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AN INTERACTIVE FUZZY SATISFICING METHOD FOR MULTIOBJECTIVE NONLINEAR PROGRAMMING PROBLEMS

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May 1984
CP-84-18

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An Interactive Fuzzy Satisficing Method for Multiobjective Nonlinear Programming Problems

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Abstract

In this paper, we present a new interactive fuzzy satisficing method for solving multiobjective nonlinear programming problems by considering that the decision maker (DM) has fuzzy goals for each of the objective functions. The fuzzy goals of the DM are quantified by eliciting corresponding membership functions through the interaction with the DM. After determining the membership functions, if the DM specifies his reference membership values, the Tchebycheff norm problem is solved and the DM is supplied with the corresponding Pareto optimal solution and the trade-off rates between the membership functions. Then by considering the current values of the membership functions as well as the trade-off rates, the DM responds by updating his reference membership values. In this way the satisficing solution for the DM can be derived efficiently from among a Pareto optimal solution set by updating his reference membership values. On the basis of the proposed method, a time-sharing computer program is written and an illustrative numerical example is demonstrated together with the computer outputs.

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1. Introduction

An application of fuzzy approach to multiobjective linear programming problems was first presented by Zimmermann [14] and further studied by Leberling [7] and Hannan [5]. Following the maximizing decision proposed by Bellman and Zadeh [1] together with linear, hyperbolic or piecewise linear membership functions, they proved that there exists an equivalent linear programming problem.

However, suppose that the interaction with the decision maker (DM) establishes that the first membership function should be linear, the second hyperbolic, the third piecewise linear and so forth. In such a situation, the resulting problem becomes a nonlinear programming problem and cannot be solved by a linear programming technique.

In order to overcome such difficulties, Sakawa [9] has proposed a new method by combined use of bisection method and linear programming method together with five types of membership functions; linear, exponential, hyperbolic, hyperbolic inverse and piecewise linear functions. This method was further extended for solving multiobjective Linear fractional and nonlinear programming problems [10, 11].

In this paper, assuming that the DM has fuzzy goal for each of the objective functions in multiobjective nonlinear programming problems, we present a new interactive fuzzy satisficing method. After determining the membership functions for each of the objective functions through the interaction with the DM, if the DM specifies his reference membership values, the Tchebycheff norm problem is solved and the DM is supplied with the corresponding Pareto optimal solution and the trade-off rates between the membership functions. Then by considering the current values of the membership functions together with the trade-off rates, the DM responds by updating his reference membership values and the satisficing solution for the DM can be derived efficiently from among a Pareto optimal solution set. On the basis of the proposed method, a time-sharing computer program is written in FORTRAN and an illustrative numerical example is demonstrated along with the computer outputs.

2. Interactive fuzzy satisficing decision making

In general, the multiobjective nonlinear programming (MONLP) problem is represented as

$$\min f(x) \triangleq (f_1(x), f_2(x), \ldots, f_k(x))^T \tag{1}$$

$$\text{subject to} \quad x \in X \subseteq E^n$$

where $f_1, \ldots, f_k$ are k distinct objective functions of the decision vector x and X is the feasible set of constrained decisions. Here, it is assumed that all f_i, $i=1, \ldots, k$ are convex and differentiable and constraint set X is convex and compact.

Fundamental to the MONLP is the Pareto optimal concept, also known as a noninferior solution. Qualitatively, a Pareto optimal solution of the MONLP is one where any improvement of one objective function can be achieved only at the expense of another.

Usually, Pareto optimal solutions consist of an infinite number of points, and some kinds of subjective judgement should be added to the quantitative analyses by the DM. The DM must select his compromise or satisficing solution from among Pareto optimal solutions.

In order to determine the compromise or satisficing solution of the DM, there are three major approaches:

(1) goal programming [2, 6],

(2) interactive approach [3, 4, 8, 12, 13, 15],

(3) fuzzy approach [5, 7, 9-11, 14].

Each of these approaches has its own advantages and disadvantages relative to the other approaches. Therefore, in this paper, we propose a new interactive fuzzy satisficing method by incorporating the desirable features of both the goal programming methods and the interactive approaches into the fuzzy approaches.

