RR-75-3

DECISION THEORETICAL REMARK ON SENSITIVITY ANALYSIS

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February 1975

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Abstract

The sensitivity of dynamic systems to changes of the parameter values can be estimated by two first order methods: either by solving the sensitivity system, or by solving the original system twice with slightly different parameter values. In this paper the problem investigated is which method should be preferred in order to minimize losses due to deviations of those first order approximations from the actual sensitivity. It is shown that the method using two different parameter values is to be preferred if one is to find out whether the solution of the dynamic system could deviate from the nominal solution by more than a prescribed standard. A problem of optimal control is described for which the other method turned out to be preferable. Some other problems are mentioned for which the preferability of one of the two methods could be proved.

I. Introduction

The changes of the state variables of a dynamic system

$$\frac{dx}{dt} = f(x,t,p) \qquad \text{(x state vector, t time, f vector-valued function, p parameter)} \tag{1}$$

due to deviations of the parameter value from the "nominal" value p_0 can be estimated by two first order methods [1]:

A. One can compute the functions

$$s_1(t) = \frac{\partial x}{\partial p}\Big|_{p = p_0} \tag{2}$$

which are the solution of the so-called sensitivity system

*The helpful discussions with S. Rinaldi regarding this problem are gratefully acknowledged.

$$\frac{ds_1}{dt} = \frac{\partial f}{\partial x}\Big|_{p = p_0} \cdot s_1 + \frac{\partial f}{\partial p}\Big|_{p = p_0} \qquad (3)$$

$$s_1(0) = \frac{\partial x(0,p)}{\partial p}\Big|_{p = p_0} ,$$

where $\frac{\partial f}{\partial x}$ is the Jacobian matrix of f.
System (3) is a linear system, whose solution requires the solution of the original system (1).

B. One can also solve the original system twice with the parameter values p_0 and $p_0 + \Delta p$ and consider the function

$$s_2(t) = \frac{x(t,p_0+\Delta p) - x(t,p_0)}{\Delta p} \qquad (4)$$

as an estimate for the sensitivity of x with respect to p. The value of Δp is normally related in some way to the expected variations of p.

With both methods p may be also an initial value of an x-component. In the following we refer to the first method as "differential sensitivity analysis" and to the second one as "finite sensitivity analysis". The criterion for choosing between the two methods is usually the computational effort. Often differential sensitivity is preferred because the linearity of the sensitivity system allows general statements to be derived without numerical calculation. Another important aspect, which seems not to have been taken into account so far, is the losses connected with wrong predictions of the methods about system sensitivity. In the following both methods are considered only with regard to this aspect.

If, for instance, we evaluate the differential sensitivity (2) of the system

$$\begin{cases} \dot{x}_1 = \left(\dfrac{x_2}{p + x_2} - 0.2 - 7.021\, x_1\right) \cdot x_1 \\ \dot{x}_2 = (1 - 0.5\, x_2 - 0.6\, x_3) \cdot x_2 - \dfrac{x_2}{p + x_2} \cdot x_1 \\ \dot{x}_3 = (1 - 0.6\, x_2 - 0.5\, x_3) \cdot x_3 \end{cases} \tag{5}$$

with respect to the parameter p we get the functions which are shown in Figure 1 together with the nominal solution of System (5) for $p_0 = 0.97$.[1] The initial values are

$$x_1(0) = 0.002 \quad , \qquad x_2(0) = 1.0 \quad , \qquad x_3(0) = 0.9 \quad .$$

If we actually increase the parameter p by 5%, however, we get the solution shown in Figure 2. It shows that the differential sensitivity would lead to a completely wrong prediction about the model behavior in the case of "small" parameter changes. But examples could be given where finite sensitivity is similarly misleading. The problem arises which method should be preferred in order to minimize losses due to such cases. The result depends on what conclusions are drawn from the results of the sensitivity analysis.

