SHRINKING LARGE-SCALE POPULATION PROJECTION MODELS BY AGGREGATION AND DECOMPOSITION

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1. Introduction

During the past two decades social scientists have come to model dynamic socioeconomic systems of growing size and complexity. Despite a heavy reliance on ever more sophisticated high-speed digital computers, however, their capacity for handling such systems has not kept pace with the growing demands for more detailed information.

> "As a consequence, it is becoming more and more important to secure information on the nature of those aspects of a system which, when present, enable us to treat a part of it separately from the rest or to deal with the relationship among particular subsystems as though it were independent of the structures within those subsystems. The latter question is that of aggregation, while the former is...one of partition ...." (Ando and Fisher, 1963, p.92).

An increasing number of social scientists currently find themselves in the somewhat frustrating position of being asked to provide accurate projections at very fine levels of detail with resources that are scarcely sufficient for carrying out such projections at much more aggregate levels of resolution. Prominent among them are demographers who are called upon to produce consistent projections of regional populations disaggregated by age, color, race, sex, and such indicators of class and welfare as employment category and income. Since the computational requirements of this task are staggering, the need for developing improved methods for "shrinking" population projection models by reducing their dimensionality is an urgent one, and the two most obvious methods for effecting such a reduction are aggregation and partitioning, or more appropriately, decomposition.

1.1 Aggregation

The need to use aggregates arises out of sheer necessity in most social science research involving large systems. Theoretical abstract reasoning and numerical empirical computation both rely on the conceptual clarity and efficient manipulation of variables afforded by aggregation. In economic modeling, for example, the many producers and consumers of a national or regional economy are aggregated into a relatively small number of sectors, and the interaction among these sectors is then studied as though it were free of influences arising from intrasectoral interaction. A typical example of this occurs in input-output analysis, and indeed it was the increasing world-wide numerical application of such models that first stimulated much of the interest in aggregation among social scientists (e.g., Ara, 1959; Fisher, 1969; Rogers, 1969).

Aggregation generally introduces inconsistencies between the outputs of the disaggregated and aggregated models. The conditions for aggregation without such inconsistencies, i.e., for perfect aggregation, are very severe and therefore are almost never met in practice. However, since any model is at best only an approximate description of reality, we remain interested in establishing the conditions under which perfect aggregation may be carried out. These conditions suggest the criteria, or rules, for selecting which variables to aggregate and help to identify the circumstances under which such an aggregation will yield results that are consistent with those of the original disaggregated model.

Aggregation of large scale problems, therefore, has two fundamental aspects. The first is the process of consolidation itself. Here the two sets of variables that are connected by a system of relations are grouped into aggregates and a new smaller system of relations is developed which connects the two sets of aggregates. The second fundamental aspect of the aggregation process is the selection of the consolidation scheme that most closely satisfies the conditions necessary for perfect aggregation, while at the same time meeting whatever informational requirements and additional constraints that may have been specified a priori. In short, consolidation is an operation that expresses a set of "new" variables as weighted averages of the set of original "old" variables, such that there are fewer new variables than old variables. Criteria for perfect aggregation, on the other hand, are rules that indicate which variables to consolidate, for example, the rule that variables which always move together may be consolidated into a single variable without introducing an aggregation error.

Two particular forms of aggregation are frequently employed in demographic analysis. The first is a consolidation across age groups. When carried out over all age groups, this form of consolidation transforms a cohort-survival model into a components-of-change model (Rogers, 1971, Ch.1). We shall, therefore, refer to aggregations of this sort as components-of-change aggregations. Such aggregations retain the geographical areal units of the original cohort-survival model but sacrifice all age-specific details.

The second form of aggregation that is frequently used is a division of a multiregional population system into two regions: a particular region under study and "the rest of the world." Such consolidations will be called bi-regional aggregations in this paper. They sacrifice considerable geographical information but preserve details about age compositions. However, if applied in sequence to each and every region of a multiregional system, they permit a collection of aggregated projections to completely preserve the levels of detail found in the original unconsolidated projection.

1.2 Decomposition

The idea of decomposing a large and complex problem into several smaller subproblems in order to simplify its solution is not new and indeed has been used for well over a century in the physical and social sciences, as well as in engineering.

However, the development and use of high-speed computers to solve these problems during the past two decades has stimulated a focused interest in decomposition techniques in such various fields of application as process control, structural engineering, systems optimization, electrical network theory, and a wide variety of seemingly unrelated problems in economics, mathematics, design, and operations research (e.g., Himmelblau, 1973; Rose and Willoughby, 1972, Tewarson, 1973, and Theil, 1972).

The central principle of decomposition analysis is that the solution of a large systems problem, involving many interacting elements, often can be broken up and expressed in terms of the solutions of relatively independent subsystem problems of lower dimensionality. The solutions of the subsystem problems then can be combined and, if necessary, modified to yield the solution of the original large-system problem. A well-known illustration of this approach is provided by the Dantzig and Wolfe decomposition algorithm in mathematical programming (Dantzig and Wolfe, 1960). This algorithm breaks up a large linear programming problem into several smaller linear programming problems and imposes additional constraints on each of the latter in order to ensure that their solutions combine to yield the optimal solution for the large scale problem.

Decompositions of large-scale problems generally proceed in two stages. First there is the partitioning stage in which a large system of variables and relations is rearranged and reordered in a search for disjoint subsystems, that is, subsets of relations which do not contain any common variables. If such subsystems exist, then each one can be treated independently of the rest. In this way the relational structure of the original large-scale problem can be exploited to produce a more efficient solution method.

Systems that can be partitioned into independent (disjoint) subsystems are said to be completely decomposable, and their matrix expression can be transformed into what is known as a block-diagonal form. The rearrangement and reordering of the relations to identify and delineate the disjoint subsystems is called permutation, and the actual separation of the large system into disjoint subsystems is called partitioning.

Partitioning of a large system into disjoint subsystems obviously cannot be accomplished if each relation in the system contains every variable. Such systems are said to be indecomposable. Fortunately, the relations in most mathematical models of socioeconomic phenomena contain only a few common variables. Moreover, when complete decomposition cannot be achieved, a partial decomposition that rearranges and reorders the relations into a block-triangular form may still be possible.

A block triangular structure defines an information flow that is serial and without loops. Causal sequences in such systems, therefore, run one-way and permit feedbacks only upward in the triangular hierarchy. An example of such a structure is afforded by a hierarchy of migration flows in which people migrate only to larger urban regions. If the regions are ordered according to their size in the population projection process, then the growth matrix assumes a block-triangular form.

Once a large system of variables and relations has been either completely or partially decomposed into indecomposable subsystems, further simplification of the problem can only be achieved by a process called tearing. This is the second stage of the decomposition procedure and consists of deleting variables from one or more of the relations in which they appear. Thus tearing represents an attempt to solve a system problem by a "forced" partitioning of that system into supposedly disjoint subsystems. The partitioning is forced because the subsystems are not truly disjoint and are rendered so only through a disregard of certain connecting relationships which are held to be insignificant. If the impacts of these connecting relationships are not completely disregarded but are allowed somehow to affect the solution of the system problem, then we have an instance of compensated tearing.

1.3 Numerical Illustrations

Imagine a multiregional population distributed among four regions called, respectively, the North, South, East, and West regions. Assume that the multiregional population is a closed system which experiences internal migration but is undisturbed by external migration flows. Moreover, assume that every year one-half of the populations of the North and South regions and three-quarters of the populations of the East and West regions, respectively, outmigrate in equal proportions to the remaining three regions. Finally, to further simplify matters, let the number of births equal the number of deaths in each region, so that natural increase is zero in each region.

Starting with an initial multiregional population of 480 individuals distributed equally among the four regions, the above regime of growth and change would produce the following population distribution one year later:

$$\underline{\text{North}}: \quad 140 = \overbrace{1/2(120)}^{\text{non-migrants}} + \overbrace{1/6(120) + 1/4(120) + 1/4(120)}^{\text{in-migrants}}$$

$$\underline{\text{South}}: \quad 140 = 1/6(120) + \overbrace{1/2(120)}^{\text{non-migrants}} + 1/4(120) + 1/4(120)$$

$$\underline{\text{East}}: \quad 100 = 1/6(120) + 1/6(120) + \overbrace{1/4(120)}^{\text{non-migrants}} + 1/4(120)$$

$$\underline{\text{West}}: \quad 100 = 1/6(120) + 1/6(120) + 1/4(120) + \overbrace{1/4(120)}^{\text{non-migrants}}$$

This projection process can be expressed conveniently in matrix form as follows:

$$\begin{bmatrix} 140 \\ 140 \\ 100 \\ 100 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/6 & 1/4 & 1/4 \\ 1/6 & 1/2 & 1/4 & 1/4 \\ 1/6 & 1/6 & 1/4 & 1/4 \\ 1/6 & 1/6 & 1/4 & 1/4 \end{bmatrix} \begin{bmatrix} 120 \\ 120 \\ 120 \\ 120 \end{bmatrix} \quad . \qquad (1)$$

Let us now "shrink" our components-of-change population projection model to a fourth of its original size by aggregating the North and South regions into one region and the East and West regions into another. The corresponding consolidation of (1) then yields

$$\begin{bmatrix} 280 \\ 200 \end{bmatrix} = \begin{bmatrix} 2/3 & 1/2 \\ 1/3 & 1/2 \end{bmatrix} \begin{bmatrix} 240 \\ 240 \end{bmatrix} \quad . \tag{2}$$

An alternative consolidation scheme is to treat one region as interacting with the rest of the system. For example, a focus on the interaction between the North region and the aggregate of all other regions gives

$$\begin{bmatrix} 140 \\ 340 \end{bmatrix} = \begin{bmatrix} 1/2 & 2/9 \\ 1/2 & 7/9 \end{bmatrix} \begin{bmatrix} 120 \\ 360 \end{bmatrix} \tag{3}$$

Note that this particular spatial consolidation is an example of bi-regional aggregation, and observe that by repeating this procedure with each of the original four regions we can obtain a population projection for every one of them.