In a minimization problem, a fuzzy goal stated by the DM may be to achieve "substantially less" than A. This type of statement can be quantified by eliciting a corresponding membership function.

In order to elicit a membership function $\mu_{f_i}(x)$ from the DM for each of the objective functions $f_i(x)$, we first calculate the individual minimum f_i^{min} and maximum f_i^{max} of each objective function $f_i(x)$ under given constraints. By taking account of the calculated individual minimum and maximum of each objective function, the DM must determine his subjective membership function $\mu_{f_i}(x)$, which is strictly monotone decreasing fucntion with respect to $f_i(x)$. Here, it is assumed that $\mu_{f_i}(x) = 0$ if $f_i(x) \geq f_i^0$ and $\mu_{f_i}(x) = 1$ if $f_i(x) \leq f_i^1$, where f_i^0 is a worst acceptable level for $f_i(x)$ and f_i^1 is a totally desirable level for $f_i(x)$ within f_i^{min} and f_i^{max}.

After determining the membership functions for each of the objective functions, the DM is asked to specify his reference membership values for all the membership functions. For the DM's reference membership values $\hat{\mu}_{f_i}$, i=1,2,...,k, the corresponding Pareto optimal solution which is in a sense close to his requirement (or better, if the reference membership values are attainable) is obtained by solving the following Tchebycheff norm problem.

$$\min_{x \in X} \max_{i=1,\dots,k} \{\hat{\mu}_{f_i} - \mu_{f_i}(x)\}, \tag{2}$$

or equivalently

$$\min_{v,\, x \in X} v$$

$$\text{subject to} \quad \hat{\mu}_{f_i} - \mu_{f_i}(x) \leqq v, \quad i=1,2,\dots,k. \tag{3}$$

The relationships between the optimal solutions of the Tchebycheff norm problem and the Pareto optimal concept of the MONLP can be characterized by the following theorems.

Theorem 1. If x^* is a unique optimal solution to the Tchebycheff norm problem (3), then x^* is a Pareto optimal solution to the MONLP.

Theorem 2. If $x^* \in X$ is a Pareto optimal solution to the MONLP with $0 < \mu_{f_i}(x^*) < 1$ holding for all i, then x^* is a unique optimal solution to the Tchebycheff norm problem (3).

If x^*, an optimal solution to (3), is not unique, then we can test the Pareto optimality for x^* by solving the following problem:

$$\max_{x \in X} \sum_{i=1}^{k} \varepsilon_i$$

$$\text{subject to} \quad f_i(x) + \varepsilon_i = f_i(x^*), \quad \varepsilon_i \geqq 0 \quad (i=1,\dots,k). \tag{4}$$

Let $\bar{x}$ be an optimal solution to (4). If all $\varepsilon_i = 0$, then x^* is a Pareto optimal solution. If at least one $\varepsilon_i > 0$, it can easily be shown that $\bar{x}$ is a Pareto optimal solution.

The DM must now either satisfy with the current Pareto optimal solution, or update his reference membership values. In order to help the

DM express his degree of preference, trade-off information between a standing membership function $\mu_{f_1}(x)$ and each of the other membership functions is very useful. Such a trade-off between $\mu_{f_1}(x)$ and $\mu_{f_i}(x)$ for each $i=1,2,\ldots,k$ is easily obtainable since it is closely related to the strict positive Lagrange multipliers of the Tchebycheff norm problem. Let the Lagrange multipliers associated with the constraints of the Tchebycheff norm problem be denoted by λ_i, $i=1,2,\ldots,k$. If all $\lambda_i > 0$ for each i, then it can be proved that the following expression holds.

$$-\partial\mu_{f_i}(x)/\partial\mu_{f_1}(x) = \lambda_1/\lambda_i, \quad i=2,\ldots,k. \tag{5}$$

So far we have considered a minimization problem and consequently assumed that the DM has a fuzzy goal such as "$f_i(x)$ should be substantially less than a_i".

