section C below, thus may be critical in connecting ecology (and the natural sciences) to economics.

One of the peculiar features of Volterra's conservative associations is the fact that for the equilibrium to exist the association must contain an even number of different species ([8], p. 174). This fundamental difference between the ecological associations containing even and odd number of species is difficult to accept from an ecological viewpoint; because of this, Volterra's conservative ecological associations were critically considered by biologists. In Part II the conservative associations with zero growth will be analyzed; for these relative growth ecological associations the disturbing effect of even and odd number of species disappears. Before, however, entering this topic, we close Volterra's analysis by summarizing the findings regarding Volterra's variational principle.

C. The principle of stationary action for the multiple species conservative associations by Volterra.

Analogical to the single species case, Volterra considered a multiple species conservative association embedded within the integral (which we shall call the "cumulative action"):

$$E = \int_0^T G\,(X_i,\ \dot{X}_i,\ t)\ dt \tag{3.1}$$

where the integrand G (which we shall call the "current action") is given by:

$$G = \sum_i b_i\dot{X}_i \ln \dot{X}_i + \frac{1}{2} \sum_i \sum_j a_{ij}\dot{X}_i X_j + \sum_i a_i b_i X_i \tag{3.2}$$

where $X_i = \int_0^t x_i(t)\,dt$. Again, the element of cumulative entropy is present. The expression $\sum_i b_i\dot{X}_i \ln \dot{X}_i$ is Volterra's "total infinitesimal vital action"; thus, the current action (3.2) is divided into three parts, each connected correspondingly with: vital action (the entropy measure of the association), interaction, and demographic energy (the total quantity of life) at a single time period. This ecological interpretation of action can obtain a deeper meaning for an association under relative spatial growth conditions (see Part II).

In Appendix A, in a more expository manner than Volterra, we prove that the integrand G from (3.2) under integral E of (3.1) produce (2.1) as its Euler condition. The first order (Euler) condition defines the principle of stationary action, since:

$$\frac{\partial G}{\partial X_i} = \frac{d}{dt}\left(\frac{\partial G}{\partial \dot{X}_i}\right), \quad i = 1, 2, \ldots, I . \tag{3.3}$$

Having so apt an expression (3.1,2) for the cumulative action, Volterra constructed the corresponding Hamiltonian H and the canonical system of differential equations equivalent to (2.1). He used the canonical (co-state) variables X_i and p_i, where:

$$p_i = \frac{\partial G}{\partial \dot{X}_i}, \tag{3.4}$$

and introduced the Hamiltonian H in the form, with G given by (3.2):

$$H = \sum_i p_i\dot{X}_i - G . \tag{3.5}$$

The system (3.3) now has the following canonical form:

$$\frac{dX_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{\partial p_i}{\partial t} = - \frac{\partial H}{\partial X_i}, \quad i = 1,2,\ldots, I . \tag{3.6}$$

Expression (3.4) implies that:

$$p_i = \frac{\partial G}{\partial \dot{X}_i} = b_i(\ln \dot{X}_i + 1) + \frac{1}{2} \sum_j a_{ij}X_j \ , \tag{3.7}$$

from which we obtain:

$$\dot{X}_i = \exp \left[\frac{1}{b_i} (p_i - b_i - \frac{1}{2} \sum_j a_{ij}X_j) \right] \tag{3.8}$$

and, using (3.5),

$$\begin{aligned} H = & \sum_i (b_i \ln \dot{X}_i + b_i) \dot{X}_i + \frac{1}{2} \sum_i \sum_j a_{ij} X_j \dot{X}_i \\ & - \sum_i b_i \dot{X}_i \ln \dot{X}_i - \frac{1}{2} \sum_i \sum_j a_{ij}\dot{X}_i X_j - \sum_i a_i b_i X_i = \\ & = \sum_i b_i \dot{X}_i - \sum_i a_i b_i X_i \ , \end{aligned} \tag{3.9}$$

or

$$H = V + P. \tag{3.10}$$

Thus, the Hamiltonian H coincides with total demographic energy, and, therefore, the principle of the conservation of demographic energy is met since H = V + P = Const. H is the first integral of the system (3.3). In the canonical variables X_i, p_i, the Hamiltonian obtains the form, from (3.8):

$$H = V + P = \sum_i b_i \exp \frac{1}{b_i} (p_i - b_i - \frac{1}{2} \sum_j a_{ij}X_j) - \sum_i a_i b_i X_i \ . \tag{3.11}$$

D. The principle of "least vital action" for the multiple species conservative associations by Volterra.

In this section, a special case of multi-species interaction is presented (the case where the demographic work is zero.) We will draw from this special case for our inter-urban spatial dynamics in section C of Part II. Volterra's definition of the total vital action for a conservative multispecies association is the integral:

$$A = \int_0^T \left(\sum_{i=1}^{I} b_i \dot{X}_i \ln \dot{X}_i \right) dt \; . \tag{4.1}$$

The first and second variations of the vital action A will be (in a manner equivalent to the one shown in Appendix A):

$$\delta A = - \int_0^T \left(\sum_i b_i \frac{\ddot{X}_i}{\dot{X}_i} h_i \right) dt \tag{4.2}$$

and

$$\delta^2 A = \int_0^T \left(\sum_i \frac{b_i}{\dot{X}_i} \dot{h}_i^{\,2} \right) dt \tag{4.3}$$

where h_i are the variations of X_i such that $h_i(0) = h_i(T) = 0$. Let X_1, $X_2, \ldots, X_I$ be the quantities of life for each kind of species. Therefore the X_i's from (2.1) satisfy the system of equations:

$$\ddot{X}_i = \dot{X}_i (a_i + \frac{1}{b_i} \sum_j a_{ji} \dot{X}_j) = \alpha_i \dot{X}_i, \tag{4.4}$$

where:

$$\alpha_i = a_i + \frac{1}{b_i} \sum_j a_{ji} \dot{X}_i \tag{4.5}$$

are Volterra's "demographic coefficients" or "effective coefficients of increase" ([8], p. 239). The substitution of (4.4) into (4.2) gives the following expression for the first variation δA of the vital action A:

$$\delta A = - \int_0^T \{ \sum_i (a_i b_i + \sum_j a_{ji} \dot{X}_j) h_i \} dt = - \int_0^T (\sum_i b_i \alpha_i h_i) dt \tag{4.6}$$

while h_i are the variations of quantities of life X_i such that $h_i(0) = h_i(T) = 0$. The expression

$$\sum_i b_i \; \alpha_i \; h_i \tag{4.7}$$

is, by Volterra, the "work of growth", or the "virtual demographic work" for the variations $h_1, h_2, \ldots, h_I$.