Another round of projections using the growth models in (1), (2), and (3) reveals that the first consolidation is an example of perfect aggregation inasmuch as it forecasts the same total population as does the original unconsolidated model in (1). The bi-regional consolidation in (3), however, is an example of imperfect aggregation and projects a slightly higher population for the North region then the one generated by the unconsolidated model. The first consolidation satisfies the sufficient condition for perfect aggregation which asserts that two populations exhibiting identical rates of birth, death, and outmigration to the rest of the multiregional system may be consolidated without thereby introducing an error into the projection process (Rogers, 1969).

Assume now that the migration flows from the North and South regions to the East and West regions and the corresponding flows in the reverse direction are ignored. The projection matrix in (1) then becomes completely decomposable and assumes a block-diagonal form:

$$\begin{bmatrix} 80 \\ 80 \\ \hline 60 \\ 60 \end{bmatrix} = \left[\begin{array}{cc|cc} 1/2 & 1/6 & 0 & 0 \\ 1/6 & 1/2 & 0 & 0 \\ \hline 0 & 0 & 1/4 & 1/4 \\ 0 & 0 & 1/4 & 1/4 \end{array}\right] \begin{bmatrix} 120 \\ 120 \\ \hline 120 \\ 120 \end{bmatrix} . \qquad (4)$$

The resulting population projection becomes an example of uncompensated tearing and, of course, produces an erroneous population forecast. Consequently, we may wish to introduce an adjustment to the model by including the ignored migration flows in the diagonal elements of the projection matrix in the form of <u>net</u> migration rates, thereby illustrating the process of compensated tearing. This gives

$$\begin{bmatrix} 140 \\ 140 \\ \hline 100 \\ 100 \end{bmatrix} = \left[\begin{array}{cc|cc} 1 & 1/6 & 0 & 0 \\ 1/6 & 1 & 0 & 0 \\ \hline 0 & 0 & 7/12 & 1/4 \\ 0 & 0 & 1/4 & 7/12 \end{array}\right] \begin{bmatrix} 120 \\ 120 \\ \hline 120 \\ 120 \end{bmatrix} \qquad (5)$$

The advantage of a block-diagonal decomposition of the kind set out in (5) is the shrinking that it achieves. The larger system projection can be partitioned and torn into independent subsystems, each of which can then be projected separately. For example, in place of the "large-scale" population projection described in (1), we may instead carry out the two "smaller" projections:

$$\begin{bmatrix} 140 \\ 140 \end{bmatrix} = \begin{bmatrix} 1 & 1/6 \\ 1/6 & 1 \end{bmatrix} \begin{bmatrix} 120 \\ 120 \end{bmatrix} \tag{6}$$

and

$$\begin{bmatrix} 100 \\ 100 \end{bmatrix} = \begin{bmatrix} 7/12 & 1/4 \\ 1/4 & 7/12 \end{bmatrix} \begin{bmatrix} 120 \\ 120 \end{bmatrix} , \tag{7}$$

respectively.

For our final numerical illustration of decomposition, let us now instead ignore only the outmigration flows from the North and South regions to the East and West regions, respectively. The projection matrix in (1) then becomes partially decomposable and assumes a block-triangular form:

$$\begin{bmatrix} 140 \\ 140 \\ \hline 60 \\ 60 \end{bmatrix} = \left[\begin{array}{cc|cc} 1/2 & 1/6 & 1/4 & 1/4 \\ 1/6 & 1/2 & 1/4 & 1/4 \\ \hline 0 & 0 & 1/4 & 1/4 \\ 0 & 0 & 1/4 & 1/4 \end{array}\right] \begin{bmatrix} 120 \\ 120 \\ \hline 120 \\ 120 \end{bmatrix} . \tag{8}$$

Modifying the above projection matrix to take the ignored flows into account, we obtain:

$$\begin{bmatrix} 140 \\ 140 \\ \hline 100 \\ 100 \end{bmatrix} = \left[\begin{array}{cc|cc} 1/2 & 1/6 & 1/4 & 1/4 \\ 1/6 & 1/2 & 1/4 & 1/4 \\ \hline 0 & 0 & 7/12 & 1/4 \\ 0 & 0 & 1/4 & 7/12 \end{array}\right] \begin{bmatrix} 120 \\ 120 \\ \hline 120 \\ 120 \end{bmatrix} \quad (9)$$

Observe that the block-triangular decomposition in (9) also permits some shrinking of the original "large-scale" model, and note that decomposition with tearing, like aggregation, generally introduces errors into the projection process.

Figure 1 summarizes the principal points of our numerical examples by illustrating the structures of the various projection matrices used in them.

2. Shrinking by Aggregation

Aggregation in demographic analysis may be carried out by consolidating:

(1) population characteristics, e.g., combining several sex, color, or age groups;

(2) time units, e.g., dealing with five-year intervals of time instead of annual ones; and

(3) spatial units, e.g., aggregating the fifty states of the U.S.A. into its 9 Census Divisions.

In each case, the consolidated projection produces results that are coarser with regard to levels of detail than those

$$\begin{bmatrix} 1/2 & 1/6 & 1/4 & 1/4 \\ 1/6 & 1/2 & 1/4 & 1/4 \\ 1/6 & 1/6 & 1/4 & 1/4 \\ 1/6 & 1/6 & 1/4 & 1/4 \end{bmatrix} \Rightarrow \begin{bmatrix} 2/3 & 1/2 \\ 1/3 & 1/2 \end{bmatrix}$$

a. An arbitrary aggregation.

$$\begin{bmatrix} 1/2 & 1/6 & 1/4 & 1/4 \\ 1/6 & 1/2 & 1/4 & 1/4 \\ 1/6 & 1/6 & 1/4 & 1/4 \\ 1/6 & 1/6 & 1/4 & 1/4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1/2 & 2/9 \\ 1/2 & 7/9 \end{bmatrix}$$

b. A bi-regional aggregation.

$$\begin{bmatrix} 1/2 & 1/6 & 1/4 & 1/4 \\ 1/6 & 1/2 & 1/4 & 1/4 \\ 1/6 & 1/6 & 1/4 & 1/4 \\ 1/6 & 1/6 & 1/4 & 1/4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1/6 & 0 & 0 \\ 1/6 & 1 & 0 & 0 \\ 0 & 0 & 7/12 & 1/4 \\ 0 & 0 & 1/4 & 7/12 \end{bmatrix}$$

c. Complete (compensated) decomposition into block-diagonal form.

$$\begin{bmatrix} 1/2 & 1/6 & 1/4 & 1/4 \\ 1/6 & 1/2 & 1/4 & 1/4 \\ 1/6 & 1/6 & 1/4 & 1/4 \\ 1/6 & 1/6 & 1/4 & 1/4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1/2 & 1/6 & 1/4 & 1/4 \\ 1/6 & 1/2 & 1/4 & 1/4 \\ 0 & 0 & 7/12 & 1/4 \\ 0 & 0 & 1/4 & 7/12 \end{bmatrix}$$

d. Partial (compensated) decomposition into (upper) block-triangular form.

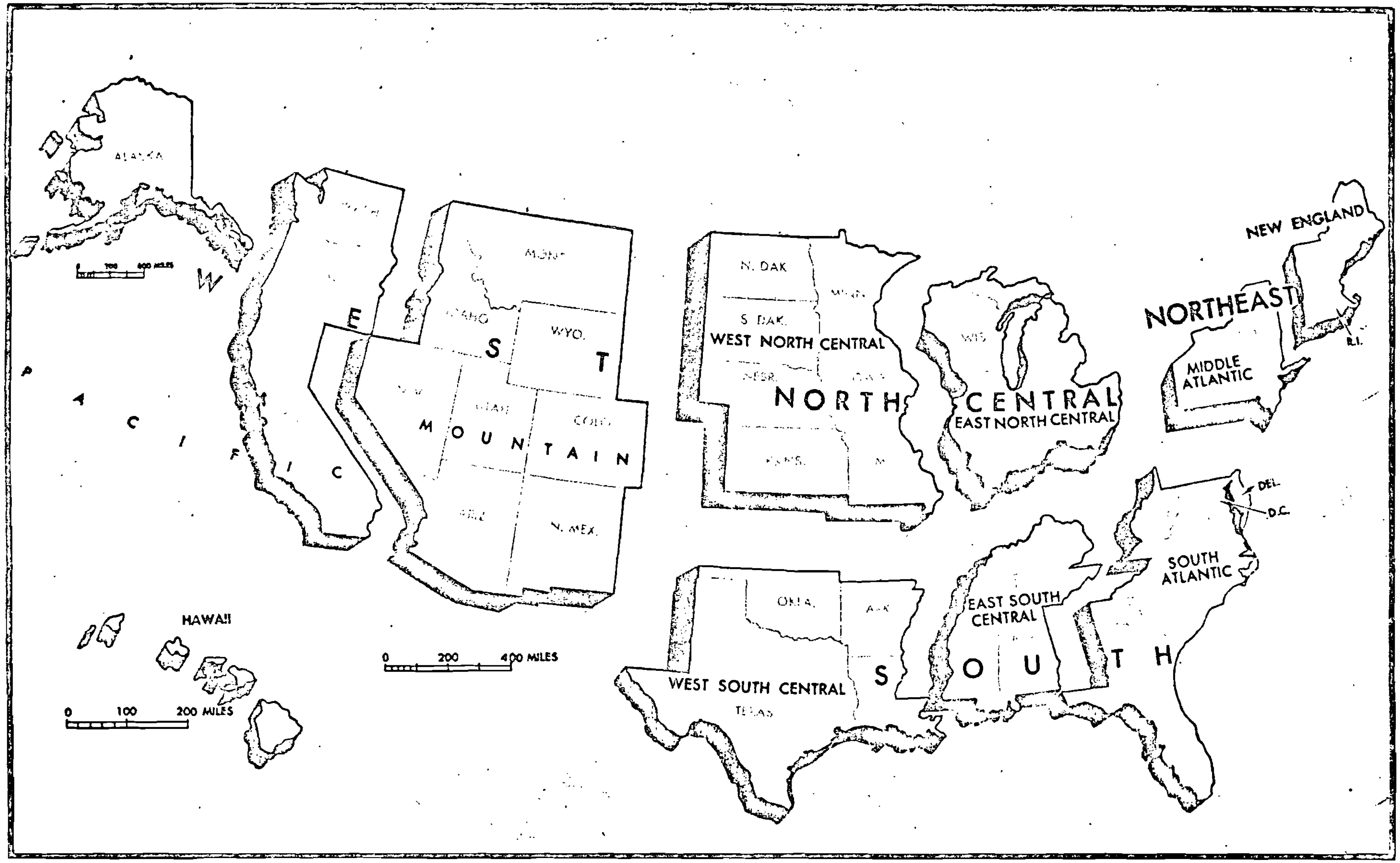
Figure 1

An illustration of the principal means of shrinking population projection matrices.

provided by the original unconsolidated model.