In the followings, we further consider a more general case where the DM has two types of fuzzy goals, namely fuzzy goals expressed in words such as "$f_i(x)$ should be in the vicinity of b_i" (fuzzy equal) as well as "$f_i(x)$ should be substantially less than a_i" (fuzzy min) are assumed. Therefore, the problem to be solved is

$$\begin{aligned} &\text{fuzzy min} && f_i(x) \quad (i \in I) \\ &\text{fuzzy equal} && f_i(x) \quad (i \in \bar{I}) \\ &\text{subject to} && x \in X \end{aligned} \tag{6}$$

where $I \cup \bar{I} = \{1,2,\ldots,k\}$.

In order to elicit a membership function from the DM for a fuzzy goal like "$f_i(x)$ should be in the vicinity of b_i", it is obvious that we can use different functions to the left and right sides of b_i. After determining the membership functions for two types of fuzzy goals, if the DM specifies his reference membership values, the Tchebycheff norm problem is solved.

Now, we introduce the concept of M-Pareto optimal solutions which are defined in terms of membership functions instead of objective functions.

Definition 1. A decision x^* is said to be an M-Pareto optimal solution to (6), if and only if there does not exist another $x \in X$ so that $\mu_{f_i}(x) \geq \mu_{f_i}(x^*)$, i=1,...,k, with strict inequality holding for at least one i.

Note that the set of Pareto optimal solutions is a subset of the set of M-Pareto optimal solutions.

Using the concept of M-Pareto optimality, the following theorem, which is similar to Theorem 1 and 2, can be obtained under slightly different conditions.

Theorem 3. x^* X is an M-Pareto optimal solution to (6), if and only if x^* is a unique optimal solution to (3).

Similar to the minimization case, a numerical test of M-Pareto optimality for x^* can be peformed by solving the following problem:

$$\max_{x \in X} \sum_{i=1}^{k} \varepsilon_i$$

subject to (7)

$$\mu_{f_i}(x) - \varepsilon_i = \mu_{f_i}(x^*), \quad \varepsilon_i \geq 0 \quad (i=1,\ldots,k).$$

Let $\bar{x}$ be an optimal solution to (7). If all $\varepsilon_i = 0$, then x^* is an M-Pareto optimal solution. If at least one $\varepsilon_i > 0$, $\bar{x}$ becomes an M-Pareto optimal solution.

Following the above discussions, we can now construct the interactive algorithm in order to derive the satisficing solution for the DM from among the (M-) Pareto optimal solution set. The steps marked with an asterisk involve interaction with the DM.

Step 1. Calculate the individual minimum and maximum of each objective function under given constraints.

Step 2*. Elicit a membership function from the DM for each of the objective functions.

Step 3. Set all the initial reference membership values equal 1, i.e., $\hat{\mu}_{f_i} = 1$ $(i=1,2,\ldots,k)$.

Step 4. For the reference membership values specified by the DM, the Tchebycheff norm problem is solved and the (M-) Pareto optimality test is performed.

Step 5*. The DM is supplied with the corresponding (M-) Pareto optimal solution and the trade-off rates between the membership functions. If the DM is satisfied with the current membership values of the (M-) Pareto optimal solution, stop. Otherwise, the DM must update his reference membership values by considering the current values of the membership functions together with the trade-off rates between the membership functions and return to Step 4. Here it should be stressed for the DM that any improvement of one membership function can be achieved only at the expense of at least one of the other membership functions.

3. An interactive computer program and an illustrative example

Fuzzy satisficing decision making processes for multiobjective nonlinear programming problems include eliciting a membership function for each of the objective functions and reference membership values from the DM. Thus, interactive utilization of computer facilities is highly recommended. Based on the method described above, we have developed a new interactive computer program. Our new package includes graphical representations by which the DM can figure the shapes of his membership functions, and he can find incorrect assessments or inconsistent evaluations promptly, revise them immediately and proceed to the next stage more easily.

Our program is composed of one main program and several subroutines. The main program calls in and runs the subprograms with commands indicated by the user (DM). Here we give a brief explanation of the major commands prepared in our program.

(1) MINMAX: Displays the calculated individual minimum and maximum of each of the objective functions under the given constraints.

(2) MF: Elicits a membership function from the DM for each of the objective functions.

(3) GRAPH: Depicts graphically the shape of the membership function for each of the objective functions.