Let us assume that for some infinitesimal variations of the quantities of life the virtual demographic work is equal to zero:

$$\sum_i (a_i b_i + \sum_j a_{ji} X_j)\, h_i = \sum_i b_i \alpha_i h_i = 0 \qquad (4.8)$$

then the first variation of the total vital action will be zero $\delta A = 0$.

Simultaneously, due to positivity of the "average weights" of species $b_i > 0$ and the positivity of populations, the second variation (4.3) of the total vital action is strongly positive $\left(\delta^2 A > 0.\right)$. The increment of the total vital action ΔA, due to the expression (3.5), will be

$$\Delta A = \delta A + \delta^2 A + \ldots > 0 \ . \qquad (4.9)$$

Thus, any infinitesimal variation $h_1, h_2, \ldots, h_I$ of the quantities of life $X_1, X_2, \ldots, X_I$ with zero virtual demographic work (4.8) will determine an increase of the total vital action (4.1). Thus, Volterra proposed an integral A to govern the evolution of the multiple species interaction over a time horizon which attains a minimum. This statement summarizes Volterra's main variational principle problems of least vital action for conservative ecological association with ecological dynamics as in (4.4).

Part II. Conservation Conditions in Urban Dynamical Systems.

In this Part we deal with relative growth dynamics, a problem Volterra never addressed. Equivalences are drawn between the absolute and relative growth which is defined under different conditions, in Section A. The necessary and sufficient conditions for the existence of an equilibrium in relative growth are demonstrated in Section B with their advantages over Volterra's conservative systems exposed. Finally, the variational principles and their entropic nature are provided in Section C, together with their interpretation for spatial systems.

A. Properties of spatial dynamic interaction.

Volterra's absolute growth multiple species ecology necessitates converting each species population to the conserved quantity V. This is done through the use of the b_i's (what Volterra called "weight equivalents"). As we mentioned in Part I this set of parameters can be interpreted as factor prices in economic production theory where different input factors are involved which are heterogeneous, for example, labor and capital. However, when dealing with human population distributed over space, or any other homogeneous geographical variable (income, capital, etc.) under relative growth conditions, the conserved quantity (whatever thay may be) does not need conversion factors. Whereas, Volterra avoided the consideration of particular conservative conditions (for example, [8], p. 170 on zero self-growth) partly because he dealt with the heterogeneity of biological species, there is no need for such restrictions to be imposed on urban spatial dynamics.

In what follows, we set up the urban relative spatial dynamics under various growth conditions. We demonstrate that spatial relative dynamics correspond to particular absolute growth ecological conservation dynamics of

Volterra. Then, we proceed to show their stability properties and discuss the competitive exclusion condition, a result applicable to all relative spatial dynamics. This discussion has direct implications upon the appropriateness of the differential equations assumed to hold for spatial dynamics; also upon the acceptance of a general "exclusionary principle," implying total agglomeration into a single site, in location theory. Finally, we propose a corresponding integrand (equivalent to Volterra's least vital action principle), and we show it to attain a maximum as it governs our interurban evolution. We show it to be a maximum cumulative entropy measure of the relative spatial interaction.

Assume a homogeneous geographical variable (population) distributed over different locations i ($i = 1,2,\ldots,I$) at any time period, t, so that the total population V is independent of time:

$$V = \sum_i y_i = \text{Const} > 0 \,. \tag{5.1}$$

Along Volterra's lines, V can be interpreted as the total "value of the distribution." Condition (5.1) is more appropriate for urban systems, as it is a less restrictive conservation condition than Volterra's definition of V. One can extend the current analysis by examining the case where V is a function of time. The special case of spatial relative dynamics where $V = 1$ will be referred to as <u>normalized</u> dynamics.

Volterra's absolute growth conservative ecology dynamics shown in (2.1) are now transformed into the relative growth urban (spatial) conservative dynamics by a system of ordinary differential equations:

$$\dot{y}_i = (a_i + \frac{1}{b_i} \sum_j a_{ji} y_j)\, y_i, \quad i = 1,2,\ldots,I \tag{5.2}$$

subject to (5.1). The validity of such an association for inter-urban dynamic spatial interaction rests on theoretical grounds, Dendrinos (with Mullally)

[5], as well as on empirical verification. On the theoretical front it identifies in its community matrix of coefficients a set of all possible geographies (predatory, competitive, cooperative, isolative, amensal, commensal) among various urban settings. Recent extensive empirical evidence seems to support the proposition that aggregate urban dynamics can be efficiently described by models drawing from mathematical ecology and population dynamics. Although these empirical findings are mostly reported for single city-nation interactions, Dendrinos (with Mullally) [5], testing of multi-urban interactions currently underway provides support for such modeling effort. These findings are reported in forthcoming papers, for example Dendrinos [4].

Two qualitative features of the model in (5.2) are well known: first, if the system has an equilibrium solution, it is unstable, or (at best) neutrally stable; second, for it to have a solution, certain conditions must hold connecting the model's coefficients in the community interaction matrix. We address next the qualitative properties of this model and their theoretical implications.

The analytical aspects of the model are shown in Appendix B, where, it is proven that the model's parameters must satisfy a special antisymmetry condition, under zero-self growth for it to have a solution. Empirical testing of validity of such inter-urban association allows, among other things, the examination of any correlation between spatial relative impedance (or relative accessibility) and the magnitude and/or sign of the interaction coefficients. On a theoretical level it enables us to detect (in case of a solution) the underlying integrands governing urban spatial relative dynamics. This antisymmetric property is very informative; it shows that, under specification (5.2), and when solution exists, spatial conservative dynamics are purely competitive. It also implies that there is no friction due to agglomeration in relative dynamics, to damp the dynamic equilibrium, thus producing exclusionary allocations.

B. Equilibrium states of relative spatial dynamics and discussion.

Although Volterra did not examine zero self-growth conservative associations, from the point of view of urban spatial dynamics these associations are insightful to model relative distribution dynamics. Under a relative framework internal growth and net migration are intertwined so that their combined effect is modeled. An important example of a zero self-growth normalized ecological association was first studied in the theory of temporal diffusion of competitive innovations, Sonis [10].

In the case of relative growth depicted by (5.2) the stability properties of the equilibrium differ significantly from Volterra's conservative ecological associations (2.1), (2.5): there is no need for an even number of species to interact for a stable equilibrium to exist. In agreement with Volterra, however, if a solution exists the equilibrium is stable only under competitive exclusion. This implies that only the concentration of the whole (homogeneous) geographical substance (e.g. population, capital, income, etc.) in one of the regions can be stable asymptotically. Asymptotical stability of the equilibrium state $(y_1^*, y_2^*, \ldots, y_I^*)$ means that, for any small perturbation, the perturbed state $(y_1, y_2, \ldots, y_I)$ exhibits the dynamic property:

$$\lim_{t \to \infty} y_i = y_i^*, \quad i = 1,2,\ldots,I .$$

Proof of this statement for spatial competitive exclusion is supplied in Appendix B.