Consider, for example, the two multiregional population systems illustrated in Figure 2: the 9 Census Divisions of the U.S. and the corresponding 4 Census Regions. Spatial expectations of life at birth and migration levels for the 9-region population system are set out in Table 1, and a cohort-survival population projection carried out using 5-year age groups produces the aggregate results that appear in Table 2. A spatial consolidation of the 9 Census Divisions into the 4 Census Regions permits a considerable shrinkage of the original model, but the process introduces some aggregation error and, more importantly, leads to population projections that are less detailed geographically than those obtained from the unconsolidated model. This can be seen by examining Tables 3 and 4, which give the 4-region counterparts of the 9-region results set out in Tables 1 and 2, respectively.

Collectively, the four tables illustrate the following important features of aggregation. First, aggregated demographic measures are weighted averages of the corresponding disaggregated measures. Second, spatial aggregation necessarily reduces the level of interregional migration, since a part of what previously was defined to be interregional migration becomes intraregional migration in the consolidated model. Finally, aggregation normally introduces an aggregation bias or error into the consolidated population projections.



SOURCE: U.S. BUREAU OF THE CENSUS

Figure 2

Regions and Geographic Divisions of the United States

TABLE 1

EXPECTATIONS OF LIFE AT BIRTH AND MIGRATION LEVELS BY DIVISION OF RESIDENCE AND DIVISION OF BIRTH: UNITED STATES TOTAL POPULATION, 1958.

A. EXPECTATIONS OF LIFE AT BIRTH: ${}_ie_j(0)$

DIVISION OF BIRTH	DIVISION OF RESIDENCE									TOTAL
	1	2	3	4	5	6	7	8	9	
1. New England	44.75	6.16	3.03	1.04	6.46	0.82	1.52	1.16	5.06	70.00
2. Middle Atlantic	2.50	48.71	3.58	0.89	6.70	0.87	1.31	1.05	4.07	69.68
3. East North Central	0.89	2.56	47.14	2.61	5.16	2.05	2.08	1.85	5.82	70.17
4. West North Central	0.79	1.75	6.32	39.56	3.45	1.20	3.98	4.13	9.57	70.75
5. South Atlantic	1.58	5.16	4.82	1.28	45.39	2.57	2.31	1.23	4.46	68.81
6. East South Central	0.77	2.27	8.94	1.68	8.36	37.48	3.81	1.28	4.25	68.83
7. West South Central	0.76	1.76	3.85	3.16	3.98	2.25	41.90	3.39	8.48	69.54
8. Mountain	0.97	2.00	3.87	3.89	3.47	1.17	5.28	33.22	15.90	69.78
9. Pacific	1.03	2.10	3.35	2.55	3.72	1.08	3.56	4.19	48.65	70.21

TABLE 1 (Continued)

EXPECTATIONS OF LIFE AT BIRTH AND MIGRATION LEVELS BY DIVISION OF RESIDENCE AND DIVISION OF BIRTH: UNITED STATES TOTAL POPULATION, 1958.

B. MIGRATION LEVELS: ${}_{i}\theta_{j}$

DIVISION OF BIRTH	DIVISION OF RESIDENCE									TOTAL
	1	2	3	4	5	6	7	8	9	
1. New England	0.6393	0.0880	0.0433	0.0149	0.0923	0.0117	0.0217	0.0166	0.0723	1.00
2. Middle Atlantic	0.0357	0.6991	0.0514	0.0128	0.0962	0.0125	0.0188	0.0151	0.0584	1.00
3. East North Central	0.0127	0.0365	0.6718	0.0372	0.0735	0.0292	0.0296	0.0264	0.0829	1.00
4. West North Central	0.0112	0.0248	0.0893	0.5592	0.0488	0.0170	0.0563	0.0584	0.1353	1.00
5. South Atlantic	0.0230	0.0750	0.0700	0.0186	0.6596	0.0373	0.0336	0.0179	0.0648	1.00
6. East South Central	0.0112	0.0330	0.1299	0.0244	0.1215	0.5445	0.0554	0.0186	0.0617	1.00
7. West South Central	0.0109	0.0253	0.0554	0.0454	0.0572	0.0324	0.6025	0.0487	0.1219	1.00
8. Mountain	0.0139	0.0287	0.0555	0.0557	0.0497	0.0168	0.0757	0.4761	0.2279	1.00
9. Pacific	0.0147	0.0299	0.0477	0.0363	0.0530	0.0154	0.0507	0.0597	0.6929	1.00

TABLE 2

MULTIREGIONAL PROJECTIONS TO STABILITY:
UNITED STATES TOTAL POPULATION, 1958, NINE-REGION PROJECTION

Projections and Stable Growth Parameters	DIVISION OF RESIDENCE 1 New England	2 Middle Atlantic	3 East North Central	4 West North Central	5 South Atlantic	6 East South Central	7 West South Central	8 Mountain	9 Pacific	TOTAL
K (1958)	9,911,000	33,181,000	35,763,000	15,114,000	24,749,000	11,769,000	16,177,000	6,349,000	19,141,000	172,154,000
% (1958)	0.0576	0.1927	0.2077	0.0878	0.1438	0.0684	0.0940	0.0369	0.1112	1.0000
K (2008)	21,644,039	59,187,140	80,761,069	31,173,278	68,283,065	24,394,274	40,446,886	22,805,818	73,166,573	421,862,143
% (2008)	0.0513	0.1403	0.1914	0.0739	0.1619	0.0578	0.0959	0.0541	0.1734	1.0000
r (∞)					0.02184					
% (∞)	0.0447	0.1013	0.1719	0.0727	0.1535	0.0492	0.1024	0.0680	0.2362	1.0000

TABLE 3

EXPECTATIONS OF LIFE AT BIRTH AND MIGRATION LEVELS BY REGION OF RESIDENCE AND REGION OF BIRTH: UNITED STATES TOTAL POPULATION, 1958.

A. EXPECTATIONS OF LIFE AT BIRTH: ${}_{i}e_{j}(0)$

REGION OF BIRTH	REGION OF RESIDENCE				TOTAL
	1	2	3	4	
1. Northeast	50.90	4.49	8.88	5.50	69.76
2. North Central	3.18	48.45	9.10	9.60	70.32
3. South	4.58	7.52	49.21	7.67	68.98
4. West	3.18	6.60	8.95	51.22	69.94

B. MIGRATION LEVELS: ${}_{i}\theta_{j}$

REGION OF BIRTH	REGION OF RESIDENCE				TOTAL
	1	2	3	4	
1. Northeast	0.7295	0.0643	0.1273	0.0788	1.00
2. North Central	0.0452	0.6889	0.1294	0.1365	1.00
3. South	0.0664	0.1091	0.7134	0.1111	1.00
4. West	0.0454	0.0944	0.1279	0.7322	1.00

TABLE 4

MULTIREGIONAL PROJECTIONS TO STABILITY:
UNITED STATES TOTAL POPULATION, 1958, FOUR-REGION PROJECTION

Projections and Stable Growth Parameters	REGION OF RESIDENCE 1 NORTHEAST	2 NORTH CENTRAL	3 SOUTH	4 WEST	TOTAL
K (1958)	43,092,000	50,877,000	52,695,000	25,490,000	172,154,000
% (1958)	0.2503	0.2955	0.3061	0.1481	1.0000
K (2008)	80,383,757	112,077,195	132,843,209	96,955,108	422,259,268
% (2008)	0.1904	0.2654	0.3146	0.2296	1.0000
r (∞)	0.02192				
% (∞)	0.1431	0.2491	0.3046	0.3032	1.0000

These three features may be illustrated with the numerical data set out in Tables 1 through 4. For example, Table 1 shows that a baby born in the New England Division of the U.S. and subjected to the multiregional regime of mortality and migration that prevailed in 1958 would have a life expectancy of 70 years (${}_1e(0) = 70.00$), over a third of which would be lived outside of the Division of birth ($\sum_{j \neq 1} {}_1\theta_j = 0.3607$). The corresponding life expectancy of a baby born in the Middle Atlantic Division is 69.68 years. Aggregation of the 9 Divisions into the 4 Regions consolidates these two cohorts of babies, according them an average life expectancy of 69.76 years (Table 3A).

The levels of interregional migration in the 9-region system may be measured by summing the off-diagonal elements in each row of the matrix in Table 1B. These sums define, for each regional cohort, the average fraction of a lifetime that is expected to be lived outside the region of birth. Such a summation results in values of 0.3607 and 0.3009, respectively, for the New England and Middle Atlantic Divisions of the U.S., for example. The same computation for the larger Northeastern region, however, gives the lower value of 0.2705.

Finally, a comparison of the population projections summarized in Tables 2 and 4 indicates the magnitudes of the aggregation errors that are introduced by the consolidation of the 9 Divisions into the 4 Regions. For the U.S. as a whole one finds, for example, that a 50-year projection of the 1958 population to the year 2008, on the assumption of an unchanging growth regime, produces an over-projection of almost 400,000 people. But, curiously enough, further projection of the same population until stability does not appreciably alter the intrinsic rate of growth (r) of the multiregional system. A difference of 0.00008 is all that distinguishes the intrinsic rate of growth of the 4-region projection from that of the 9-region projection.