(4) GO: Derives the satisficing solution for the DM from among the (M-) Pareto optimal solution set by updating the reference membership values.

(5) STOP: Exists from the program.

In our computer program, the DM can select his membership function in a subjective manner from among the following five types of functions; linear, exponential, hyperbolic, hyperbolic inverse and piecewise linear functions. Then the parameter values are determined through the interaction with the DM. Here, except for hyperbolic functions, it is assumed that $\mu_{f_i}(x) = 0$ if $f_i(x) \geqq f_i^0$ and $\mu_{f_i}(x) = 1$ if $f_i(x) \leqq f_i^1$, where f_i^0 is a worst acceptable level for $f_i(x)$ and f_i^1 is a totally desirable level for $f_i(x)$.

(1) Linear membership function:

$$\mu_{f_i}(x) = [f_i(x) - f_i^0]/[f_i^1 - f_i^0]. \tag{8}$$

The linear membership function can be determined by asking the DM to specify the two points f_i^0 and f_i^1 within f_i^{max} and f_i^{min}.

(2) Exponential membership function:

$$\mu_{f_i}(x) = a_i[1 - \exp\{- b_i(f_i(x) - f_i^0)/(f_i^1 - f_i^0)\}] \tag{9}$$

The exponential membership function can be determined by asking the DM to specify the three points f_i^0, $f_i^{0.5}$ and f_i^1 within f_i^{max} and f_i^{min}, where f_i^a represents the value of $f_i(x)$ such that the degree of membership function $\mu_{f_i}(x)$ is a.

(3) Hyperbolic membership function:

$$\mu_{f_i}(x) = (1/2) \tanh((f_i(x) - b_i)\alpha_i) + (1/2). \tag{10}$$

The hyperbolic membership function can be determined by asking the DM to specify the two points $f_i^{0.25}$ and $f_i^{0.5}$ within f_i^{max} and f_i^{min}.

(4) Hyperbolic inverse membership function:

$$\mu_{f_i}(x) = a_i \tanh^{-1}((f_i(x) - b_i)\alpha_i) + (1/2). \tag{11}$$

The hyperbolic inverse membership function can be determined by asking the DM to specify the three points f_i^0, $f_i^{0.25}$ and $f_i^{0.5}$ within f_i^{max} and f_i^{min}.

(5) Piecewise linear membership function:

$$\mu_{f_i}(x) = \sum_{j=1}^{N_i} \alpha_{ij}|f_i(x) - g_{ij}| + \beta_i f_i(x) + \gamma_i. \tag{12}$$

Here, it is assumed that $\mu_{f_i}(x) = t_{ir} f_i(x) + s_{ir}$ for each segment $g_{ir-1} \leqq f_i(x) \leqq g_{ir}$. The piecewise linear membership function can be determined by asking the DM to specify the degree of membership in each of several values of objective functions within f_i^{max} and f_i^{min}.

We now demonstrate the interaction processes using our computer program by means of an illustrative example which is designed to test the program.

Consider the following multiobjective decision making problem.

fuzzy min $\quad f_1(x) = x_1{}^2 + (x_2+5)^2 + (x_3-60)^2$

fuzzy min $\quad f_2(x) = (x_1+20)^2 + (x_2-55)^2 + (x_3+20)^2$

fuzzy equal $\quad f_3(x) = (x_1-20)^2 + (x_2-10)^2 + (x_3-30)^2$

subject to $\quad x \in X = \{(x_1,x_2,x_3)|x_1^2 + x_2^2 + x_3^2 \leqq 100,\ 0 \leqq x_i \leqq 10,\ i=1,2,3\}$

In applying our computer program to this problem, suppose that the interaction with the hypothetical DM establishes the following membership functions and corresponding assessment values for the three objective functions.

f_1: linear, $(f_1^0, f_1^1) = (3700, 2525)$

f_2: hyperbolic, $(f_2^{0.25}, f_2^{0.5}) = (3800, 3500)$

$$f_3 \begin{cases} \text{left: linear, } (f_3^0, f_3^1) = (800, 1100) \\ \text{right: exponential, } (f_3^0, f_3^{0.5}, f_3^1) = (1300, 1250, 1100) \end{cases}$$

In Appendix, the interaction processes using our computer program are shown with the aid of some of the computer outputs. In this example, at the second iteration, the satisficing solution of the DM is derived.