Under relative growth conditions depicted by expression (5.2), at best neutrally stable solutions are obtained and in all likelihood only exclusive allocation of the homogeneous geographic variable onto one locale will result. Inter-urban interactions, viewed within the framework of a specific environment (the nation or a region) used to normalize the urban size, imply asymptotically stable total agglomeration of the geographic variable onto one site. This is the result of pure competition and absence of friction found in the antisymmetric properties of the interaction matrix. In modeling spatial dynamics in a relative framework, thus, fundamental instability is built into the system.

One by looking at the empirical evidence produced so far, Dendrinos (with Mullally) [5], finds stable patterns of spatial growth when cities are viewed in isolation and in reference to the nation as the environment over which their relative size is computed. This juxtaposition has certain implications: the relative community interaction matrix of (5.2) may be applicable for certain selective environments (i.e., regions), the U.S. as a whole not being one of them for its urban areas. In theory, there is no apparent reason why one cannot find an environment with respect to which cities could exhibit dynamics of the type found in (5.2). Further, the stability properties of particular spatial systems may vary as one changes the broader environment within which these systems are viewed. This may lead the spatial analyst in certain instances to formulating inter-urban interdependencies in a different manner than (5.2). For instance, one may wish to introduce stronger forms of inter-urban interconnectance by including cubic terms in the state variables, Sonis [10].

The information of an active environment can be accomplished with the help of a stochastic matrix $S = ||s_{ij}||$ which describes the process of redistribution of population among the various regions due to this intervention. This redistribution process acts in addition to the ecological dynamics present in the interaction matrix. The active environment smooths out the extreme action of the competitive exclusion principle and leads to a more balanced final asymptotically stable distribution of population among regions. Clearly, further extensive empirical search is needed to ascertain the validity of any of the above theoretical conjectures.

C. Variational principles for urban/regional relative growth dynamics.

In this section the main finding of the paper is presented, namely the derivation of an integral which governs the evolution of spatial relative dynamics as assumed in (5.2) when they possess a solution. It is closely related to spatial cumulative entropy from which it draws its interpretation. This theoretical implication supports the evidence of an entropy principle in spatial dynamics. Consider a normalized spatial dynamic system given by:

$$\dot{y}_i = \left(\sum_j b_{ij} y_j \right) y_i, \quad i = 1,2,\ldots,I \tag{6.1}$$

$$\sum_i y_i = 1 \tag{6.2}$$

with the antisymmetric matrix $B = (b_{ij})$: $b_{ij} = - b_{ji}$; $b_{ii} = 0$. Denoting as:

$$Y_i(t) = \int_0^t y_i(t)\, dt, \quad \dot{Y}_i = y_i, \ \ddot{Y}_i = \dot{y}_i , \tag{6.3}$$

one has the dynamical system:

$$\ddot{Y}_i = \dot{Y}_i \sum_j b_{ij} \dot{Y}_j . \tag{6.4}$$

$$\sum_i \dot{Y}_i = 1. \tag{6.5}$$

In this zero aggregate and self-growth conservative spatial dynamics the value of the whole association V is equal to one and the potential inter-urban demographic energy P is zero, (see (2.2) and (2.4). Therefore, one can use an appropriate modification of the Volterra integrand (3.2) for the construction of a variational problem (equivalent to his integral of cummulative action), which generates the system (6.4, 5) as its Euler condition. We propose the following integrand:

$$\phi = -2 \sum_i \dot{Y}_i \ln \dot{Y}_i + \sum_i \sum_j b_{ij} \dot{Y}_i Y_j \tag{6.6}$$

and the associated cumulative action

$$E' = \int_0^T \phi(Y_i, \dot{Y}_i)\, dt \tag{6.7}$$

we interpret as the urban fitness function. The variational principle of stationary cumulative action means that the first variation of the integral E' vanishes, giving rise to the system of Euler differential equations

$$\frac{\partial\phi}{\partial Y_i} = \frac{d}{dt}\left(\frac{\partial\phi}{\partial\dot{Y}_i}\right), \quad i = 1,2,\ldots,I \,. \tag{6.8}$$

Direct calculation gives

$$\frac{\partial\phi}{\partial Y_i} = \sum_j b_{ji} \dot{Y}_i \,, \tag{6.9}$$

$$\frac{\partial\phi}{\partial\dot{Y}_i} = -2(\ln \dot{Y}_i + 1) + \sum_j b_{ij} Y_j \,. \tag{6.10}$$

Therefore the integrand we propose through, (6.8) implies that through (6.9, 10) and the time derivative of (6.10) the original condition (6.4) is obtained, since:

$$\sum_j b_{ji} \dot{Y}_i = -2 \frac{\ddot{Y}_i}{\dot{Y}_i} + \sum_j b_{ij} \dot{Y}_j \tag{6.11}$$

whereas, the antisymmetry of the interaction matrix $B = (b_{ij})$ implies (6.5).

The stationary value of the cumulative action E' turns out to be the cumulative entropy for normalized spatial dynamic systems:

$$E' = \int_0^T \phi(Y_i, \dot{Y}_i, t)\, dt = \int_0^T [-2 \sum_i \dot{Y}_i \ln \dot{Y}_i + \sum_i \dot{Y}_i (\sum_j b_{ij} Y_j)]\, dt =$$

$$= \int_0^T [-2 \sum_i y_i \ln y_i + \sum_i y_i \int_0^t (\sum_j b_{ij}\, y_j)\, dt]\, dt =$$

$$= \int_0^T [-2 \sum_i y \ln y_i + \sum_i y_i \int_0^t \dot{y}_i/y_i\, dt]\, dt =$$

$$= \int_0^T [-2 \sum_i y_i \ln y_i + \sum_i y_i \ln y_i]\, dt =$$

$$= \int_0^T (- \sum_i y_i \ln y_i)\, dt \qquad (6.12)$$

Thus,

$$\int_0^T \phi\, (Y_i, \dot{Y}_i, t)\, dt = \int_0^T (- \sum_i y_i \ln y_i)\, dt \qquad (6.13)$$

and, therefore the "stationary cumulative action" for a normalized spatial distribution dynamics is the cumulative entropy of the population distribution during the time horizon T. This is our main finding.

Contrasting Volterra's integrand G, (3.2), in his conservative ecological dynamics, with our integrand ϕ in relative urban dynamics, one sees that the three terms in the ecological conservative systems (constituting total infinitesimal vital action) collapse in spatial dynamics into a single term; vital action, interaction and demographic energy merge into a single entity, namely cumulative spatial entropy.