Aggregation over regions preserves age-specific details at the expense of geographic details. If the latter are of greater interest than the former, one may instead consolidate all age groups into a single variable and retain the original set of geographical areas. The application of such an aggregation to the cohort-survival model associated with Tables 1 and 2 yields the components-of-change projection process illustrated in Figure 3 and produces the multiregional projections in Table 5.

Table 5 reveals that a components-of-change aggregation of the original cohort-survival model leads to a substantial underprojection of total population growth, but a relatively accurate projection of the spatial distribution of that growth. The total U.S. population in the year 2008, for example, is underprojected by over 51 million people, and the intrinsic rate of growth is underprojected by more than 6 per 1000. Yet the Pacific Division is allocated approximately 17 per cent of the total population in the year 2008 by both models.

1959											1958
10,022,829		1.001728	0.001205	0.000342	0.000284	0.000775	0.000270	0.000289	0.000444	0.000436	9,911,000
33,457,706		0.002935	1.002820	0.001049	0.000580	0.002710	0.000812	0.000625	0.000823	0.000827	33,181,000
36,216,395		0.001106	0.001430	1.004586	0.003253	0.002303	0.005235	0.001613	0.001838	0.001328	35,763,000
15,249,522		0.000349	0.000268	0.001297	0.999266	0.000507	0.000714	0.001712	0.002556	0.001202	15,114,000
25,261,427	=	0.003269	0.003430	0.002605	0.001362	1.005136	0.004931	0.001740	0.001499	0.001549	24,749,000
11,892,775		0.000253	0.000279	0.001073	0.000486	0.001524	0.999640	0.001315	0.000530	0.000427	11,769,000
16,429,159		0.000511	0.000408	0.000792	0.001978	0.001053	0.002181	1.004362	0.003391	0.001647	16,177,000
6,518,501		0.000426	0.000395	0.000887	0.002591	0.000507	0.000471	0.002060	0.996787	0.002626	6,349,000
19,678,904		0.002081	0.001548	0.002471	0.004574	0.001823	0.001545	0.004025	0.010701	1.005854	19,141,000

Figure 3

The multiregional components-of-change population projection model: United States total population, 1958, nine-region projection

TABLE 5

MULTIREGIONAL PROJECTIONS TO STABILITY:

UNITED STATES TOTAL POPULATION, 1958, NINE-REGION PROJECTIONS

Projections and Stable Growth Parameters	DIVISION OF RESIDENCE 1 New England	2 Middle Atlantic	3 East North Central	4 West North Central	5 South Atlantic	6 East South Central	7 West South Central	8 Mountain	9 Pacific	TOTAL
K (1958)	9,911,000	33,181,000	35,763,000	15,114,000	24,749,000	11,769,000	16,177,000	6,349,000	19,141,000	172,154,000
% (1958)	0.0576	0.1927	0.2077	0.0878	0.1438	0.0684	0.0940	0.0369	0.1112	1.0000
K (2008)	17,927,349	53,159,821	68,434,148	25,822,107	62,159,432	21,199,129	35,493,951	19,076,175	61,336,572	364,608,685
% (2008)	0.0492	0.1458	0.1877	0.0708	0.1705	0.0581	0.0973	0.0523	0.1682	1.0000
r (∞)	0.01554									
% (∞)	0.0360	0.0897	0.1516	0.0631	0.1748	0.0490	0.1107	0.0717	0.2533	1.0000

The divergence between the projections in Tables 2 and 5 increases exponentially over time. Figure 4 shows that the two models project similar population totals during the first decade, start to diverge shortly thereafter, and then grow increasingly further apart. This suggests that shrinking by components-of-change aggregation is most effective for short-run projections.

We have seen that aggregation is generally accompanied by loss of detail. This, however, need not always be the case. One can, for example, obtain a bi-regionally aggregated population projection for every region of a multiregional system and thereby retain the same level of detail in the resulting collection of consolidated projections as originally existed in the single unconsolidated model. By way of illustration, consider the 9 sets of 2 x 2 regional life expectancies and migration levels that appear in Table 6. They were obtained using 9 bi-regional aggregations of the data set that produced Table 1. The projection model that produced Table 2 was similarly aggregated, and the collection of 9 bi-regional projections yielded the results set out in Table 7. A comparison of the projections in Table 7 with those in Table 2 suggests that an exhaustive collection of bi-regional aggregations is a reasonably accurate substitute for a large-scale population projection model.

Although bi-regional aggregations may be applied with some success to shrink a large model, they can be

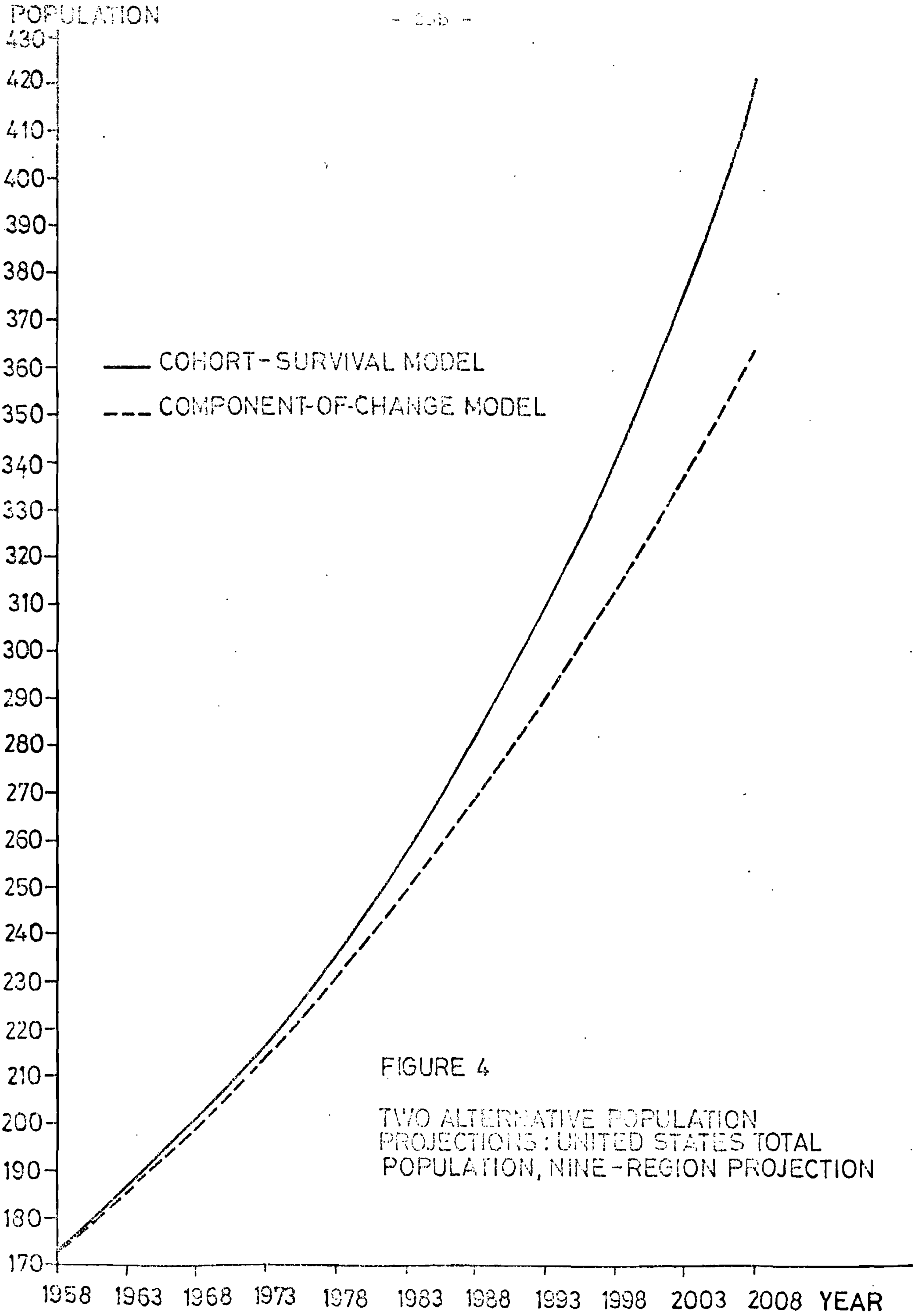


FIGURE 4

TWO ALTERNATIVE POPULATION PROJECTIONS : UNITED STATES TOTAL POPULATION, NINE-REGION PROJECTION

TABLE 6

EXPECTATIONS OF LIFE AT BIRTH AND MIGRATION LEVELS BY PLACE OF RESIDENCE AND PLACE OF BIRTH: UNITED STATES TOTAL POPULATION, 1958.

A. EXPECTATIONS OF LIFE AT BIRTH: ${}_{i}e_{j}(0)$

PLACE OF BIRTH	PLACE OF RESIDENCE 1	2	PLACE OF BIRTH	PLACE OF RESIDENCE 1	2	PLACE OF BIRTH	PLACE OF RESIDENCE 1	2
1. New England	44.70	25.28	1. West North Central	39.33	31.10	1. West South Central	41.64	27.67
2. Rest of the U.S.	1.36	68.07	2. Rest of the U.S.	2.03	67.32	2. Rest of the U.S.	2.62	66.88
	1	2		1	2		1	2
1. Middle Atlantic	48.55	21.14	1. South Atlantic	45.39	23.37	1. Mountain	32.68	36.74
2. Rest of the U.S.	3.12	66.38	2. Rest of the U.S.	5.60	64.04	2. Rest of the U.S.	2.16	67.34
	1	2		1	2		1	2
1. East North Central	47.13	22.90	1. East South Central	37.36	31.39	1. Pacific	47.96	22.09
2. Rest of the U.S.	4.86	64.51	2. Rest of the U.S.	1.69	67.84	2. Rest of the U.S.	6.31	63.18

TABLE 6 (Continued)

EXPECTATIONS OF LIFE AT BIRTH AND MIGRATION LEVELS BY PLACE OF RESIDENCE AND PLACE OF BIRTH: UNITED STATES TOTAL POPULATION, 1958.

