SMOOTH AND HEAVY SOLUTIONS TO CONTROL PROBLEMS

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ABSTRACT

We introduce the concept of viability domain of a set-valued map, which we study and use for providing the existence of smooth solutions to differential inclusions.

We then define and study the concept of heavy viable trajectories of a controlled system with feedbacks. Viable trajectories are trajectories satisfying at each instant given constraints on the state. The controls regulating viable trajectories evolve according a set-valued feedback map. Heavy viable trajectories are the ones which are associated to the controls in the feedback map whose velocity has at each instant the minimal norm. We construct the differential equation governing the evolution of the controls associated to heavy viable trajectories and we state their existence.

DEDICATION

I would have liked to find an original way to dedicate this lecture to Professor Ky Fan, but I did not see any better solution than to simply confess that it is both an honor and a pleasure to have been invited to this conference held in his honor.

I have been deeply influenced by the theorems discovered and proved by Professor Ky Fan, and, in particular, by his 1968 famous inequality. Let me just repeat what I tell my students when I begin to teach the Ky Fan inequality. I tell them a lot of stories, how the young Ky Fan came to Paris in 1939 for one year with only a metro map, how he had to survive during the darkest years of the history of my country, how he met Fréchet and worked with him, etc. But most important, I choose the Ky Fan inequality as the best illustration of the concept of "labor value" of a theorem.

Indeed, most of the theorems of nonlinear functional analysis are equivalent to the Brouwer fixed point theorem. But when we prove that statement (A) is equivalent to statement (B), there is always one implication, say "A implies B", that is more difficult to prove than the other one. We then can say that statement (B) "incorporates" more labor value than statement (A). An empirical law shows that the more labor value a theorem incorporates, the more useful it is. And my point is that among all the theorems equivalent to the Brouwer fixed point theorem I know, the Ky Fan inequality is one which is the most valuable.

SMOOTH AND HEAVY VIABLE SOLUTIONS TO CONTROL PROBLEMS

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1. VIABLE SOLUTIONS TO A CONTROL PROBLEM

Let $X = \mathbb{R}^n$, $U : X \rightrightarrows X$ be a set-valued map with closed graph and $f : \text{Graph } U \to X$ be a continuous map. We consider the control problem with feedbacks

$$(1.1) \quad \begin{cases} \text{i)} & x'(t) = f(x(t), u(t)) \quad , \\ \text{ii)} & \text{for almost all } t \geq 0, \quad u(t) \in U(x(t)) \\ \text{iii)} & x(0) = x_0 \text{ given in Dom } U \quad . \end{cases}$$

Instead of selecting a solution $x(\cdot)$ to (1) which minimizes a given functional, as in optimal control theory(1), we are only selecting solutions which are viable in the sense that, given a closed subset $K \subset X$

$$(1.2) \qquad \forall t \geq 0 \ , \quad x(t) \in K \quad .$$

A first issue is to provide necessary and sufficient conditions linking the dynamics of the system (described by f and U) and the constraints bearing on the system (described by the closed subset K) such that the viability property

$$(1.3) \qquad \begin{array}{l} \forall x_0 \in K \text{ , there exists a solution to (1)} \\ \text{viable in } K \end{array}$$

holds true. This allows us to describe the evolution of the viable controls $u(\cdot)$, (the controls which govern viable solutions).

A second issue is to provide conditions for having smooth viable solutions to a control problem, in the sense that the viable control function is absolutely continuous instead of being simply measurable.

A third issue is to give a mathematical description of the "heavy viable solutions" of the control system which we observe in the evolution of large systems arising in biology and economic and social sciences. Such large systems keep the same control whenever they can and change them only when the viability is at stakes, and do that as slowly as possible. In other words, heavy viable solutions are governed by those controls who minimize at each instant the norm of the velocity of the viable controls. In the case when $f(x,u) = u$, system (1) reduces to the differential inclusion $x'(t) \in U(x(t))$, $x(0) = x_0$: heavy (viable) solutions to this system minimize at each instant the norm of the acceleration of viable solutions; in other words, they evolve with maximal inertia. Hence the name heavy viable solutions (or inert viable solutions).[2]

For solving this problem, we need to introduce and study two concepts: viability domains of differential inclusions and derivatives of set-valued maps.