4. Conclusion

In this paper, we have proposed an interactive fuzzy satisficing method in order to deal with the fuzzy goals of the DM in multiobjective nonlinear programming problems. In our interactive scheme, after determining the membership functions, the satisficing solution of the DM can be derived by updating his reference membership values based on the current values of the membership functions together with the trade-off rates between the membership functions. Furthermore, (M-) Pareto optimality of the generated solution in each iteration is guaranteed. Based on the proposed method, the time-sharing computer program has been written to facilitate the interaction processes. An illustrative numerical example demonstrated the feasibility and efficiency of both the proposed technique and its interactive computer program under the hypothetical DM. However, applications to real-word problems must be carried our in cooperation with a person actually involved in decision making. From such experiences the proposed technique and its computer program must be revised.

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```
COMMAND:
=GO

--------------------< ITERATION 1 >--------------------

INITIATES AN INTERACTION WITH ALL THE INITIAL REFERENCE
MEMBERSHIP VALUES ARE 1

( KUHN-TUCKER CONDITIONS SATISFIED )

OPTIMAL SOLUTION TO THE TCHEBYCHEFF NORM PROBLEM
FOR INITIAL REFERENCE MEMBERSHIP VALUES
----------------------------------------------------------
         MEMBERSHIP          I     OBJECTIVE FUNCTION
-----------------------------+----------------------------
   M(F1) = 0.5756141320D+00  I  F(1) = 0.3023653395D+04
   M(F2) = 0.5756141320D+00  I  F(2) = 0.3416769249D+04
   M(F3) = 0.5756141320D+00  I  F(3) = 0.9726842396D+03
----------------------------------------------------------
   X( 1) = 0.                  X( 2) = 0.6774159957D+01
   X( 3) = 0.6287594057D+01
----------------------------------------------------------

M-PARETO OPTIMALITY TEST
( KUHN-TUCKER CONDITIONS SATISFIED )
  EPS( 1)= 0.
  EPS( 2)= 0.
  EPS( 3)= 0.

----------------------------------------------------------
TRADE-OFFS AMONG MEMBERSHIP FUNCTIONS
 -DM(F2)/DM(F1)  = 0.2853582811D+01
 -DM(F3)/DM(F1)  = 0.2109354632D+01
----------------------------------------------------------
             GRAPH OF THE TRADE-OFFS
                      OOOOO : -DM(F2)/DM(F1)
                      XXXXX : -DM(F3)/DM(F1)
 1.00+---------!---------!---------!---------+
    !                            O          !
    !                                       !
    !                            X          !
    !                                       !
    !                         O             !
    !                         X             !
    !                                       !
    !                                       !
    !                      #                !
 0.50-                                       -
    !                                       !
    !                   X                   !
    !                   O                   !
    !                                       !
    !                X                      !
    !                                       !
    !                O                      !
    !             X                         !
    !                                       !
 0.  +---------!---------!---------!---------+
    0.       0.25      0.50      0.75      1.00

ARE YOU SATISFIED WITH THE CURRENT MEMBERSHIP VALUES OF
THE PARETO OPTIMAL SOLUTION ?
=NO
```

....

ARE YOU SATISFIED WITH THE CURRENT MEMBERSHIP VALUES OF
THE PARETO OPTIMAL SOLUTION ?
=YES

THE FOLLOWING VALUES ARE YOUR SATISFICING SOLUTION :

```
---------------------------------------------------------------
          MEMBERSHIP           I      OBJECTIVE FUNCTION
-------------------------------+-------------------------------
   M(F1) = 0.5490868946D+00  I  F(1) = 0.3054822899D+04
   M(F2) = 0.6190868946D+00  I  F(2) = 0.3367376043D+04
   M(F3) = 0.5990868946D+00  I  F(3) = 0.9797260684D+03
---------------------------------------------------------------
   X( 1) = 0.                   X( 2) = 0.7179716029D+01
   X( 3) = 0.6088244174D+01
---------------------------------------------------------------
```

COMMAND:
=STOP