II. The Problem of Maximum Permissible Deviation

A common problem for sensitivity analysis is to find out whether the solution of a System (1) can deviate from the nominal solution by more than a prescribed standard A if the parameter can vary over a certain range $(p_0 - \Delta p_0, p_0 + \Delta p_0)$. If the standard is exceeded certain actions are taken: if a sys-

[1] System (5) can be looked upon as the description of a three species ecological system with x_1 = rabbit density, x_2 = density of grass eaten by the rabbits, x_3 = density of plants not eatable by rabbits but interacting with the grass.

tem is to be designed, the component which causes the excess has to be replaced by a component with higher specifications. Or, if the System (1) is to describe a natural situation, more accurate measurements may be necessary.

The values of $S_1 = \underset{t}{\mathrm{Max}} (s_1 \Delta p_0)$ or $S_2 = \underset{t}{\mathrm{Max}} (x(p_0+\Delta p_0) - x(p_0))$ can be taken as indicators of whether the supplementary motion exceeds the prescribed limits or not. For the sake of simplicity and without loss of generality, we consider a one variable system and disregard the time dependence of the sensitivity. Let S_0 be the maximum deviation of x from the normal solution over the interval $(p_0-\Delta p_0, p_0+\Delta p_0)$. Then the payoff table for the choice between the two methods of sensitivity analysis has the following form:

Alternatives / Events	Differential Sensitivity (i = 1)		Finite Sensitivity (i = 2)	
	Loss	Probability	Loss	Probability
$S_0 > A \cap S_i > A$	0	P_{11}	0	P_{21}
$S_0 > A \cap S_i < A$	L_1	P_{12}	L_1	P_{22}
$S_0 < A \cap S_i < A$	0	P_{13}	0	P_{23}
$S_0 < A \cap S_i > A$	L_2	P_{14}	L_2	0

L_1 is the loss sustained if it is not recognized that the stanard is violated, L_2 is the loss in the case where sensitivity is overestimated. The alternative with the minimum expected loss is to be chosen. From the payoff table it can be seen that the decisive event for this choice is the second one, so that we must compare P_{12} with P_{22}:

We have

$$P_{i2} = P(S_0 > A \cap S_i < A)$$

$$= P(S_0 > A) \cdot P(S_i < A \mid S_0 > A)$$

$$= P(S_0 > A) \cdot \left(1 - \frac{P(S_i > A) \cdot P(S_0 > A \mid S_i > A)}{P(S_0 > A)}\right) ,$$

because for any two events a and b, we have

$$P(b) = P(a) \cdot P(b|a) + P(\bar{a}) \cdot P(b|\bar{a})$$

$$= P(b) \cdot P(a|b) + P(\bar{a}) \cdot P(b|\bar{a}) \quad .$$

By definition of S_2 we have $P(S_0 > A|S_2 > A) = 1$. Furthermore, we can under very weak assumptions prove that $P(S_2 > A) \geq P(S_1 > A)$: we can set

$$S_1 = |B| \text{ and } S_2 = |B + R| \quad ,$$

where B and R are random variables which are reasonably assumed to be independent. (One can think of B and R as the first order term and the rest of a Taylor expansion.) Let $\nu_B(b)$, $\nu_R(r)$, and $\nu_Z(z)$ be the frequency functions of B, R and B + R, respectively, and let us assume that for both ν_B and ν_R the following conditions are fulfilled:

$$\begin{cases} \nu_X(x) = \nu_X(-x) \quad , \\ \nu_X(x_1) \leq \nu_X(x_2) \quad , \quad \text{if } |x_1| > |x_2| \quad . \end{cases} \tag{6}$$