Let me mention that these results were obtained in collaboration with Halina Frankowska and Georges Haddad.

2. VIABILITY DOMAINS AND INVARIANT SETS OF A SET-VALUED MAP

In this section, we consider a set-valued map F from X to X satisfying once and for all

$$(2.1)\quad \begin{cases} \text{i)} & \text{the graph of } F \text{ is nonempty and closed} \\ \text{ii)} & \forall x \in \mathrm{Dom}(F)\ ,\quad \sup\limits_{v \in F(x)} \|v\| =: \|F(x)\| \leq a\|x\| + b \end{cases}$$

(This implies that F is upper semicontinuous with compact images.) We propose to extend the concept of invariant subspace K by a single-valued map f, defined by

$$(2.2)\qquad \forall x \in K\ ,\quad f(x) \in K \quad \text{or} \quad f(K) \subset K\ .$$

When we think about the extension of the concept of invariant subset K, we have the choice of using either the property $f(K) \subset K$ or the property

$$(2.3)\qquad \forall x \in K\ ,\quad f(x) \in T_K(x)$$

because a vector subspace K is always the tangent space to every points of K. ($T_K(x) = K$ for all $x \in K$.)

When K is any subset, there are many ways to introduce "tangent cones" $T_K(x)$ to K at x which coincide with the tangent space when K is a smooth manifold and to the tangent cone of convex analysis when K is convex. The 1943 Nagumo theorem shows that we have to choose the contingent cone introduced by Bouligand in the thirties. The contingent cone $T_K(x)$ to K at x is defined by

$$(2.4)\qquad T_K(x) := \left\{ v \in X \;\middle|\; \liminf_{h \to 0+} \frac{d(x+hv,K)}{h} = 0 \right\}$$

Nagumo's theorem states that if a continuous map f satisfies property (2.3), then for any $x_0 \in K$, there exists a viable solution to the differential equation $x'(t) = f(x(t))$, $x(0) = x_0$.

When we consider the differential inclusion

$$(2.5)\qquad x'(t) \in F(x(t)),\quad x(0) = x_0$$

there are two ways of adapting property (2.3).

Definition 2.1

We shall say that a subset $K \subset \mathrm{Dom}\, F$ is a viability domain of F if

$$(2.6) \qquad \forall x \in K\,, \quad F(x) \cap T_K(x) \neq \phi$$

and is invariant by F if

$$(2.7) \qquad \forall x \in K\,, \quad F(x) \subset T_K(x) \quad .$$

These definitions are motivated by the following theorems.

Theorem 2.2 (G. Haddad, 1981)

If F has convex values and if $K \subset \mathrm{Dom}(F)$ is a closed viability domain of F, then for any $x_0 \in K$, there exists a viable solution to the differential inclusion (2.5) (viability property)

Theorem 2.3 (F.H. Clarke, 1975)

If F is Lipschitz and if $K \subset \mathrm{Dom}\, F$ is a closed invariant subset by F, then for any $x_0 \in K$, all the solutions to the differential inclusion (2.5) are viable (invariance property).

The concept of invariance in the above sense requires the knowledge of F outside K. Let us mention a more intrinsic result.

Theorem 2.4 (J.P. Aubin and F.H. Clarke, 1977)

If F is continuous and if $K \subset \mathrm{Dom}\, F$ is a closed invariant subset by F, then the viability property holds true.

We now provide an example of viability domains.
Let us consider "limit sets"

$$(2.8) \qquad L(x(\cdot)) := \bigcap_{T>0} \overline{x([T,\infty[)}$$

of solutions $x(\cdot)$ to the differential inclusion $x'(t) \in F(x(t))$.