B. MIGRATION LEVELS: ${}_{i}\theta_{j}$

PLACE OF BIRTH	PLACE OF RESIDENCE		PLACE OF BIRTH	PLACE OF RESIDENCE		PLACE OF BIRTH	PLACE OF RESIDENCE	
	1	2		1	2		1	2
1. New England	0.6388	0.3612	1. West North Central	0.5584	0.4416	1. West South Central	0.6008	0.3992
2. Rest of the U.S.	0.0196	0.9804	2. Rest of the U.S.	0.0293	0.9707	2. Rest of the U.S.	0.0378	0.9622
	1	2		1	2		1	2
1. Middle Atlantic	0.6967	0.3033	1. South Atlantic	0.6601	0.3399	1. Mountain	0.4708	0.5292
2. Rest of the U.S.	0.0449	0.9551	2. Rest of the U.S.	0.0804	0.9196	2. Rest of the U.S.	0.0311	0.9689
	1	2		1	2		1	2
1. East North Central	0.6730	0.3270	1. East South Central	0.5435	0.4565	1. Pacific	0.6847	0.3153
2. Rest of the U.S.	0.0700	0.9300	2. Rest of the U.S.	0.0243	0.9757	2. Rest of the U.S.	0.0909	0.9091

TABLE 7

MULTIREGIONAL PROJECTIONS TO STABILITY:

UNITED STATES TOTAL POPULATION, 1958, NINE BI-REGIONAL PROJECTIONS

Projections and Stable Growth Parameters	DIVISION OF RESIDENCE									TOTAL
	1 New England	2 Middle Atlantic	3 East North Central	4 West North Central	5 South Atlantic	6 East South Central	7 West South Central	8 Mountain	9 Pacific	
K (1958)	9,911,000	33,181,000	35,763,000	15,114,000	24,749,000	11,769,000	16,177,000	6,349,000	19,141,000	172,154,000
% (1958)	0.0576	0.1927	0.2077	0.0878	0.1438	0.0684	0.0940	0.0369	0.1112	1.0000
K (2008)	22,420,802	60,240,640	83,052,968	31,136,660	70,878,872	24,837,796	40,472,448	22,355,426	73,141,824	428,537,436
% (2008)	0.0523	0.1406	0.1938	0.0727	0.1654	0.0580	0.0944	0.0522	0.1707	1.0000
r (∞)	0.02157	0.02181	0.02157	0.02154	0.02155	0.02155	0.02157	0.02162	0.02159	----
% (∞)	0.0513	0.1070	0.1890	0.0663	0.1737	0.0513	0.0933	0.0565	0.2118	1.0000

computationally demanding if it is necessary that they be applied as many times as the number of regions in a multiregional system. In such instances, a more efficient and effective shrinking technique often can be developed using decomposition methods.

3. Shrinking by Decomposition

Decomposition procedures have been used often in demographic analysis, although they have not been specifically identified by that name. Perhaps their most common application is manifested in representations of multiregional population systems by collections of single-region models which assume that each regional population is undisturbed by migration. Such an assumption is, of course, equivalent to the premise that the multiregional population system is completely decomposable into independent single-region sub-systems arranged in block-diagonal form. A modification of the no-migration assumption is often introduced into the single-region model by including the impact of net migration in the survivorship proportions, i.e., by treating an out-migrant as a "death" and an in-migrant as a replacement for a death. Such a modification of the complete single-region decomposition was adopted to derive the projections in Table 8.

Table 8 presents the summary results of 9 single-region cohort-survival population projections. The regions

TABLE 8

MULTIREGIONAL PROJECTIONS TO STABILITY: UNITED STATES TOTAL POPULATION, 1958, NINE SINGLE-REGION DECOMPOSITIONS WITH NET MIGRATION

Projections and Stable Growth Parameters	DIVISION OF RESIDENCE 1 New England	2 Middle Atlantic	3 East North Central	4 West North Central	5 South Atlantic	6 East South Central	7 West South Central	8 Mountain	9 Pacific	TOTAL
K (1958)	9,911,000	33,181,000	35,763,000	15,114,000	24,749,000	11,769,000	16,177,000	6,349,000	19,141,000	172,154,000
% (1958)	0.0576	0.1927	0.2077	0.0878	0.1438	0.0684	0.0940	0.0369	0.1112	1.0000
K (2008)	21,361,806	54,784,164	80,574,344	27,888,196	72,708,288	21,538,842	38,569,232	27,877,196	105,479,992	450,782,060
% (2008)	0.0474	0.1215	0.1787	0.0619	0.1613	0.0478	0.0856	0.0618	0.2340	1.0000
r (∞)	0.02027	0.01451	0.02049	0.01638	0.02379	0.01400	0.02034	0.03207	0.03907	------
% (∞)	0.0543	0.1856	0.2025	0.0861	0.1549	0.0750	0.0976	0.0393	0.1047	1.0000

are those delineated in Figure 2, and the results correspond to the ones set out earlier in Table 2. Thus Table 8 may be viewed as the output produced by a particular shrinking of the "large-scale" population projection model associated with Table 2. The discrepancies between the two sets of results may be attributed largely to the representation of interregional migration as net migration in the decomposed model.

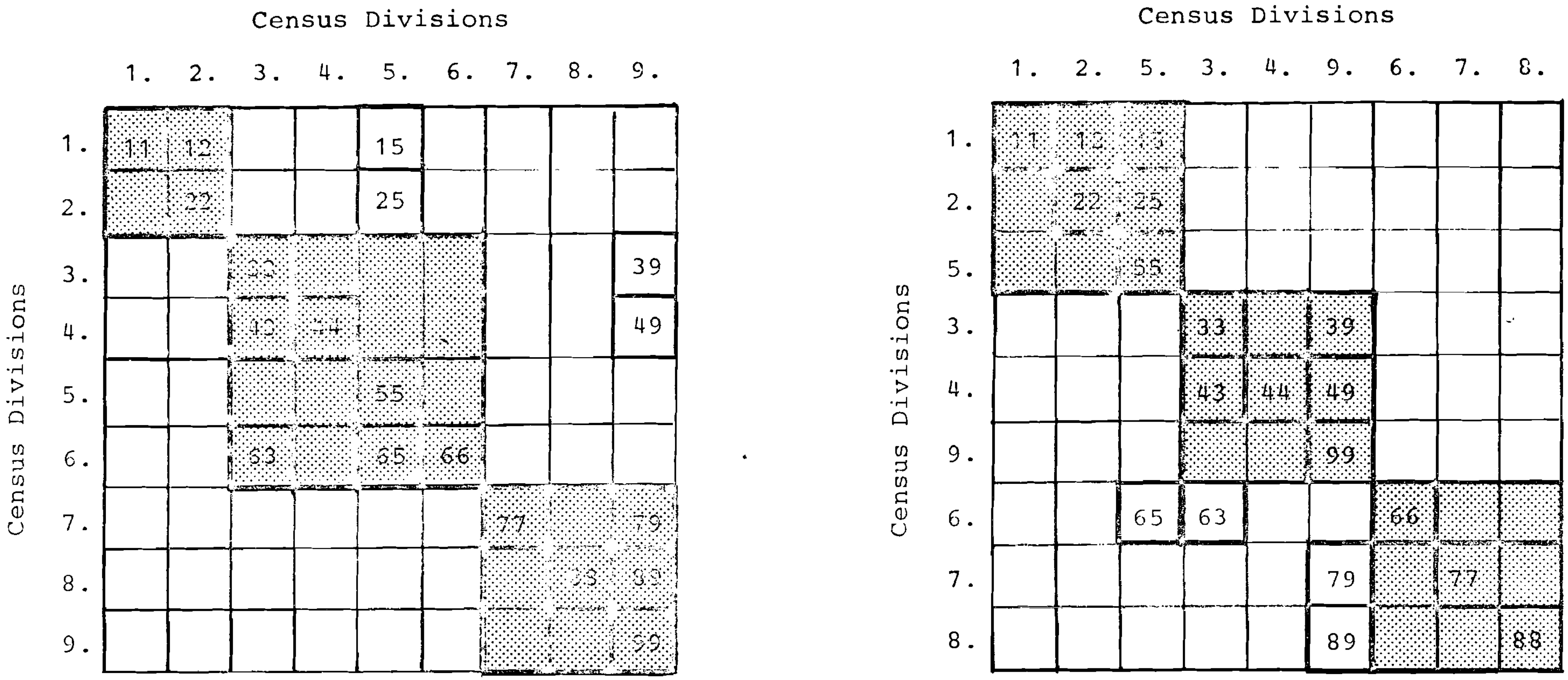
Table 8 reveals that the representation of internal migration as a net flow can introduce serious errors into the population projection process. Net migration is defined with respect to the particular regional population being projected. If that population is currently experiencing an excess of in-migrants over out-migrants, this feature will be built-in as part of the projection process, and its effects will multiply and increase cumulatively over time. The converse applies, of course, to regions experiencing net out-migration. In short, regional populations with a positive net migration rate are likely to be overprojected and those with a negative net migration rate are likely to be underprojected. The projections in Table 8 support this argument. Only the populations of the three Census Divisions that experienced a positive net migration in 1958 are overprojected in the year 2008 (i.e., the South Atlantic, the Mountain, and the Pacific Divisions); the populations of the remaining six Census Divisions are underprojected.

The original 9-region population projection model and its complete single-region decomposition represent opposite extremes of the decomposition spectrum. A large number of alternatives lie in between, two of which appear in Figure 5.

Figure 5 describes two complete decompositions of the 9-region population system. Both decompositions reflect the particular structure of interregional migration levels described in Table 1A, and both were defined by an essentially arbitrary decision to delete interregional linkages that exhibited migration levels below eight percent. Since in both cases this procedure still did not produce a complete decomposition, four additional migration levels (those lying outside of the block-diagonal submatrices in Figure 5) were also deleted in each decomposition.