With these weak assumptions we have

$$
\begin{aligned}
P(S_2 > A) &= P(|B+R| > A) \\
&= 2 \int_A^{\infty} \nu_Z(z)\, dz \\
&= 2 \int_A^{\infty} \int_{-\infty}^{+\infty} \nu_B\,(z-r) \cdot \nu_R\,(r)\, dr\, dz \\
&= 2 \int_{-\infty}^{+\infty} \nu_R\,(r) \cdot \int_A^{\infty} \nu_B\,(z-r)\, dz\, dr \\
&= 2 \left\{ \int_{-\infty}^{0} \nu_R\,(r) \cdot \int_A^{\infty} \nu_B\,(z-r)\, dz\, dr \right. \\
&\qquad \left. + \int_{-\infty}^{0} \nu_R(r) \cdot \int_A^{\infty} \nu_B(z+r)\, dz\, dr \right\} \\
&= 2 \int_{-\infty}^{0} \nu_R(r) \cdot \left(1 - \int_{-A}^{+A} \nu_B(z-r)\, dz \right) dr \\
&= 1 - \int_{-A}^{+A} \nu_B(z-\xi)\, dz \text{ with } -\infty < \xi \leq 0 \quad .
\end{aligned}
$$

The last integral is maximal for $\xi = 0$. In this case the integral is equal to $P(|B| < A)$. Therefore we have

$$
P(S_2 > A) \geq P(S_1 > A) \quad .
$$

This means that for the problem at hand the finite sensitivity analysis ought to be preferred to the differential sensitivity analysis.

The same arguments can be applied to the case where

$$S_0 = \operatorname*{Max}_p (x) - \operatorname*{Min}_p (x) \quad ,$$

$$S_1 = \left| 2 \cdot \Delta p_0 \cdot \frac{\partial x}{\partial p}\bigg|_{p = p_0} \right| \quad ,$$

$$S_2 = |x(p_0 + \Delta p_0) - x(p_0 - \Delta p_0)| \quad .$$

III. The Problem of Optimally Sensitive Control

Another problem, for which the differential sensitivity turns out to be preferable, is the design of a system feedback which corrects an optimal open loop control $\tilde{u}_0$ such that the performance index

$$\int_{t_1}^{t_2} L\ (x,u,t)\ dt$$

remains as close as possible to the minimum if a system parameter deviates from its nominal value. Let us again assume a system with only one state variable, one control variable, and one parameter. Then a first order approximation to the solution of the problem is to add the feedback

$$\Delta u_1 = \frac{\partial \tilde{u}}{\partial p}\bigg|_{p = p_0} \cdot \left(\frac{\partial x}{\partial p}\bigg|_{p = p_0} \right)^{-1} \cdot (x - x_0) \tag{7}$$

to the nominal optimal control $\tilde{u}_0$ [2]. Here x_0 is the nominal motion of the state variable. Another possibility would be to feedback

$$\Delta u_2 = \frac{\Delta \tilde{u}}{\Delta p_0} \cdot \left(\frac{\Delta x}{\Delta p_0}\right)^{-1} \cdot (x - x_0) \quad , \tag{8}$$

where Δp_0 could be, for instance, the variance of p. By simple geometrical reasoning, one can prove that using expression (8) with finite sensitivities gives a greater expected value of the shortest distance between any point $\{\tilde{u}_0 + \Delta u_2, x\}$ and the curve $\tilde{u}(x)$ if the parameter values are symmetrically distributed around the nominal value. Figure 3 illustrates for a certain time the relationship between the two approximations and the optimal control curve, for which a second order approximation was chosen. In general, greater distance from the optimal control means a greater value of the performance index. Therefore feedback (7) should be preferred in order to minimize expected costs.

IV. Possible Extensions

A possible generalization of the problem in section 2 is to ask whether expression (2) or (4) is more appropriate for estimating the probability that a parameter value is drawn for which the state variable exceeds a prescribed limit. Though it has not yet been proved, some numerical experiments indicate that, for this problem also, finite sensitivity is to be preferred. Figure 4 shows such an experimental result: for a great number of functions f(p), with p normally distributed around the nominal value, those probabilities are computed and compared with the estimates according to (2) (Fig. 4a) and (4) (Fig. 4b). (The estimates have been computed with $s_1 \cdot (p - p_0)$ and $S_2 \cdot (p - p_0)$ as approximations to f(p).) Δp in (4) was chosen to be equal to the variance of the parameter. The functions f were 5th order polynomials with coefficients randomly selected from normal distributions with zero mean and the variance decreasing as the order of the polynomial term increases. The critical cases are again those in which the probability of exceeding the standard is greater than a certain decision threshold, say 5%, while the sensitivity estimate remains below this threshold. The figure clearly shows that there are more cases in the critical (shaded) area with the differential sensitivity (Fig. 4a) than with the finite sensitivity (Fig. 4b).