Theorem 2.5

If F has convex values, the limit sets of the solutions $x(\cdot)$ to the differential inclusion $x'(t) \in F(x(t))$ are closed viability domains.

This theorem provides many examples of viability domains. Equilibria (solutions to $0 \in F(\bar{x})$), trajectories of periodic solutions, etc., are closed viability domains. The question arises whether there exists a largest closed viability domain. Such a largest closed viability domain would then contain all the interesting features of the differential inclusion,

Theorem 2.6

If F has convex values, there exists a largest closed viability domain of F.

Let us mention a consequence of the "coincidence theorem" due to Ky Fan.

Theorem 2.7 (Ky Fan)

If F has convex values, any compact convex viability domain of F contains an equilibrium.

We also observe that for the set-valued analogues of linear operators the concepts of closed viability convex cones and invariant closed convex cones are "dual".

We recall that closed convex processes A are the set-valued maps whose graphs are closed convex cones.

When P is a cone, we denote by P^+ its (positive) polar cone, defined by

$$(2.9) \qquad P^+ := \{p \in X^* \mid \forall x \in P , \langle p,x \rangle \geq 0\} \quad .$$

We can "transpose" closed convex processes in the following way: A^*, the transpose of A, is defined by

$$(2.10) \qquad \begin{aligned} p \in A^*(q) &\iff \forall x \in \mathrm{Dom}\ A,\ \forall y \in A(x),\ \langle p,x \rangle \leq \langle q,y \rangle \\ &\iff (-p,q) \in (\mathrm{Graph}\ A)^+ \end{aligned}$$

Theorem 2.8 (J.P. Aubin, H. Frankowska, C. Olech, 1985)

Let A be a closed convex process defined on the whole space X. The two following properties are equivalent

$$(2.11)\quad \begin{cases} \text{i)} & \text{a closed convex cone } P \text{ is invariant by } A \\ \text{ii)} & P^+ \text{ is a viability domain of } A^*. \end{cases}$$

This result plays a crucial role in the study of controllability of the differential inclusion

$$(2.12)\qquad x'(t) \in A(x(t)) \quad ,$$

and the observability of the adjoint differential inclusion

$$(2.13)\qquad -q'(t) \in A^*(q(t)) \quad .$$

It plays also a role in existence theorems of eigenvalues and eigenvector, as a consequence of Ky Fan's theorem.

Theorem 2.9 (J.P. Aubin, H. Frankowska, C. Olech, 1985).

Let A be a closed convex process defined on the whole space X and P be a closed convex cone with nonempty interior. If P is invariant by A, the two following equivalent conditions

$$(2.14)\quad \begin{cases} \text{i)} & \exists\, \lambda \in \mathbb{R} \text{ such that } \operatorname{Im}(A-\lambda I) \neq X \\ \text{ii)} & \exists\, q \neq 0,\ q \in P^+ \text{ such that } \lambda q \in A^*(q) \end{cases}$$

holds true.

We can say that a solution λ to (2.14)i) is an eigenvalue of A and that a solution q of (2.14)ii) is an eigenvector of A^*.

3. SMOOTH SOLUTIONS TO CONTROL PROBLEMS

Let us return now to our control problem (1.1), which reduces to the differential inclusion $x' \in F(x)$ where F is the set-valued map defined by

$$F(x) := f(x, U(x)) \quad .$$

Let us introduce the feedback map R associated to a subset $K \subset \operatorname{Dom} U$ in the following way:

(3.1) $R(x) := \{u \in U(x) \mid f(x,u) \in T_K(x)\}$.

Then Viability Theorem 2.1 implies the following theorem.

Theorem 3.1

Let us assume that U has a closed graph and compact values, that $f : \text{Graph } U \to X$ is continuous, that

(3.2) $\forall x \in \text{Dom } U \ , \quad \sup_{u \in U(x)} \|f(x,u)\| \leq a\|x\| + b$

and that the subsets $f(x,U(x))$ of velocities are convex. Let K be a closed subset of Dom U. Then the viability property holds true if and only if

(3.3) $\forall x \in K \ , \quad R(x) \neq \phi$.