Figure 5A illustrates a decomposition of the 9-region population model into three smaller multiregional models containing two, four, and three regions, respectively. Internal migration is treated as a place-to-place flow among regions within each diagonal block and as a net flow elsewhere. Thus we have here an example of compensated tearing in which the conceptual approaches at both extremes of the decomposition spectrum are represented. Table 9 summarizes the multiregional population projections produced by this particular model.

Figure 5B depicts an alternative decomposition. In this instance, a permutation of the rows and columns of the



A. Complete Decomposition A

B. Complete Decomposition B

Figure 5

Two alternative decompositions of a multiregional system

TABLE 9

MULTIREGIONAL PROJECTIONS TO STABILITY:
UNITED STATES TOTAL POPULATION, 1958, DECOMPOSITION A

Projections and Stable Growth Parameters	DIVISION OF RESIDENCE									TOTAL
	1 New England	2 Middle Atlantic	3 East North Central	4 West North Central	5 South Atlantic	6 East South Central	7 West South Central	8 Mountain	9 Pacific	
K (1958)	9,911,000	33,181,000	35,763,000	15,114,000	24,749,000	11,769,000	16,177,000	6,349,000	19,141,000	172,154,000
% (1958)	0.0576	0.1927	0.2077	0.0878	0.1438	0.0684	0.0940	0.0369	0.1112	1.0000
K (2008)	20,818,862	55,406,756	79,776,664	28,969,902	69,440,440	23,452,330	42,158,288	27,528,572	93,899,880	441,451,694
% (2008)	0.0472	0.1255	0.1807	0.0656	0.1573	0.0531	0.0955	0.0624	0.2127	1.0000
r (∞)	0.01664	0.01664	0.02026	0.02026	0.02026	0.02026	0.03289	0.03289	0.03289	-----
% (∞)	0.0979	0.1299	0.2036	0.0588	0.2301	0.0544	0.0312	0.0372	0.1570	1.0000

migration level matrix and a decision to delete a different set of four migration levels yields a different connectivity structure and associated decomposition. This decomposition partitions the 9-region system into three 3-region subsystems and results in the projections set out in Table 10.

The two alternative decompositions both overproject the total U.S. population in 2008. The individual regional shares of this total population follow the general pattern exhibited by the single-region decomposition of Table 8. That is, regional populations experiencing positive net migration in 1958 are accorded a larger than warranted regional share, and vice-versa. This pattern arises out of the particular method of compensated tearing used in the projections, i.e., compensation by means of net migration, and reflects the same biases that were found in the single-region decomposition.

Another contributor to the discrepancies between the results of the two decomposed models and those of the original model is the insufficiently weak degree of connectivity between the various sets of multiregional subsystems. Recall that, for illustrative purposes, we arbitrarily deleted internal migration flows associated with migration levels below 8 per cent. It is likely that this is much too high a value for a threshold level, and its adoption undoubtedly contributed something to the overall projection error. That contribution, however, is surely small compared to the one introduced by the representation of internal

TABLE 10

MULTIREGIONAL PROJECTIONS TO STABILITY:

UNITED STATES TOTAL POPULATION, 1958, DECOMPOSITION B

Projections and Stable Growth Parameters	DIVISION OF RESIDENCE									TOTAL
	1 New England	2 Middle Atlantic	3 East North Central	4 West North Central	5 South Atlantic	6 East South Central	7 West South Central	8 Mountain	9 Pacific	
K (1958)	9,911,000	33,181,000	35,763,000	15,114,000	24,749,000	11,769,000	16,177,000	6,349,000	19,141,000	172,154,000
% (1958)	0.0576	0.1927	0.2077	0.0878	0.1438	0.0684	0.0940	0.0369	0.1112	1.0000
K (2008)	21,162,692	57,420,652	82,082,112	30,588,244	69,149,768	22,266,124	39,261,660	25,469,752	87,833,784	435,234,788
% (2008)	0.0486	0.1319	0.1886	0.0703	0.1589	0.0512	0.0902	0.0585	0.2018	1.0000
r (∞)	0.02018	0.02018	0.02900	0.02900	0.02018	0.02555	0.02555	0.02555	0.02900	-----
% (∞)	0.0637	0.1385	0.0566	0.0298	0.2924	0.0111	0.0481	0.0943	0.2656	1.0000

migration as a net flow. Both sources of error are, of course, interrelated. The level of compensation which is required in the form of net migration is intimately related to the amount of net migration which is to be treated in that way, and this amount in turn depends on the volume of migration that falls below the threshold level.

Aggregation and decomposition techniques are not mutually exclusive methods of shrinking a large-scale population model. They can, of course, be combined in various ways to reduce the dimensionality of such a model without incurring a major sacrifice in accuracy or level of detail in the process. We now turn to an examination of one of the more obvious ways in which they may be combined and compare its empirical performance with that of an equally obvious alternative.

4. Aggregation and Decomposition Combined

The idea that it might be useful to model different parts of a large system at different levels of detail received one of its first formal mathematical treatments two decades ago in a seminal paper read by Herbert Simon and Albert Ando at the meetings of the Econometric Society in December of 1956 and subsequently published in *Econometrica* five years later (Simon and Ando, 1961)[1].

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[1]A recent revival of interest in this fundamental idea has produced several interesting articles, one of which specifically suggests an application to migration modeling (Batty and Masser, 1975).

The essence of their basic argument is neatly captured by the following physical illustration:

"Consider a building whose outside walls provide perfect thermal insulation from the environment. The building is divided into a large number of rooms, the walls between them being good, but not perfect, insulators. Each room is divided into a number of offices by partitions. The partitions are poor insulators. A thermometer hangs in each of the offices. Suppose that at time t_0 the various offices within the building are in a state of thermal disequilibrium--there is a wide variation in temperature from office to office and from room to room. When we take new temperature readings at time t_1, several hours after t_0, what will we find? At t_1 there will be very little variation in temperature among the offices within each single room, but there may still be large temperature variations among rooms. When we take readings again at time t_2, several days after t_1, we find an almost uniform temperature throughout the building; the temperature differences among rooms have virtually disappeared.

A temperature equilibrium within each room will be reached rather rapidly, while a temperature equilibrium among rooms will be reached only slowly,...as long as we are not interested in the rapid fluctuations in temperature among offices in the same room, we can learn all we want to know about the dynamics of this system by placing a single thermometer in each room--it is unnecessary to place a thermometer in each office." (Simon and Ando, 1961, pp. 70-71).

4.1 The Simon-Ando Theorem

Recognizing that complete decomposability is relatively rare in socioeconomic systems, Simon and Ando (1961) examine the behavior of linear dynamic systems with "nearly" completely decomposable subsystems. They show that, in the short-run, such systems behave almost as though they were in fact completely decomposable and that, in the middle-run, their behavior can be studied by consolidating the variables of each subsystem into a single variable and ignoring the interrelationships within each subsystem[2].

The crux of the Simon-Ando theorem is the assertion that the equilibrium of a nearly completely decomposable dynamic linear system may be viewed as a composite growth process which evolves in three temporal phases. During the first phase, the variables in each subsystem arrive at equilibrium positions determined by the completely decomposed system. After a longer time-period, the system enters its second phase, at which point the variables of each subsystem, maintaining their proportional relationships, move together as a block toward equilibrium values established by the third phase of the growth process. In this final phase, all variables approach the rate of growth defined by the largest characteristic root of the matrix associated with the original nearly completely decomposable system.

The Simon-Ando theorem suggests a shrinking procedure for large-scale population projection models that combines aggregation and decomposition in a particularly appealing way. One begins by partitioning the large multiregional

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[2]In a subsequent paper, Ando and Fisher (1963) extend the Simon-Ando theorem to nearly block-triangular (i.e., nearly partially decomposable) linear systems. Although we do not consider such systems in the rest of this paper, it should be clear that our exposition could be appropriately expanded to cover this more general case of near decomposability.

system projection model into smaller submodels in a way that effectively exploits any weak interdependencies revealed by indices such as migration levels. The growth of the original multiregional system then may be projected by appropriately combining (1) the results of disaggregated intra-subsystem projections, in which within subsystem interactions are represented at a relatively fine level of detail, with (2) the results of aggregate inter-subsystem projections, in which the between subsystem interactions are modeled at a relatively coarse level of detail. For example, within each multiregional subsystem, the projection model could focus on the full age composition of every regional population and examine its evolution over time; between each multiregional subsystem, the projection model would suppress the regional age compositions and would deal only with total populations. In the short-run, the within subsystem interactions would dominate the behavior of the system; in the long-run, the between subsystem interactions would become increasingly important and ultimately would determine the behavior of the entire system.