There are certainly still more problems solved by means of sensitivity functions for which one can prove that one of the two methods is in general "better" (in the sense described in the introduction). It would also be of interest to compare both methods for higher order sensitivities. This means, for instance, finding out whether it is better in a certain situation to know the values of a function at three points or to have at one point the first three terms of the Taylor expansion of that function.

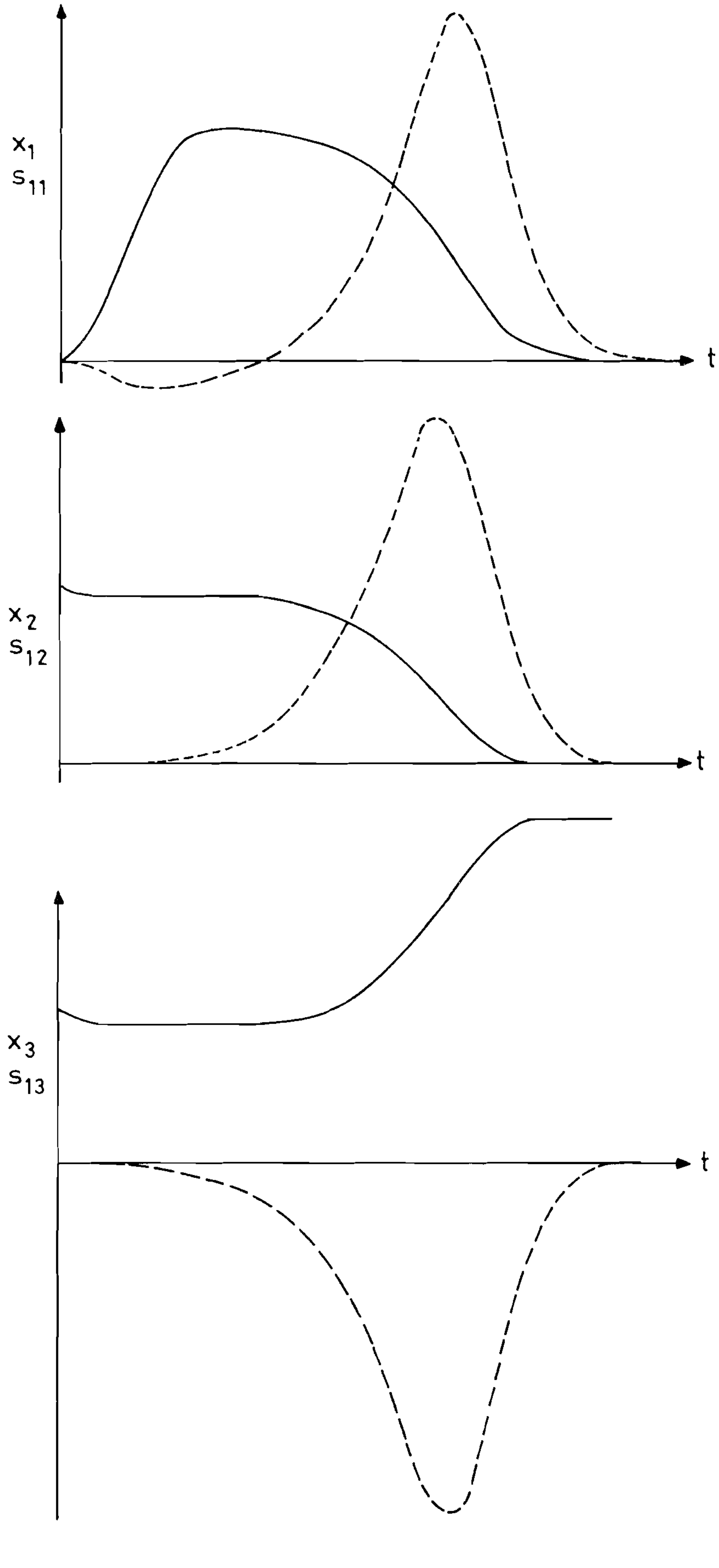

Figure 1. Nominal solutions x_i (———) and sensitivity functions S_{1i} (-----) for System (5).