When this tangential condition is satisfied, viable controls evolve according to the law

(3.4) for almost all t, $u(t) \in R(x(t))$.

The measurable selection theorem allows to state that we can find such viable controls which are measurable.

Since the definition of heavy viable solutions involves the derivatives of viable controls, we have to find sufficient conditions for having absolutely continuous viable controls. For that purpose, we can think to impose an a priori bound on the velocity of the viable controls, requiring for instance that

(3.5) for almost all t, $\|u'(t)\| \leq c(\|x(t)\| + \|u(t)\| + 1)$.

Theorem 3.5

Let us assume that the graph of U is closed and that $f : \text{Graph } U \to X$ is continuous and satisfies for some $c_0 \in \mathbb{R}_+$:

(3.6) $\forall (x,u) \in \text{Graph}(F) \ , \quad \|f(x,u)\| \leq c_0(\|x\| + \|u\| + 1)$.

Then we can associate with any $c \geq c_0$ a set-valued map $R_c \subset U$ having the following property:

$\forall x_0 \in K$, $\forall u_0 \in R_c(x)$, there exists a smooth solution to the control problem

$$(3.7) \quad \begin{cases} \text{i)} \quad x'(t) = f(x(t),u(t)) \quad , \\ \text{ii)} \quad u'(t) \in c(\|x(t)\|+\|u(t)\|+1)B \text{ , B is the unit ball} \end{cases}$$

which are viable in the sense that

$$(3.8) \qquad \forall t \geq 0 \text{ , } \quad x(t) \in K \quad \text{and} \quad u(t) \in R_c(x(t)) \quad .$$

Furthermore, R_c is the largest of the set-valued maps satisfying the above property.

If we introduce the set-valued map G_c defined by

$$(3.9) \qquad G_c(x,u) := \begin{cases} \{f(x,u)\} \times c(\|x\|+\|u\|+1)B & \text{if } x \in K \\ & \text{and } u \in U(x) \\ \phi & \text{if not} \end{cases}$$

Then the graph of set-valued map R_c satisfying properties (3.7) and (3.8) is the largest closed viability domain of this set-valued map G_c.

We observe that if $c_1 \leq c_2$, then

$$(3.10) \qquad R_{c_1}(x) \subset R_{c_2}(x) \subset R(x)$$

and that the set-valued map $c \to \text{Graph } R_c$ is upper semicontinuous.

Hence smooth viable solutions of the control problem are governed by controls $u(t)$ evolving according to the feedback law

$$(3.11) \qquad \text{for all } t \geq 0 \text{ , } \quad u(t) \in R_c(x(t)) \quad .$$

Since the definition of heavy viable solutions to control problems involves the knowledge of the derivative $u'(t)$ of the controls $u(t)$ governing (smooth) viable solutions, we are led to "differentiate" the feedback law (3.11) and, for that purpose, to "differentiate" the set-valued map R_C.

4. CONTINGENT DERIVATIVES OF SET-VALUED MAPS

We choose the concept of contingent derivatives (see Aubin (1981), Aubin and Ekeland (1984)). When F is a set-valued map from a Banach space X to a Banach space Y and when (x,y) belongs to the graph of F, then we define the contingent derivative DF(x,y) as the closed process from X to Y whose graph is equal to the contingent cone to Graph(F) at (x,y):

$$\text{(4.1)} \qquad \operatorname{Graph} DF(x,y) := T_{\operatorname{Graph}(F)}(x,y) \quad .$$

In other words,

$$\text{(4.2)} \qquad v \in DF(x,y)(u) \iff (u,v) \in T_{\operatorname{Graph}(F)}(x,y) \quad .$$

We can check that

$$\text{(4.3)} \qquad v \in DF(x,y)(u) \iff \liminf_{\substack{h \to 0+ \\ u' \to u}} d\left(v \, , \, \frac{F(x+hu')-y}{h}\right) = 0 \quad .$$

This concept of contingent derivative captures many of the properties of the Gâteaux derivative of single-valued differential maps. We just mention here the "chain rule" property which is relevant to our problem.