It is important to point out that the first term of integrand ϕ which is Volterra's "infinitesimal vital action", represents the Shannon entropy of population distribution (Shannon [9], p. 396). Volterra naturally did not give such an interpretation of the "infinitesimal vital action," (or what we defined as "current vital action" earlier), as Shannon's work appeared in 1948 in the context of communication theory. The second term of integrand Φ in (6.6) represents the interaction between different parts of the homogeneous geographical substance (population). The analogue of the Volterra "least vital action" principle obtains for the relative growth dynamics the form of dynamic maximum cumulative entropy principle.

In general one cannot determine whether (6.12) has a minimum or a maximum. However, for a special case to be addressed below one can show that the stationary value of (6.12) attains a maximum. Let us consider first the integral (i.e., the cumulative entropy) for the general case (6.12) which can also be written as

$$E' = - \int_0^T \left(\sum_i \dot{Y}_i \ln \dot{Y}_i\right) dt \tag{6.14}$$

The first and second variations of the cumulative entropy E' are

$$\delta E' = \int_0^T \left(\sum_i \frac{\ddot{Y}_i}{\dot{Y}_i} h_i\right) dt \tag{6.15}$$

$$\delta^2 E' = - \int_0^T \left(\sum_i \frac{\dot{h}_i^2}{\dot{Y}_i}\right) dt \tag{6.16}$$

where h_1, h_2, ..., h_I are the variations of the cumulative populations, $Y_i = \int_0^t y_i \, dt$, such that $h_i(0) = h_i(T) = 0$ (see Appendix A). Since the cumulative populations Y_i satisfy the system (6.4, 5), the first variation is

$$\delta E' = \int_0^T \sum_i \left(\sum_j b_{ij} \dot{Y}_j\right) h_i dt \tag{6.17}$$

It is possible to interpret the expressions

$$\beta_i = \sum_j b_{ij} \dot{Y}_j \tag{6.18}$$

as "coefficients of population increase due to the interaction," or <u>relative</u>

migration (i.e., relative transfer of population) coefficients. In a similar manner the expression $\sum_i \beta_i h_i$ can be interpreted as the virtual work of urban growth due to the inter-urban interaction.

Next we draw from the special case of multispecies interaction by Volterra, when the virtual work of urban growth is zero to show that (6.12) obtains a maximum. If for some choice of the variations $h_1, h_2, \ldots, h_I$ of the cumulative populations $Y_1, Y_2, \ldots, Y_I$ condition (6.3) is equal to zero, $\sum_i \beta_i h_i = \sum_i (\sum_j b_{ij} \dot{Y}_j) h_i = 0$, then the first variation (6.17) of cumulative entropy is equal to zero $\delta E' = 0$, and, simultaneously, due to the positivity of the populations $y_i = \dot{Y}_i > 0$, the second variation (6.16) of the cumulative entropy is strongly negative, $\delta^2 E' < 0$.

Therefore, for the same variations h_i which imply the vanishing of the virtual work of urban growth due to inter-urban interaction the increment of the cumulative entropy

$$\Delta E' = \delta E' + \delta^2 E' + \ldots \tag{6.20}$$

is strongly negative, which implies a decrease in the cumulative entropy, q.o.d.

In conclusion, a dynamic <u>maximum</u> entropy problem is uncovered to generate the relative dynamic urban model of spatial adaptation of a homogeneous geographical substance (population). This fitness function is equivalent to a least effort principle found in Volterra's ecology.

A final remark on the postulated integrand for spatial associations: the proposed integrand must not necessarily be unique. Either a class of equivalent integrands can possibly exist, the one proposed here being merely their canonical form; or quite different ones might also produce the same result. For a case in point see Gelfand and Fomin [6], p. 36. This, of course, would imply that there are multiple objective functions which can produce an adaptation path. These are potentially interesting questions for future theoretical speculation and empirical research.

D. The Hamiltonian of urban conservative relative dynamics.

We now turn to the search for a possible governing Hamiltonian of the normalized spatial dynamics. Introducing the co-state variables:

$$p_i = \frac{\partial \Phi}{\partial \dot{Y}_i}, \quad i = 1,2,\ldots,I, \tag{7.1}$$

and the new Hamiltonian:

$$H = \Phi - \sum_i p_i \dot{Y}_i \tag{7.2}$$

one can obtain:

$$\dot{Y}_i = \exp\left[-\frac{1}{2}(p_i - \sum_j b_{ij} Y_j) - 1\right] \tag{7.3}$$

and the canonical system of differential equations:

$$\frac{dY_i}{dt} = -\frac{\partial H}{\partial p_i}; \quad \frac{dp_i}{dt} = \frac{\partial H}{\partial Y_i}. \tag{7.4}$$

Stating the Hamiltonian H as a function of the co-state variables, due to (7.1, 2) we obtain:

$$H = -2 \sum_i \dot{Y}_i \ln \dot{Y}_i + \sum_i \sum_j b_{ij} \dot{Y}_i Y_j - \sum_i \dot{Y}_i(-2 \ln \dot{Y}_i - 2 + \sum_j b_{ij} Y_j) =$$

$$= 2 \sum_i \dot{Y}_i = 2 \sum_i \exp\left[-\frac{1}{2}(p_i - \sum_j b_{ij} Y_j) - 1\right]. \tag{7.5}$$

Due to the condition $\sum_i \dot{Y}_i = 1$ the value of the Hamiltonian is $H = 2$ on the extremals Y_i, and, therefore, it is a first integral of the normalized spatial dynamics.

It is interesting to note that one can easily obtain from (2.17) the first integral for the normalized spatial dynamical system as a Cobb-Douglas (production) function $\prod_i y_i^{y_i^*}$, where y_i^*, $i = 1,2,\ldots,I$, is some equilibrium state of the system. This expression is also similar to the objective function of a geometric programming problem.

As it was mentioned earlier, Volterra found the sufficient conditions for the presentation of the solution of system (2.1) in quadratures. In the case of the normalized spatial dynamics, the analogical condition allows us to obtain the explicit formulas of solutions in elementary functions. This allows us to draw some links between spatial dynamics and the logit model. It is done in Appendix C. Here, one may note the following:[$] condition (7.3) can also be written as

$$\dot{Y}_i = \exp\left((-\frac{1}{2}p_i + 1) + \frac{1}{2}\sum_j b_{ij}Y_j\right) / \sum_k \exp\left((-\frac{1}{2}p_k + 1) + \frac{1}{2}\sum_j b_{kj}Y_j\right) . \tag{7.6}$$

Expression (7.6) corresponds to the logit model of random utility choice with utility functions

$$U_i = (-\frac{1}{2}p_i + 1) + \frac{1}{2}\sum_j b_{ij}Y_j . \tag{7.7}$$

These U contain separable effects; one is a negative association with p_i and the other is a positive association with b_{ij} and Y_j.