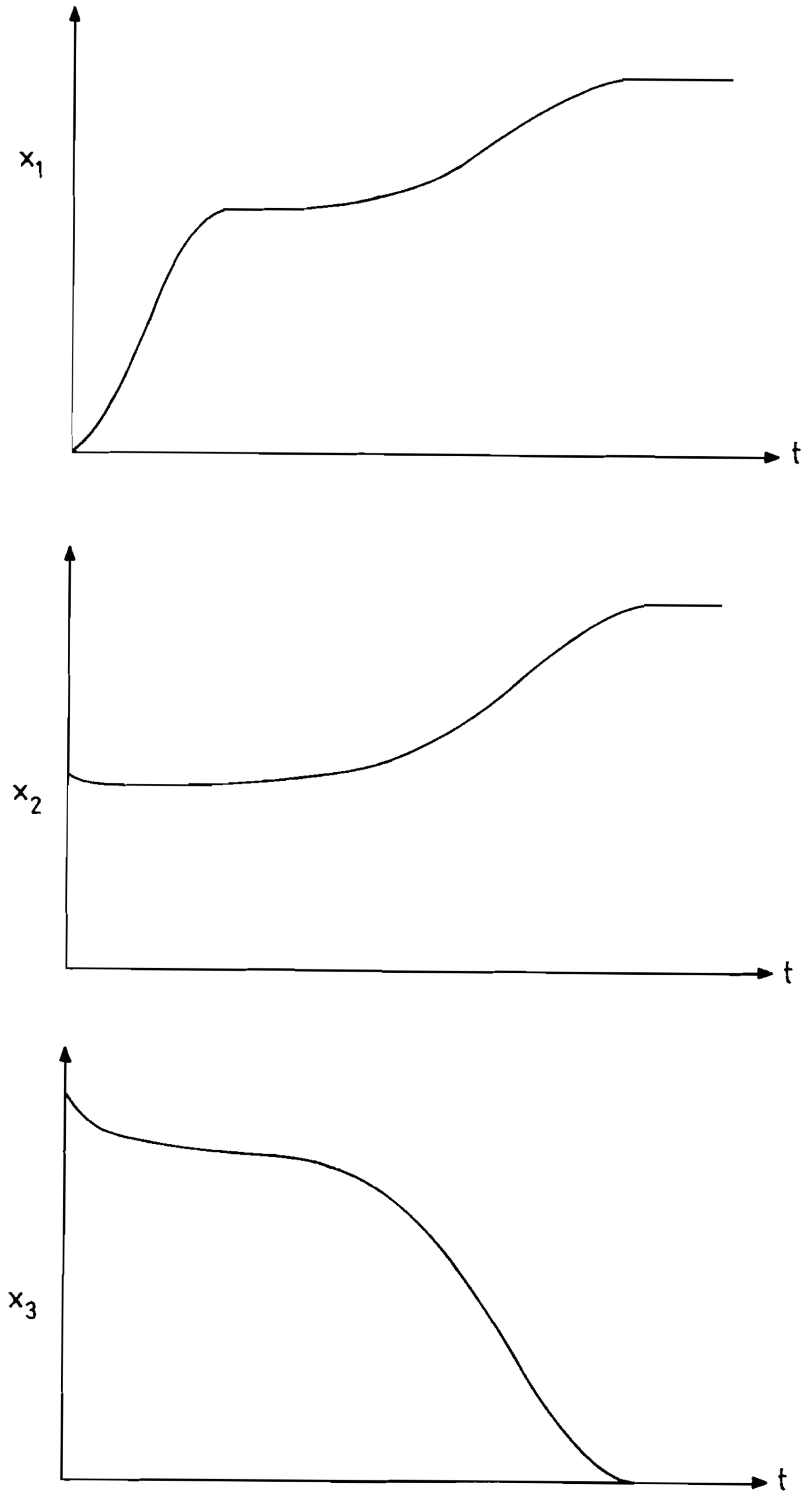

Figure 2. Solution of System (5) with $p = 1.05 \cdot p_0$.