Let $x(\cdot)$ and $y(\cdot)$ be two absolutely continuous functions of t satisfying the relation

$$\text{(4.4)} \qquad \text{for all } t, \quad y(t) \in F(x(t)) \quad .$$

Then

$$\text{(4.5)} \qquad \text{for almost all } t, \quad y'(t) \in DF(x(t),y(t))(x'(t)) \quad .$$

5. HEAVY VIABLE SOLUTIONS TO A CONTROL PROBLEM

Since smooth viable solutions $x(t)$ to the control problem (1.1) and (3.5) are governed by absolutely continuous controls $u(t)$ obeying the feedback law (3.11) we know that the velocity $u(t)$ obeys the law

(5.1) $\quad$ for almost all $t \geq 0$, $u'(t) \in DR_C(x(t),u(t))(f(x(t),u(t)))$.

Therefore, heavy viable solutions $x(t)$ are governed by controls $u(t)$ which are solutions to the differential inclusion

(5.2) for almost all $t \geq 0$, $u'(t) \in m(DR_C(x(t),u(t))(f(x(t),u(t))))$,

where, when A is a subset of a vector space,

(5.2) $\quad m(a) := \{x \in A \mid \|x\| = \inf_{y \in A} \|y\|\}$

Theorem 5.1 (Aubin-Frankowska)

Heavy viable solutions to the control problem (1.1) and (3.5) are solutions to the differential inclusions

(5.3) $\quad \begin{cases} \text{i)} \quad x'(t) = f(x(t),u(t)) \\ \text{ii)} \quad u'(t) \in d(0,DR_C(x(t),u(t))(f(x(t),u(t)))) \end{cases}$

which are viable in the sense that

(5.4) $\quad \forall t \geq 0, \quad u(t) \in R_C(x(t))$.

If we assume that

(5.5) $\quad (x,u,v) \to DR_C(x,u)(v)$ is lower semicontinuous,

then for any $x_0 \in \text{Dom } R_C$ and any $u_0 \in R_C(x_0)$, there exists a heavy viable solution to the control problem (1.1) and (5.2).

Footnotes

(1)

Optimal control theory does assume implicitly

(1) the existence of a decision-maker operating the controls of the system (there may be more than one decision-maker in a game-theoretical setting)

(2) the availability of information (deterministic or stochastic) on the future of the system; this is necessary to define the costs associated with the trajectories

(3) that decisions (even if they are conditional) are taken once and for all the initial time.

(2)

Palaeontological concepts such as punctuated equilibria proposed by Elredge and Gould are consistent with the concept of heavy viable trajectories.

Indeed, for the first time, excavations at Kenya's Lake Turkana have provided clear fossil evidence of evolution from one species to another. The rock strata there contain a series of fossils that show every small step of an evolutionary journey that seems to have proceeded in fits and starts. Williamson (1981) examined 3.300 fossils showing how thirteen species of molluscs changed over several million years. What the record indicated was that the animals stayed much the same for immensely long stretches of time. But twice, about 2 million years ago and then again 700.000 years ago, the pool of life seemed to explode - set off, apparently, by a drop in the lake's water level. In an instant of geologic time, as the changing lake environment allowed new types of molluscs to win the race for survival, all of the species evolved into varieties sharply different from their ancestors. That immediate forms appeared so quickly, with new species suddenly evolving in 5.000 to 50.000 years after millions of years of constancy, challenges the traditional theories of Darwin's disciples since the fossils of Lake Turkana don't record any gradual change; rather, they seem to reflect eons of stasis interrupted by brief evolutionary "revolutions".

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