One may wish to interpret p_i as the relative marginal cost of fitness for urban setting i's population (the equivalent to shadow prices in micro-economics). At equilibrium $U_i = \bar{U} = \frac{1}{2}H$, so that the Hamilitonian can be

[$]We thank Giorgio Leonardi from IIASA for pointing to us this rewriting of (7.9), since the denominator equals one by definition.

viewed as a relative utility function. Utility thus increases as the marginal cost of fitness declines, and increases as the relative inter-urban interaction ($\sum_j b_{ij} Y_j$) increases. In view of this, the interpretation of the relative fittness function Φ is insightful: it is the sum of all urban areas' current relative fitness level. E' is the cumulative total fitness of the community of urban areas over a time horizon T. Fitness is such that inter-urban interaction maximizes the cumulative entropy of the association. (cumulative action.) The term $\sum_i p_i y_i$ represents the value of effort to adapt in all settings, so that from (7.2) the current fitness level is the net sum of the utility level enjoyed and the total cost of adaptation (the effort to adapt).

One may ponder planning (decentralized control) and aggregate social welfare aspects of the proposed Hamiltonian. It must be kept in mind that this is a relative growth model and the implied controls are much more comprehensive (over space, functions and agents) than those in welfare economics. Thus, although the scheme may be similar to welfare maximization problems, the ecological base of the model proposed makes any practical (i.e., specific) use of it restrictive. Ecological models, and this is a main message of this line of work, present a different perspective on policy making, a subject more fully addressed in [5]. They provide the final outcomes from a complex and broader interplay of actions among a very large number of producers, consumers and governments.

Conclusions.

Volterra's original absolute growth conservation conditions and variational principles were reconsidered for one and multiple species ecology. They were contrasted with more specific conservation conditions for relative growth and spatial distribution in urban systems and appropriate equivalences and substantive interpretations were drawn. Volterra's stationary action multiple species integrand $G = \sum_i b_i \dot{X}_i \ln X_i + \frac{1}{2} \sum_i \sum_j a_{ij} \dot{X}_i X_j + \sum_i a_i b_i X_i$, condition (3.2), was found to have a corresponding one in relative spatial urban dynamics under pure competition of the form: $\Phi = 2 \left(- \sum_i \dot{Y}_i \ln \dot{Y}_i + \frac{1}{2} \sum_i \sum_j b_{ij} \dot{Y}_i Y_j\right)$, condition (6.6). Furthermore, we proved that the corresponding integral produces in our case $E' = \int_0^T \left(- \sum_i y_i \ln y_i\right) dt$, i.e., a stationary cumulative entropy measure, condition (6.13) of spatial dynamics.

The stationary maximum cumulative entropy integral shown to govern and produce as its solution the urban relative spatial dynamics was viewed as one (among possibly many) adaptation (fitness) function in spatial dynamics equivalent to a least effort principle in Volterra's absolute growth conservative ecological associations. The detailed comparison of Volterra's fifty year old studies involving three different classes of ecological problems with recent developments in ecological urban dynamics was shown to lend new insights toward a deeper understanding of spatial dynamic processes. In particular, basic conditions resulting in dynamic instability (competitive exclusion implying concentration of population in a single area), or neutral stability in relative urban evolution were provided and contrasted with the stable motion of urban relative dynamics viewed in isolation within the nation's environment. It set the framework for modeling multi-urban interactions.

This theoretical framework provides for empirical testing, a task which on single relative growth dynamics was partly carried out in [5]; multi-urban interactions in a relative growth framework are dealt with in forthcoming papers, example [4], where it is shown that each particular form of an inter-urban interaction matrix generates particular unstable hierarchical dynamics.

References

1. L. Curry, 1981, "Division of Labor from Geographical Competition," Annals of the Association of American Geographers, Vol. 71, No. 2: 133-165.

2. D.S. Dendrinos, 1979, "A Basic Model of Urban Dynamics Expressed as a Set of Volterra-Lotka Equations," in D.S. Dendrinos, Catastrophe Theory in Urban and Transportation Analysis, Report # DOT/RSPA/DPB-25/80/2, U.S. Department of Transportation, Washington, D.C., 1980.

3. D.S. Dendrinos, H. Mullally, 1981, "Evolutionary Patterns of Urban Populations," Geographical Analysis, Vol. 13, No. 4: 328-344.

4. D.S. Dendrinos, 1984, "Nonlinear Dynamics of Urban Hierarchies; the U.S. Mountain Region (1890-1980) "Paper to be presented at the NATO Advanced Summer Institute, 1985 Germany. Report SES-821-6620 National Science Foundation (forthcoming).

5. D.S. Dendrinos (with H. Mullally), 1984 Urban Evolution: Studies in the Mathematical Ecology of Cities, Oxford University Press (in press).

6. I.M. Gelfand, S.V. Fomin, 1963 Calculus of Variations, revised English edition, translated and edited by R.A. Silverman, Prentice-Hall.

7. G.A. Korn, T.M. Korn, 1961, Mathematical Handbook for Scientists and Engineers, McGraw-Hill, New York, Toronto, London.

8. F.M. Scudo, J.R. Ziegler (eds), 1978 The Golden Age of Theoretical Ecology: 1923-1940, Springer-Verlag, Lecture Notes in Biomathematics, Vol. 22.

9. C. Shannon, 1948, "A mathematical theory of communication", Bell System Technical Journey, Vol. 27, No.3:379-423; No.4:623-656.

10. M. Sonis, 1983, "Competition and Environment - A Theory of Temporal Innovation Diffusion," in D. Griffith, A Lea (eds) Evolving Geographic Structures, Martinus Nijhoff.

11. M. Sonis, 1984, "Dynamic choice of alternatives, innovation diffusion and ecological dynamics of Volterra-Lotka", London Papers in Regional Science, Vol. 14 (in press).