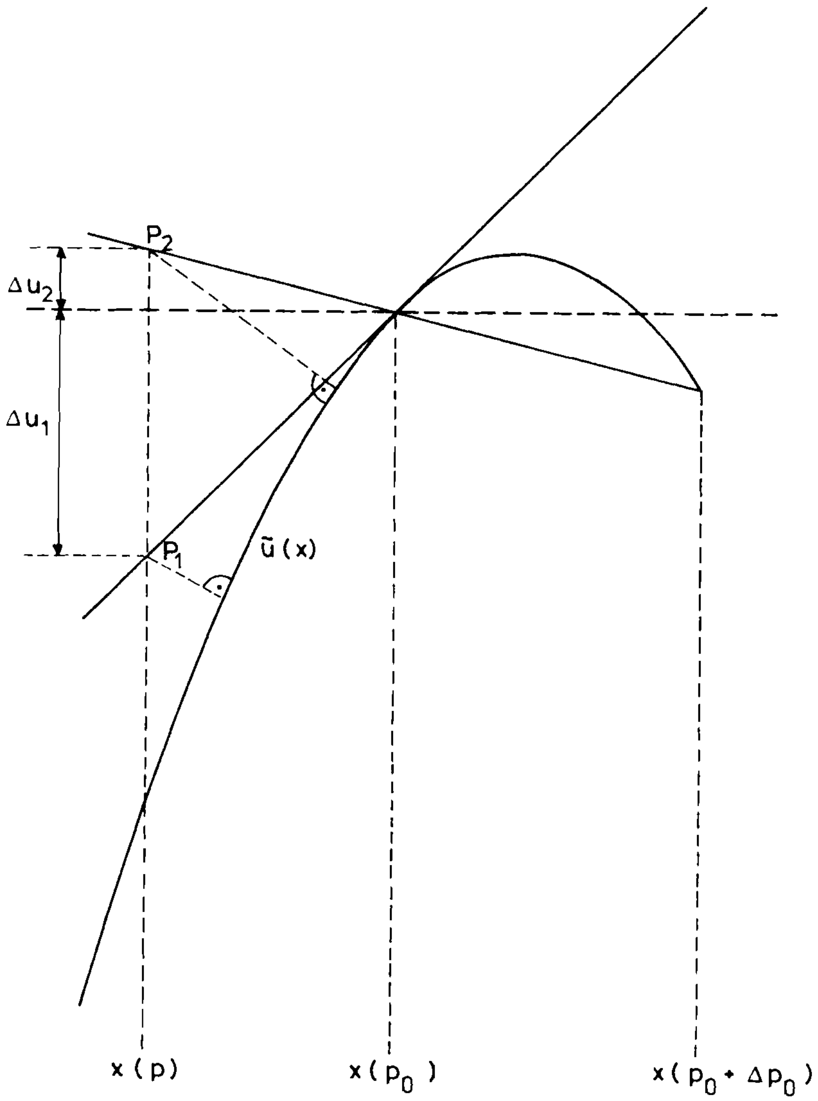

Figure 3. Illustration of the relationship between optimal feedback $\tilde{u}$ and the approximations (7) and (8) of $\tilde{u}$. (It is Dist $(P_1,\tilde{u}) \leq \frac{1}{2}$ Dist $(P_2,\tilde{u})$ if $x < x_0$.)