4.2 A Numerical Illustration

The above discussion can be illuminated with the aid of a simple numerical example drawn from the Simon and Ando paper. Recall the 4-region numerical illustration in Section 1.3, and assume that the projection matrix of that multiregional system is now taken to be the nearly completely decomposable matrix

$$\left[\begin{array}{cc|cc} 0.9700 & 0.0200 & 0 & 0.0002 \\ 0.0295 & 0.9800 & 0 & 0.0002 \\ \hline 0.0005 & 0 & 0.9600 & 0.0396 \\ 0 & 0 & 0.0400 & 0.9600 \end{array}\right] = \underset{\sim}{G} \text{ , say.} \tag{10}$$

Let the corresponding completely decomposable matrix be

$$\left[\begin{array}{cc|cc} 0.9700 & 0.0200 & 0 & 0 \\ 0.0300 & 0.9800 & 0 & 0 \\ \hline 0 & 0 & 0.9600 & 0.0400 \\ 0 & 0 & 0.0400 & 0.9600 \end{array}\right] = \left[\begin{array}{c|c} \underset{\sim}{G}_1 & \underset{\sim}{0} \\ \hline \underset{\sim}{0} & \underset{\sim}{G}_2 \end{array}\right] = \underset{\sim}{G}_d\text{, say.} \tag{11}$$

Note that $\underset{\sim}{G}_1$ is the disaggregated projection matrix for the North-South subsystem, and $\underset{\sim}{G}_2$ is the disaggregated projection matrix for the East-West subsystem.[3] The original projection matrix $\underset{\sim}{G}$ may be consolidated to give the aggregated projection matrix needed for modeling the interrelated growth of

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[3]Note that in Simon and Ando's numerical illustration the compensation for tearing is introduced in the off-diagonal elements. For example, the element 0.0005 in (10) is added to 0.0295 to give the 0.0300 in (11). Our compensation procedure would instead have added it to 0.9700.

these two subsystems:[4]

$$\hat{G} = CGD = \begin{bmatrix} 0.9998 & 0.0002 \\ 0.0002 & 0.9998 \end{bmatrix} , \qquad (12)$$

where

$$C = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 0.4 & 0 \\ 0.6 & 0 \\ 0 & 0.5 \\ 0 & 0.5 \end{bmatrix} .$$

The long-run behavior of this particular system can be studied by examining the behavior of the elements of the

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[4]The weights in the D matrix are those used by Simon and Ando. They are the proportions defined by the characteristic vector associated with the largest characteristic root of the G matrix. In most applications it is much more convenient to use the proportions defined by the observed population distribution, because such a procedure avoids the necessity of calculating the largest characteristic root and its associated characteristic vector. A compromise solution is to use the roots and vectors of the individual submatrices, which in this particular illustration leads to practically the same numerical results. (Note that the largest characteristic root of every G matrix in this illustration is unity.)

matrix $\underset{\sim}{G}$ as it is raised to higher powers. It is a simple exercise on a digital computer to show that

$$\underset{\sim}{G}^{128} = \left[\begin{array}{cc|cc} 0.390089 & 0.392503 & 0.009465 & 0.011385 \\ 0.579037 & 0.586246 & 0.013138 & 0.015999 \\ \hline 0.016631 & 0.011831 & 0.487509 & 0.485107 \\ 0.014244 & 0.009419 & 0.489888 & 0.487509 \end{array}\right] \quad (13)$$

and that

$$\underset{\sim}{G}^{(128)^2} = \underset{\sim}{G}^{128} \times \underset{\sim}{G}^{128} = \left[\begin{array}{cc|cc} 0.200776 & 0.200782 & 0.200222 & 0.200225 \\ 0.298656 & 0.298664 & 0.297829 & 0.297833 \\ \hline 0.250286 & 0.250279 & 0.250973 & 0.250970 \\ 0.250282 & 0.250275 & 0.250976 & 0.250973 \end{array}\right] . \quad (14)$$

Observe that the elements in the <u>diagonal</u> <u>submatrices</u> maintain the same proportion over the rows and independently of the columns within each submatrix while moving toward their equilibrium values. That is, both in (13) and (14) the proportional <u>within</u> subsystem allocation is one of 0.4 to 0.6 in the upper diagonal submatrix and one of 0.5 to 0.5 in the lower diagonal submatrix. The same within subsystem allocations are also defined by the completely decomposable system, i.e.,

$$\begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix} = \begin{bmatrix} 0.9700 & 0.0200 \\ 0.0300 & 0.9800 \end{bmatrix} \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix} \qquad (15)$$

$$\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0.9600 & 0.0400 \\ 0.0400 & 0.9600 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} \qquad (16)$$

The between subsystem allocations are defined by the characteristic vector associated with the largest characteristic root of $\hat{\underset{\sim}{G}}$ in (12) and may be shown to be equal to each other:

$$\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0.9998 & 0.0002 \\ 0.0002 & 0.9998 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} . \qquad (17)$$

Combining the information on within subsystem allocations with that on between subsystem allocations, we define the completely decomposable approximation of (14) to be the matrix

$$\left[\begin{array}{cc|cc} 0.20 & 0.20 & 0.20 & 0.20 \\ 0.30 & 0.30 & 0.30 & 0.30 \\ \hline 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 \end{array}\right] . \qquad (18)$$

Note that the column proportions in (18) indicate that at equilibrium (i.e., during stable growth), the multiregional population of 480 individuals will be distributed among the four regions according to the following allocations: 96 individuals in the North, 144 in the South, 120 in the East, and another 120 in the West.

4.3 Simple Shrinking by Aggregation and Decomposition

The Simon and Ando theorem suggests the following simple method for shrinking large-scale population projection models. We begin by partitioning a multiregional system into its constituent single regions and projecting their growth and change as if they were independent closed population subsystems undisturbed by migration. The first stage, therefore, corresponds to a single-region decomposition with zero net migration. We then suppress all age-specific details and project the multiregional population using a components-of-change model. The results of the latter stage determine the total multiregional population and its spatial distribution; the results of the first stage define the individual regional age compositions. In this way, within subsystem interactions (i.e., changes in age structure) are modeled at a fine level of detail, whereas between subsystem interactions (i.e., changes in spatial structure) are modeled at a course level of detail. If the original multiregional system is sufficiently close to being nearly decomposable, the approximate (two-stage) projection should produce a reasonably accurate multiregional population projection.

The shrinking procedure described above may be applied to the "large-scale" nine-region population projection model of Table 2. Table 11 sets out the principal results generated by such a shrinking of the original model. The growth of the total population and its spatial allocation are taken

TABLE 11

MULTIREGIONAL PROJECTIONS TO STABILITY: UNITED STATES TOTAL POPULATION, 1958,
NINE SINGLE-REGION (NO-MIGRATION) DECOMPOSITIONS WITH COMPONENTS-OF-CHANGE AGGREGATION

	1	2	3	4	5	6	7	8	9	TOTAL
K (1958)	9,911,000	33,181,000	35,763,000	15,114,000	24,749,000	11,769,000	16,177,000	6,349,000	19,141,000	172,154,000
% (1958)	0.0576	0.1927	0.2077	0.0878	0.1438	0.0684	0.0940	0.0369	0.1112	1.0000
K (2008)	17,927,349	53,159,821	68,434,148	25,822,107	62,159,432	21,199,129	35,493,951	19,076,175	61,336,572	364,608,685
% (2008)	0.0492	0.1458	0.1877	0.0708	0.1705	0.0581	0.0973	0.0523	0.1682	1.0000
r (∞)	0.01554									
% (∞)	0.0360	0.0897	0.1516	0.0631	0.1748	0.0490	0.1107	0.0717	0.2533	1.0000
2008: Approximate Projection										
C (0-14)	0.3544	0.3378	0.3678	0.3690	0.3546	0.3655	0.3742	0.3728	0.3560	----
C (15-64)	0.5889	0.6004	0.5778	0.5751	0.5879	0.5814	0.5725	0.5740	0.5836	----
C (65+)	0.0567	0.0618	0.0544	0.0559	0.0575	0.0532	0.0533	0.0532	0.0604	----
2008: Original Projection										
C (0-14)	0.3560	0.3367	0.3642	0.3664	0.3513	0.3621	0.3709	0.3740	0.3587	0.3581
C (15-64)	0.5873	0.5988	0.5802	0.5713	0.5840	0.5765	0.5696	0.5719	0.5865	0.5825
C (65+)	0.0567	0.0644	0.0557	0.0623	0.0647	0.0614	0.0595	0.0541	0.0548	0.0594

from the projection in Table 5; the individual regional age compositions (consolidated into three age groups for ease of presentation) were obtained by recomputing the single-region projections of Table 8 with net migration set equal to zero. The combined results indicate that regional age compositions and regional shares are projected moderately well, but that the total multiregional population is seriously underprojected. (The latter is no surprise since it already was observed and discussed in connection with Table 5.)

In applying the above shrinking procedure we adopted the regional age compositions of the single-region (no-migration) projections and the regional shares of the components-of-change projection. For the total multiregional population we chose the level projected by the latter (364,608,685); we would have done much better to have used that of the former (419,173,278). In the remainder of this paper, therefore, we shall modify the shrinking procedure accordingly and shall define the resulting modified version to be the cohort-components method of simple shrinking. This method adopts the regional age compositions and total multiregional population projected by a collection of single-region cohort-survival models that ignore migration, and then spatially allocates this total population according to the regional shares projected by a components-of-change model.

The accuracy with which the bi-regionally aggregated models of Table 7 approximated the original projection in

Table 2 suggests another method of simple shrinking, one which we shall call the cohort-biregional method of simple shrinking. In this method, the tearing occasioned by complete decompositions of the kind defined in Figure 5 are compensated not by net migration but by bi-regional aggregation. Specifically, each multiregional subsystem is augmented by an additional "rest-of-the-world" region which serves as the destination of all migration out of the subsystem and as the source of all migration into the subsystem. Table 12 presents the results produced by the application of such a method of shrinking to the projection model of Table 2. The particular decomposition scheme adopted was that of Figure 5B.

According to Table 12, cohort-biregional shrinking is a more accurate method of shrinking than cohort-components shrinking, at least with regard to the particular data set examined in this paper. The former projects regional age compositions that are virtually identical to those projected by the original large-scale model. The total multiregional population and its regional distribution are somewhat less accurately approximated, but nevertheless are, in general, closer approximations than those advanced by the cohort-components method of Table 11. Finally, the cohort-biregional shrinking can be more readily transformed into a method for approximating the intrinsic rate of growth and related stable growth parameters of the multiregional population system.