APPENDIX A

In order to fully capture the richness of Volterra's formulation, we go back to certain basic principles in the calculus of variations problems, for example Gelfand and Fomin [6]. The variation h(t) of function X(t) is the difference between a new function $\tilde{X}(t)$ and X(t) such that:

$$\delta X = h(t) = \tilde{X}(t) - X(t) \ . \qquad (A.1)$$

We replace in the integral E all functions $X_i(t)$ by the "varied" functions $X_i(t) + h_i(t)$ where $h_i(0) = h_i(T) = 0$, $i = 1,2,\ldots I$, and construct the increment $\Delta E(h_i) = E(X_i + h_i, \dot{X}_i + \dot{h}_i) - E(X_i, h_i)$. In the case of (3.1,2) the increment is:

$$\Delta E(h_i) = \int_0^T [G\,(X_i + h_i, \dot{X}_i + \dot{h}_i) - G(X_i, \dot{X}_i)]\ dt \qquad (A.2)$$

and it is possible to find, applying Taylor's formula, that:

$$\Delta E(h_i) = \int_0^T [\ \sum_i (\frac{\partial G}{\partial X_i} h_i + \frac{\partial G}{\partial \dot{X}_i} h_i) +$$

$$+ \frac{1}{2} \sum_i \sum_j (\frac{\partial^2 G}{\partial X_i \partial X_j} h_i h_j + \frac{\partial^2 G}{\partial \dot{X}_i \partial X_j} h_i \dot{h}_j + \frac{\partial^2 G}{\partial \dot{X}_i \partial \dot{X}_j} \dot{h}_i \dot{h}_j) + \ldots]\ dt \ . \qquad (A.3)$$

We now form the first and second variations of the integral E:

$$\Delta E = \delta E + \delta^2 E + \ldots \qquad (A.4)$$

The first variation is:

$$\delta E\,(X_i,\ h_i) = \int_0^T \Big[\ \sum_i \Big\{\frac{\partial G}{\partial X_i}\, h_i + \frac{\partial G}{\partial \dot{X}_i}\, \dot{h}_i\Big\}\Big]\ dt =$$

$$= \int_0^T \Big[\ \sum_i \Big\{\frac{\partial G}{\partial X_i} - \frac{d}{dt}\,\frac{\partial G}{\partial \dot{X}_i}\Big\}\, h_i\Big]\ dt \qquad \text{(A.5)}$$

and the second variation:

$$\delta^2 E(X_i,\ h_i) = \int_0^T \Big[\ \sum_i \sum_j \Big\{\frac{\partial^2 G}{\partial X_i\, \partial X_j}\, h_i h_j + \frac{\partial^2 G}{\partial X_i\, \partial \dot{X}_j}\, h_i \dot{h}_j + \frac{\partial^2 G}{\partial \dot{X}_i\, \partial \dot{X}_j}\, \dot{h}_i \dot{h}_j\Big\}\ \Big]\ dt \qquad \text{(A.6)}$$

A necessary condition for the integral E to have an extremum (maximum or minimum) is that its first variation δE vanish for all admissible variations h_i (i.e., $h_i(0) = h_i(T) = 0$):

$$\delta E = 0 \qquad \text{(A.7)}$$

which is equivalent to the fact that the functions $X_i(t)$ satisfy the Euler equations

$$\frac{\partial G}{\partial X_i} = \frac{d}{dt}\Big(\frac{\partial G}{\partial \dot{X}_i}\Big)\ , \quad i = 1,2,\ldots,I\ . \qquad \text{(A.8)}$$

The above conditions (A.8) are necessary but not sufficient for the existence of an extremum; they define the principle of stationary action. In other words, the actual trajectory of Volterra's conservative ecological association does not minimize the cumulative action E but only causes its first variation to vanish. Direct calculation provides immediately the equivalence between the system in (3.7) and the system(2)

$$a_i b_i + \frac{1}{2} \sum_j a_{ji} \dot{X}_j = b_i \ddot{X}_i / \dot{X}_i + \frac{1}{2} \sum_j a_{ij} \dot{X}_j . \tag{A.9}$$

The antisymmetry of the interaction matrix $A = (a_{ij})$ and appropriate substitutions prove that the Euler equations (A.8) are Volterra's dynamic equations (2.1).

A final analytical note on system (2.1): the Volterra differential equations is a non-linear one; therefore, it is usually impossible to reduce the integration of non-linear systems to quadratures. Given the special form (3.1), however, Volterra was able ([8] p. 256) to give conditions for the integration of (2.1) in quadratures of the form:

$$\frac{a_{ij}}{a_i a_j b_i b_j} + \frac{a_{jk}}{a_j a_k b_j b_k} + \frac{a_{ki}}{a_k a_i b_k b_i} = 0 \quad \text{for each } i,\ j,\ k . \tag{A.10}$$

But even this simple, and very restrictive, analytical form for the integrability in quadratures did not allow him to construct the explicit solution for the system (2.1) in elementary functions. It is shown, in Part II, that the case of relative growth allows for explicit solutions to be obtained in elementary functions. Further, they give the model of generalized logistic growth (Sonis [10], p. 117). Moreover, it is be shown that conditions (A.10) are only sufficient (but not necessary) for the integrability in quadratures, Appendix C.

(2)Note an error regarding the integrand on p. 240 of [8]: the parenthesis in equation C', last term.

APPENDIX B

The system (5.2) with condition of zero aggregate growth (5.1) is equivalent to the system

$$\dot{y}_i = y_i \sum_j b_{ij} y_j \ , \quad i = 1,2,\ldots,I \tag{B.1}$$

$$\sum_i y_i = V = \text{Const} > 0 \tag{B.2}$$

where

$$b_{ij} = \frac{a_i}{V} + \frac{a_{ji}}{b_i} \tag{B.3}$$

It is possible to interpret the above system as a special case of relative spatial growth dynamics with zero self growth $a_i = 0$, $i = 1,2,\ldots,I$, ("average weight" of species equal to unity: $b_i = 1$, $i = 1,2,\ldots,I$, and interaction coefficients $b_{ij} = - a_{ij}$ in an analogical to Volterra's system). Thus, the relative spatial growth model includes both zero aggregate (regional) growth (V = Const) and zero individual city growth ($a_i = 0$). We will call such dynamics "zero growth dynamics."

The introduction of new variables

$$z_i = y_i/V \tag{B.4}$$

and new interaction coefficients

$$b_{ij} = a_i + \frac{a_{ji}V}{b_i} \tag{B.5}$$

result in a zero growth normalized dynamics:

$$\dot{z}_i = z_i \sum_j b_{ij} z_j \tag{B.6}$$

$$\sum_i z_i = 1 \ . \tag{B.7}$$

It is important to note that a zero growth (normalized or not) relative spatial dynamics can be linked directly to Volterra's conservative ecological associations: the condition of zero aggregate growth (5.1) is equivalent to

$$\sum_i \sum_j b_{ij} y_i y_j = 0 , \tag{B.8}$$

i.e., to Volterra's conservation condition. This follows immediately from the equality:

$$\sum_i \sum_j b_{ij} y_i y_j = \sum_i y_i \left(\sum_j b_{ij} y_j\right) = \sum_i \dot{y}_i = \frac{dV}{dt} . \tag{B.9}$$