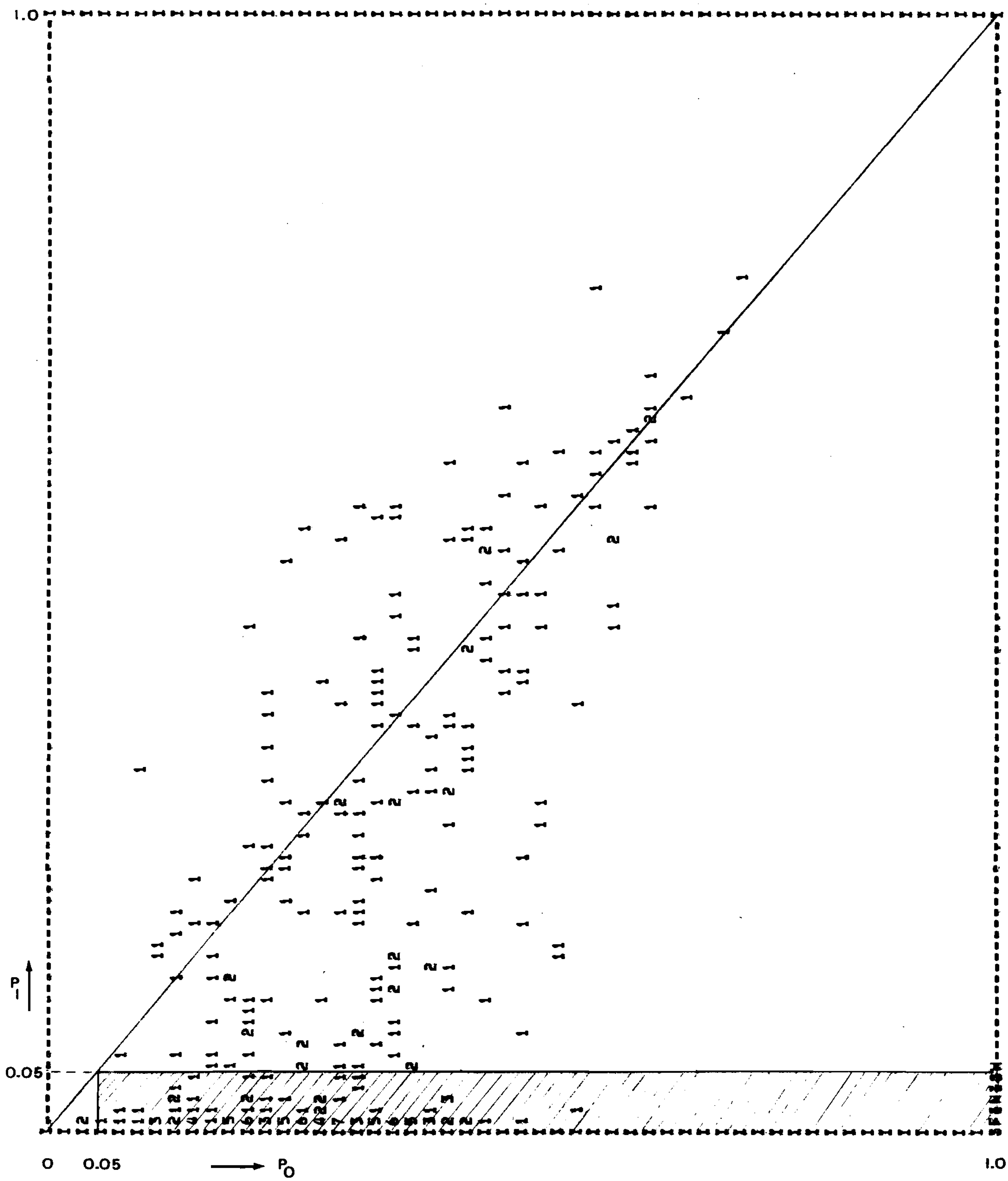

Figure 4a. Probability of exceeding a maximum permissible deviation from the function value $f(p_0)$ for a large number of functions $f(p)$, p normally distributed around p_0. Actual probability P_0 compared with the differential sensitivity estimate P_1. The numbers indicate how many points (P_0,P_1) fell into the same interval.