TABLE 12

MULTIREGIONAL PROJECTIONS TO STABILITY: UNITED STATES TOTAL POPULATION, 1958, DECOMPOSITION B WITH BI-REGIONAL AGGREGATION

Projections and Stable Growth Parameters	DIVISION OF RESIDENCE 1 New England	2 Middle Atlantic	3 East North Central	4 West North Central	5 South Atlantic	6 East South Central	7 West South Central	8 Mountain	9 Pacific	TOTAL
K (1958)	9,911,000	33,181,000	35,763,000	15,114,000	24,749,000	11,769,000	16,177,000	6,349,000	19,141,000	172,154,000
% (1958)	0.0576	0.1927	0.2077	0.0878	0.1438	0.0684	0.0940	0.0369	0.1112	1.0000
K (2008)	21,737,424	59,648,676	81,625,712	30,941,760	70,046,704	24,580,972	40,261,452	22,187,936	72,139,368	423,170,004
% (2008)	0.0514	0.1410	0.1929	0.0731	0.1655	0.0581	0.0951	0.0524	0.1705	1.0000
r (∞)	0.02186	0.02186	0.02160	0.02160	0.02186	0.02160	0.02160	0.02160	0.02160	-----
% (∞)	0.0451	0.1060	0.1870	0.0696	0.1684	0.0523	0.0969	0.0590	0.2158	1.0000
2008: Approximate Projection										
C (0-14)	0.3560	0.3369	0.3647	0.3665	0.3517	0.3627	0.3706	0.3730	0.3586	----
C (15-64)	0.5874	0.5990	0.5803	0.5713	0.5843	0.5766	0.5697	0.5719	0.5864	----
C (65+)	0.0566	0.0641	0.0550	0.0623	0.0640	0.0607	0.0597	0.0551	0.0550	----
2008: Original Projection										
C (0-14)	0.3560	0.3367	0.3642	0.3664	0.3513	0.3621	0.3709	0.3740	0.3587	0.3581
C (15-64)	0.5873	0.5988	0.5802	0.5713	0.5840	0.5765	0.5696	0.5719	0.5865	0.5825
C (65+)	0.0567	0.0644	0.0557	0.0623	0.0647	0.0614	0.0595	0.0541	0.0548	0.0594

The cohort-components and the cohort-biregional methods of simple shrinking appear to be the most desirable shrinking methods among those examined in this paper. Table 13 indicates that they are the most accurate in projecting the total multiregional population. With the possible exception of the less-efficient bi-regional aggregation method of shrinking, they also appear to be the most accurate in projecting the regional shares and age compositions of the multiregional population. The accuracy with which the cohort-biregional method projects regional age compositions is especially remarkable and is well illustrated in Table 14, which presents the alternative projections of the age composition of the Pacific Division by way of example.

5. Conclusion

Imagine a demographer faced with the problem of projecting, in a consistent manner and in age-specific detail, the future populations of the 265 Standard Metropolitan Statistical Areas (SMSAs) of the contemporary United States. Such a large-scale multiregional cohort-survival model is beyond the data processing capabilities of his digital computer and, moreover, would be needlessly cumbersome in light of certain observed weak connectivities between several subsystems of SMSAs. What findings and what approaches does this paper present that might be useful to him as he proceeds to design a population projection model?

TABLE 13

ALTERNATIVE PROJECTIONS OF THE TOTAL POPULATION AND ITS REGIONAL DISTRIBUTION IN THE YEAR 2008: UNITED STATES TOTAL POPULATION, 1958

Alternative*	DIVISIONAL SHARES OF TOTAL POPULATION (2008)									TOTAL	TOTAL POPULATION
	1	2	3	4	5	6	7	8	9		
Table 2	0.0513	0.1403	0.1914	0.0739	0.1619	0.0578	0.0959	0.0541	0.1734	1.0000	421,862,143
Table 7	0.0523	0.1406	0.1938	0.0727	0.1654	0.0580	0.0944	0.0522	0.1707	1.0000	428,537,436
Table 8	0.0474	0.1215	0.1787	0.0619	0.1613	0.0478	0.0856	0.0618	0.2340	1.0000	450,782,060
Table 9	0.0472	0.1255	0.1807	0.0656	0.1573	0.0531	0.0955	0.0624	0.2127	1.0000	441,451,694
Table 10	0.0486	0.1319	0.1886	0.0703	0.1589	0.0512	0.0902	0.0585	0.2018	1.0000	435,234,788
Table 11	0.0492	0.1458	0.1877	0.0708	0.1705	0.0581	0.0973	0.0523	0.1682	1.0000	(419,173,278)
Table 12	0.0514	0.1410	0.1929	0.0731	0.1655	0.0581	0.0951	0.0524	0.1705	1.0000	423,170,004

*Table 2--original unconsolidated model
Table 7--bi-regional aggregations
Table 8--single-region decompositions (with net migration)
Table 9--decomposition A
Table 10--decomposition B
Table 11--single-region (no-migration) decompositions with components-of-change aggregation (cohort-components shrinking)
Table 12--decomposition B with bi-regional aggregation (cohort-biregional shrinking)

TABLE 14

ALTERNATIVE PROJECTIONS OF THE PACIFIC DIVISION'S AGE COMPOSITION IN THE YEAR 2008: UNITED STATES TOTAL POPULATION, 1958

	ALTERNATIVE*						
Age	Table 2	Table 7	Table 8	Table 9	Table 10	Table 11	Table 12
0-4	0.1352	0.1403	0.1417	0.1389	0.1387	0.1335	0.1351
5-9	0.1182	0.1234	0.1218	0.1202	0.1202	0.1176	0.1182
10-14	0.1053	0.1094	0.1067	0.1058	0.1062	0.1049	0.1053
15-19	0.0955	0.0956	0.0982	0.0968	0.0970	0.0940	0.0954
20-24	0.0861	0.0838	0.0910	0.0889	0.0888	0.0846	0.0860
25-29	0.0768	0.0745	0.0818	0.0798	0.0797	0.0758	0.0768
30-34	0.0686	0.0664	0.0723	0.0710	0.0706	0.0672	0.0686
35-39	0.0603	0.0582	0.0619	0.0615	0.0610	0.0590	0.0603
40-44	0.0523	0.0504	0.0515	0.0521	0.0517	0.0518	0.0523
45-49	0.0462	0.0445	0.0439	0.0448	0.0449	0.0473	0.0463
50-54	0.0395	0.0386	0.0356	0.0371	0.0375	0.0406	0.0395
55-59	0.0339	0.0332	0.0298	0.0314	0.0317	0.0351	0.0339
60-64	0.0272	0.0269	0.0231	0.0247	0.0248	0.0282	0.0273
65-69	0.0190	0.0192	0.0150	0.0166	0.0166	0.0201	0.0191
70-74	0.0136	0.0137	0.0106	0.0118	0.0118	0.0157	0.0137
75-79	0.0102	0.0101	0.0077	0.0087	0.0087	0.0123	0.0102
80-84	0.0067	0.0067	0.0050	0.0056	0.0057	0.0085	0.0068
85+	0.0052	0.0052	0.0022	0.0043	0.0044	0.0039	0.0052
TOTAL	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
TOTAL POPULATION	73,166,573	73,141,824	105,479,992	93,899,880	87,833,784	(70,515,742)	72,139,368

*Same as in Table 13

The principal findings of this paper revolve around the various ways of shrinking a large-scale population projection model and may be summarized as follows:

1.) Components-of-change models are unreliable generators of middle and long-run projections of population totals, but seem to be reasonably accurate in projecting regional shares of such totals (Table 5).

2.) Bi-regional aggregation is an effective and relatively efficient method for shrinking projection models of a small to modest scale (Table 7).

3.) Modeling internal migration as a net flow can introduce serious biases into the projection process (Table 8). (Such biases are inevitably introduced in treating immigration and emigration as a net flow, but in most countries they tend to be relatively small.)

4.) Effective decompositions are not unique and may be difficult to identify in large systems (Tables 9 and 10). Consequently algorithms such as those discussed in Tewarson (1973) need to be adapted and applied in searches for decompositions that are in some sense "optimal".

5.) The simple cohort-components method of shrinking is a reasonably accurate procedure, is easy to apply, and has the distinct advantage of not requiring age-specific migration flow data for its implementation (Tables 11, 13, and 14). It therefore is the obvious choice for shrinking large-scale projection models of population systems for which such data are either unavailable or too costly to obtain.

6.) The simple cohort-biregional method of shrinking appears to be very accurate and seems to be an effective compromise between bi-regional aggregation and single-region decomposition, combining the best features of each (Tables 12, 13 and 14). It is especially well-suited for shrinking large-scale projection models of population systems that are comprised of several weakly connected subsystems.

The two principal approaches for shrinking examined in this paper have been aggregation and decomposition. They have been combined to define two fundamental methods of shrinking, both of which reflect the proposition that strongly interconnected regions should be modelled as separate closed subsystems using the cohort-survival model. The two methods differ in the way that they connect these subsystems together. The cohort-components method uses a components-of change model to establish such connections; the cohort-biregional method relies instead on a residual "rest-of-the-world" region. Each alternative differs with respect to data inputs and outputs, computational efficiencies, and projection accuracy. Yet little can be said about the trade-offs between these attributes in the abstract, because they depend so much on the specifics of each empirical situation. The particular connectivity structure of an observed multiregional population, the particular data availability with regard to age-specific migration flows, the particular purposes for which the projections are being generated, all are important considerations in a rational choice between the two alternatives. Yet such considerations will vary from one situation to another, and will combine in different ways to suggest the superiority of one alternative over the other. In consequence, each particular situation requires a specific evaluation.

This paper represents a first and therefore preliminary examination of shrinking large-scale population projection models. Consequently it only outlines the fundamental problem and identifies what appear to be fruitful means for dealing with it. Much more remains to be done. For example, it is likely that further research could establish conditions for "perfect decomposition" akin to those already established for perfect aggregation (Rogers, 1969 and 1975). The relative computational efficiencies of the two alternative methods in shrinking certain prototype connectivity structures could be examined profitably. More complex hierarchical extensions of the simple shrinking methods could be investigated, such as the extension of the simple cohort-components method to include several multiregional (no-migration) cohort-survival submodels, and the disaggregation of the "rest-of-the-world" region in the simple cohort-biregional method. Efficient algorithms for approximating the intrinsic rate of growth and other related stable growth measures using shrinking methods appears to be another promising direction for research. The assumption of no interregional differentials in fertility and mortality has been used before to shrink a large-scale population projection model and deserves to be reconsidered in the context of this paper (Rogers, 1968, Ch. 3). Finally, the possibility of shrinking data input requirements by means of "model" schedules also merits careful examination (United Nations, 1967, Rogers, 1975, Ch. 6).

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