We now further analyze the zero growth relative spatial dynamics:

$$\dot{y}_i = y_i \sum_j b_{ij} y_j , \quad i = 1,2,\ldots,I$$

$$V = \sum_i y_i \quad (V > 0) .$$

We prove that in the above system the matrix $B = (b_{ij})$ is antisymmetric, a fact with significant interpretational implications outlined in the main text. By introducing the condition:

$$y_I = V - \sum_{i=1}^{I-1} y_i \tag{B.10}$$

variables $y_1, y_2, \ldots, y_{I-1}$ become independent. Due to (B.8) one can derive the following:

$$\begin{aligned} 0 = \sum_i \sum_j b_{ij} y_i y_j &= \sum_{i=1}^{I-1} \sum_{j=1}^{I-1} (b_{ij} + b_{ji} - b_{iI} - b_{Ii} \\ &- b_{jI} - b_{Ij} - 2b_{II}) y_i y_j + \sum_{i=1}^{I-1} b_{ii} y_i^2 \\ &+ V \sum_{i=1}^{I-1} (b_{iI} + b_{Ii} - 2b_{II}) y_i + V^2 b_{II} \end{aligned} \tag{B.11}$$

identifying a polynomial of second degree in the independent variables $y_1, y_2, \ldots, y_{I-1}$. Condition (B.11) holds as an identity, thus requiring that all its coefficients must be zero. This, in turn, implies that:

$$V^2 \, b_{II} = 0 \; ; \quad b_{ii} = 0 \; , \quad i = 1,2,\ldots, I-1 \tag{B.12}$$

$$V \, (b_{iI} + b_{Ii} - 2b_{II}) = 0 \; , \quad i = 1,2,\ldots, I-1 \tag{B.13}$$

$$(b_{ij} + b_{ji}) - (b_{iI} + b_{Ii}) - (b_{jI} + b_{Ij}) - 2b_{II} = 0, \quad i = 1,2,\ldots,I-1. \tag{B.14}$$

Since $V > 0$, then:

$$b_{ij} = - b_{ji}, \quad b_{ii} = 0, \quad i = 1,2,\ldots,I \tag{B.15}$$

which are the antisymmetry conditions for matrix $B = (b_{ij})$. Inspite the similarities between the above and Volterra's models, nonetheless, there is a big difference in the equilibrium properties: all peculiar conditions associated with the existence of an equilibrium found in absolute ecological dynamics are absent in the case of relative spatial growth. This is proven below.

The equilibrium states y_i^*, $i = 1,2,\ldots,I$, of the dynamical system in (B.1) with the antisymmetric interaction matrix $B = (b_{ij})$ are the solutions of the following system of equations:

$$y_i^* \sum_{j=1}^{I} b_{ij} y_j^* = 0 \; , \quad i = 1,2,\ldots,I \; , \tag{B.16}$$

$$\sum_{j=1}^{I} y_j^* = V \quad (V > 0) \; . \tag{B.17}$$

A complete description of all possible types of equilibrium states, are analyzed next.

It is easy to see (by substitution) that the simplest solutions of the system are I different solutions of the type:

$$y_1^* = y_2^* = \dots = y_{r-1}^* = 0,\ y_r^* = V,\ y_{r+1}^* = \dots = y_I^* = 0,\ r = 1,2,\dots,I\ . \quad (B.18)$$

These equilibrium states represent the competitive exclusion of species or the total concentration of whole geographical substance within the r-th region ($r = 1,2,\dots,I$). Further, let us consider the equilibrium states with only r ($2 < r < I$) non-zero coordinates. For each r there exist not more than $\binom{r}{I}$ such equilibrium states. Without loss of generality, one can assume that:

$$y_1^*,\ y_2^*,\ \dots\ y_r^* > 0,\ y_{r+1}^* = \dots = y_I^* = 0\ . \quad (B.19)$$

For this type of equilibrium the system (B.16) will be:

$$\sum_{j=1}^{r} b_{ij}y_j^* = 0, \quad i = 1,2,\dots,r, \quad (B.20)$$

$$\sum_{j=1}^{r} y_j^* = V\ . \quad (B.21)$$

The matrix B_r of the linear system (B.20) is an antisymmetric $r \times r$ matrix. If its determinant is non-zero, then the system has only a zero solution which contradicts condition (B.21). This is the difference between Volterra's and our spatial relative growth conservative dynamics. This possibility holds only for even r, due to the antisymmetry of the matrix B_r. But for even r it is possible that the determinant det B_r will degenerate to zero, the necessary condition for existence of equilibrium. In the case of odd r, the determinant of the system (B.20), due to antisymmetry or B_r, is equal to zero automatically. In any case, if the determinant of the system equals zero:

$$\det B_r = 0 \, , \tag{B.22}$$

then the system (B.20,21) has a solution

$$y_j^* = \frac{V \, A_{rj}}{\sum\limits_{k=1}^{r} A_{rk}} \, , \quad j = 1,2,\ldots,r \tag{B.23}$$

where A_{rj} are the algebraic complements (or co-factors), for the elements a_{rj} in the r-th row of the matrix B_r. This statement follows from the well-known property of co-factors (Korn and Korn [7], 1.5-2):

$$\sum_{j=1}^{r} b_{ij}A_{rj} = \begin{cases} \det B_r, & i = r \\ 0 & , \; i \neq r \end{cases} \tag{B.24}$$

which in our case (B.22) is

$$\sum_{j=1}^{r} b_{ij}A_{rj} = 0 \, , \quad i = 1,2,\ldots,r \, . \tag{B.25}$$

The conservation condition (B.21) obviously holds for y_i^* from (B.23). Further, for the same y_j^*

$$\sum_{j=1}^{r} b_{ij}y_j^* = \sum_{j=1}^{r} b_{ij} \frac{V \, A_{rj}}{\sum\limits_{k=1}^{r} A_{rk}} = \frac{V}{\sum\limits_{k=1}^{r} A_{rk}} \sum_{j=1}^{r} b_{ij}A_{rj} = 0 \, . \tag{B.26}$$

Now it will be shown that only the competitive exclusion equilibrium (B.18) can be stable asymptotically. The original system of differential equations (B.1) can be rewritten, after the substitution:

$$y_r = V - \sum_{j \neq r} y_j \, , \tag{B.27}$$

is made, in the form:

$$\dot{y}_i = y_i \left[\sum_{j \neq r} (b_{ij} - b_{ir}) \, y_j + V \, b_{ir} \right] = f_i, \; i \neq r \, . \tag{B.28}$$