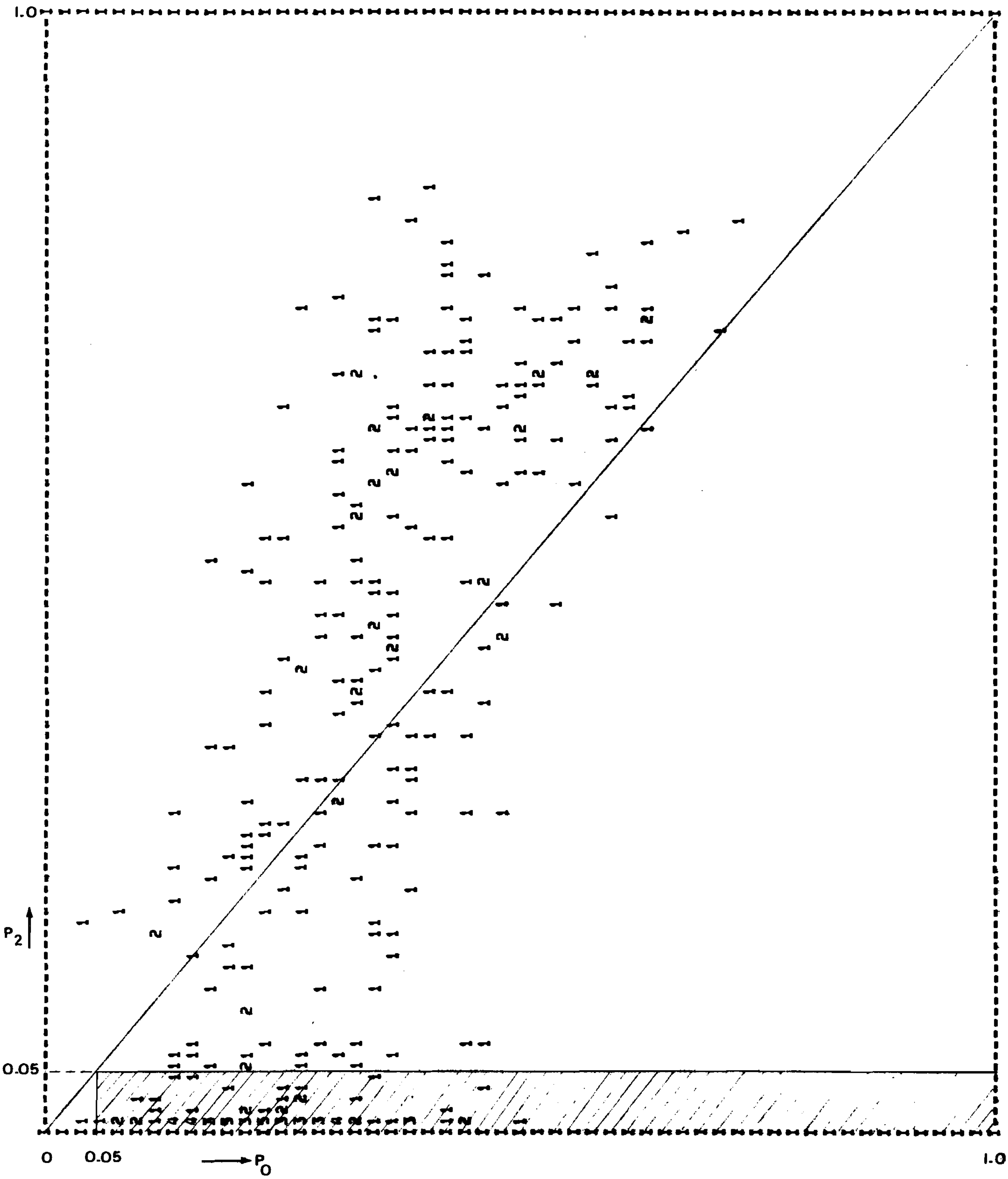

Figure 4b. As in Figure 4a, but P_0 compared with the finite sensitivity estimate P_2.

References

[1] Cruz, J.J., Ed. System Sensitivity Analysis. Stroudsburg, Pa., Dowden, Hutchinson & Ross, Inc., 1973.

[2] Kokotvic, P., J.B. Cruz, J. Heller, and P. Sannuti. "Synthesis of Optimally Sensitive Systems." Proc. IEEE, 56, 8 (1968), 1318-1324.