It is well-known (Korn and Korn [7], 9.5-4), that the equilibrium states for such a system are stable asymptotically if, and only if, all eigenvalues of the matrix $L = \left(\frac{\partial f_i}{\partial y_j} \Big|_{\substack{y_k^* \\ (k \neq r)}} \right) = (l_{ij})$ have negative real parts. In our case

$$\frac{\partial f_i}{\partial y_j} = y_i \, (b_{ij} - b_{jr}) \ , \quad i \neq j \ , \tag{B.29}$$

$$\frac{\partial f_i}{\partial y_i} = \sum_{j \neq r} (b_{ij} - b_{ir}) \, y_j + V \, b_{ir} + y_i \, (b_{ii} - b_{ir}) =$$

$$= \sum_{j=1}^{I} b_{ij} y_j - y_i b_{ir} \ , \tag{B.30}$$

and, therefore, for the equilibrium states (B.18) the matrix L has the form of a diagonal (I-1) x (I-1) matrix with its diagonal elements Vb_{ir} ($i \neq r$):

$$L = \begin{bmatrix} Vb_{1r} & 0 & \dots & 0 \\ 0 & Vb_{2r} & \dots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & \dots & Vb_{I,r} \end{bmatrix} . \tag{B.31}$$

The eigenvalues of the matrix L are its diagonal elements, and, therefore, the condition for asymptotical stability of the competitive exclusion equilibrium are that $Vb_{ir} < 0$, $i \neq r$, or :

$$b_{ir} < 0, \quad i = 1,2,\dots, r-1, r+1,\dots, I \ . \tag{B.32}$$

This means that the competitive exclusion equilibrium (B.18) is asymptotically stable if, and only if, all non-diagonal elements of the r-th column of the antisymmetric interaction matrix $B = (b_{ij})$ are strongly negative.

It follows immediately from the antisymmetry of the interaction matrix B that from I different competitive exclusion equilibrium states (B.18) only one can be asymptotically stable since the negativity of the r-th column of the matrix B implies the positivity of the r-th row, i.e., the matrix B can not include two different negative columns.

Further, for the equilibrium states of the type (B.19) with r non-zero coordinates (B.23) matrix L has components from (B.29, 30):

$$l_{ij} = (b_{ij} - b_{ir})\, y_i^* = \begin{cases} 0 & i > r \\ \dfrac{V A_{ri}}{\sum\limits_{k=1}^{r} A_{rk}} (b_{ij} - b_{ir}) & i < r \end{cases} \quad . \tag{B.33}$$

The sum of the real parts of the eigenvalues for the matrix L is the trace $\sum\limits_{j\neq r} l_{jj}$ of the matrix L. Due to (B.25), this trace is equal to zero:

$$\mathrm{Tr}L = \sum_{j\neq r} l_{jj} = \frac{V}{\sum\limits_{k=1}^{r} A_{rk}} \sum_{j=1}^{r} b_{rj}A_{rj} = 0 \ . \tag{B.34}$$

This means that the sum of the real parts of the eigenvalues is equal to zero, which is possible if either the real part of each eigenvalue is equal to zero, or there exist eigenvalues with strongly positive real parts. In both cases the equilibrium states of the type (B.23) are asymptotically unstable. In the latter case the equilibrium is unstable, while the former case of pure imaginary eigenvalues implies periodic motion.

APPENDIX C

The sufficient conditions for the integration of the normalized spatial dynamical system in elementary functions are:

$$b_{ij} + b_{jk} + b_{ki} = 0 \quad \text{for each } i,j,k \,. \tag{C.1}$$

These conditions together with the antisymmetry of the interaction matrix $B = (b_{ij})$ allow the construction of the system's explicit solution in the following way (Sonis [10], p. 116):

$$\frac{d}{dt} \ln \frac{y_i}{y_k} = \frac{\dot{y}_i}{y_i} - \frac{\dot{y}_k}{y_k} = \sum_j (b_{ij} - b_{ik})\, y_j = \sum_j b_{ik} y_j = b_{ik} \;; \tag{C.2}$$

then

$$\ln \frac{y_i}{y_k} = b_{ik} t + C_1 \,, \tag{C.3}$$

where C_1 is a constant. Therefore,

$$\frac{y_i}{y_k} = C \exp b_{ik} t \qquad \left(C = \exp C_1 = \frac{y_i(0)}{y_k(0)} \right) . \tag{C.4}$$

Thus,

$$y_i = y_k \frac{y_i(0)}{y_k(0)} \exp b_{ik} t \,. \tag{C.5}$$

One can obtain, due to the above condition:

$$1 = \sum_j y_j = \frac{y_k}{y_k(0)} \sum_j y_j(0) \exp b_{jk} t \,, \tag{C.6}$$

or

$$y_k = y_k(0) \Big/ \sum_j y_j(0) \exp b_{jk} t \,. \tag{C.7}$$

Therefore, from (C.5), for each fixed k

$$y_i = y_i(0) \exp b_{ik}t \Big/ \sum_j y_j(0) \exp b_{jk}t \ , \quad i = 1,2,\ldots,I \ . \qquad (C.8)$$

The above expression describes the generalized logistic growth and represents a temporal extension of the well-known logit random utility choice model (Sonis [11]). This extension lies at present in the inner core of the unification of ideas of urban dynamics, innovation diffusion and individual's choice behavior.

Condition (C.1) is only a sufficient but not necessary condition for the integration in quadratures. For proving this one can consider the following normalized spatial dynamic systems

$$\dot{y}_1 = y_1 \, (y_2 + y_3) \qquad (C.9.1)$$

$$\dot{y}_2 = y_2 \, (- y_1 - y_3) \qquad (C.9.2)$$

$$\dot{y}_3 = y_3 \, (- y_1 + y_2) \qquad (C.9.3)$$

$$y_1 + y_2 + y_3 = 1 \qquad (C.9.4)$$

with the antisymmetric interaction matrix

$$B = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \qquad (C.10)$$

For this matrix condition (C.1) is not true because

$$b_{12} + b_{23} + b_{31} = 1 - 1 - 1 = - 1 \neq 0 \ , \qquad (C.11)$$

but the substitutions:

$$y_2 + y_3 = 1 - y_1 \ , \qquad - y_1 - y_3 = y_2 - 1 \qquad (C.12)$$

convert the system (C.9.1-4) into the system

$$\dot{y}_1 = y_1 (1 - y_1) \tag{C.13.1}$$

$$\dot{y}_2 = - y_2 (1 - y_2) \tag{C.13.2}$$

$$y_3 = 1 - y_1 - y_2 \tag{C.13.3}$$

with an obvious solution

$$y_1 = 1 / 1 + C_1 e^{-t} , \tag{C.14.1}$$

$$y_2 = 1 / 1 + C_2 e^{t} , \tag{C.14.2}$$

$$y_3 = (C_1 C_2 - 1) / (1 + C_1 e^{-t}) (1 + C_2 e^{t}) . \tag{C.14.3}$$