LOCAL INVERTIBILITY OF SET-VALUED MAPS

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ABSTRACT

We prove several equivalent versions of the inverse function theorem: an inverse function theorem for smooth maps on closed subsets, one for set-valued maps, a generalized implicit function theorem for set-valued maps. We provide applications of the above results to the problem of local controllability of differential inclusions.

I dedicate this paper to Professor Ky-Fan, who has greatly influenced me, in particular, when I met him in CEREMADE during the fall of 1982.

1. The Inverse Function Theorem

Let X be a Banach space, $K \subset X$ be a subset of X. We recall the definition of the *tangent cone* to a subset K at x_o introduced in Clarke [1975]:

$$C_K(x_o) := \{v \in X \mid \lim_{\substack{h \to o+ \\ x \to x_o \\ x \in K}} \frac{d(x+hv,K)}{h} = 0\} \quad .$$

We state now our basic result.

Theorem 1.1.

Let X be a Banach space, Y be a finite dimensional space, $K \subset X$ be a closed subset of X and x_o belong to K. Let A be a differentiable map from a neighborhood of K to Y. We assume that A' is continuous at x_o and that the following *surjectivity assumption* holds true

(1) $$A'(x_o)C_K(x_o) = Y \quad .$$

Then $A(x_o)$ belongs to the interior of A(K) and there exist constants ρ and ℓ such that, for all

(2) $$\begin{cases} y_1, y_2 \in A(x_o) + \rho B \text{ and any solution } x_1 \in K \text{ to the} \\ \text{equation } A(x_1) = y_1 \text{ satisfying } \|x_o - x_1\| \leq \ell\rho, \text{ there} \\ \text{exists a solution } x_2 \in K \text{ to the equation } A(x_2) = y_2 \\ \text{satisfying } \|x_1 - x_2\| \leq \ell \|y_1 - y_2\|. \quad \blacktriangle \end{cases}$$

We recall

Definition

A set-valued map G from Y to X is *pseudo-Lipschitz* around $(y_o, x_o) \in \text{Graph}(G)$ if there exist neighborhoods V of y_o and W of x_o and a constant ℓ such that

$$\begin{cases} \text{i)} \quad \forall y \in V, \ G(y) \neq \emptyset \\ \text{ii)} \quad \forall y_1, y_2 \in V, \ G(y_1) \cap W \subset G(y_2) + \ell \|y_1 - y_2\| B \quad . \quad \blacktriangle \end{cases}$$

The above definition was introduced in Aubin [1982], [1984]. (See also Rockafellar [to appear] d) for a thorough study of pseudo-Lipschitz maps.)

Hence, the second statement of Theorem 1.1 reads:

(2)' $$\begin{cases} \text{the map } y \to A^{-1}(y) \cap K \text{ is pseudo-Lipschitz around} \\ (Ax_o, x_o). \end{cases}$$

Remark

If x_o belongs to the interior of K, then $C_K(x_o) = X$. Then assumption (1) states that $A'(x_o)$ is surjective, and we obtain the usual "inverse function theorem", also called the "Liusternik theorem".

We deduce a characterization of the interior of a closed subset of a finite-dimensional space given by Clarke [1983]:

$$x_0 \in \mathrm{Int}(K) \Leftrightarrow C_K(x_0) = X \quad .$$

(We take X = Y and A to be the identity). ▲

The proof of Theorem 2.1 is based on the Ekeland variational principle [1974] and is given in Aubin-Frankowska [1985].

Corollary 1.2.

We posit the assumptions of Theorem 1.1. Let $M := A^{-1}(A(x_0)) \cap K$ be the set of solutions $x \in K$ to the equation $A(x) = A(x_0)$. Then there exist a neighborhood U of x_0 and a constant ℓ such that

$$\forall x \in K \cap U, \; d(x,M) \leq \ell \, \|A(x) - A(x_0)\| \quad .$$

Furthermore

$$C_K(x_0) \cap \mathrm{Ker}\, A'(x_0) \subset C_M(x_0) \quad . \quad ▲$$

We shall derive the extension to set-valued maps of the inverse function theorem. Let X,Y be Banach spaces and F be a map from X into the subsets of Y.

The *derivative* $CF(x_0,y_0)$ of F at $(x_0,y_0) \in \mathrm{Graph}\,(F)$ is the set-valued map from X to Y associating to any $u \in X$ elements $v \in Y$ such that (u,v) is tangent to Graph (F) at (x_0,y_0):

$$v \in CF(x_0,y_0)(u) \Leftrightarrow (u,v) \in C_{\mathrm{Graph}\,(F)}(x_0,y_0)$$

Theorem 1.3.

Let F be a set-valued map from a Banach space X to a finite dimensional space Y and (x_0,y_0) belong to the graph of F. If

graph F is closed and CF (x_0,y_0) is surjective,

then F^{-1} is pseudo-Lipschitz around $(y_0,x_0) \in \mathrm{Graph}\,(F^{-1})$.

▲

Proof

We apply Theorem 1.1 when X is replaced by $X \times Y$, K is the graph of F and A is the projection from $X \times Y$ to Y. ■

Remark: A dual formulation.

Since the dimension of Y is finite, assumption (1) is equivalent to

$$A'(x_o)\, C_K(x_o) \text{ is dense in } Y$$

which can be translated as

$$\text{if } A'(x_o)^* q \text{ belongs to } C_K(x_o)^-, \text{ then } q = 0 \quad .$$

If F is a set-valued map from X to Y, we define the coderivative $CF(x_o,y_o)^*$ of F at $(x_o,y_o) \in \text{Graph}\ (F)$ as the "transpose" of $CF(x_o,y_o)$, from Y^* to X^* defined by

$$p \in CF(x_o,y_o)^*(q) \Leftrightarrow \sup_{(u,v) \in \text{Graph}\ CF(x_o,y_o)} (\langle p,u\rangle - \langle q,v\rangle) = 0 \quad .$$

Therefore, in Theorem 1.3, we can replace the surjectivity assumption by the "dual assumption"

$$CF(x_o,y_o)^{*-1}(0) = \{0\} \quad .$$

2. Applications to Local Controllability

Let us consider a set-valued map F from $\mathbb{R}^n$ into compact subsets of $\mathbb{R}^n$. We associate with F the *differential inclusion*

(3) $\quad x' \in F(x) \quad .$

A particular case of (3) is the parametrized system (also called a "control system")

(4) $\quad x' = f(x,u(t)) \quad , \quad u(t) \in U$

where U is a given set of controls; then F is defined by

$$F(x) = \{f(x,u) : u \in U\} \quad .$$

Let $T > 0$. A function $x \in W^{1,1}(0,T)$ (Sobolev space) is called a *solution of differential inclusion* (3) if

$$x'(t) \in F(x(t)) \text{ a.e. in } [0,T] \quad .$$

For a point $\xi \in \mathbb{R}^n$ denote by $S_T(\xi)$ the set of solutions to (3) starting from ξ and defined on the time interval $[0,T]$. The *reachable set* for (3) at time T from ξ is denoted by $R(T,\xi)$, i.e.

$$R(T,\xi) = \{x(T) : x \in S_T(\xi)\} \quad .$$

The system (3) is called *locally controllable* around ξ if for some time $T > 0$

(5) $$\xi \in \text{Int } R(T,\xi) \quad .$$

The purpose of this section is to provide a sufficient condition for (5) when ξ is an equilibrium of F, i.e. $0 \in F(\xi)$.

We shall apply the results of Section 1. The set of solutions $S_T(\xi)$ is closed in $W^{1,1}(0,T)$ whenever Graph (F) is closed in $\mathbb{R}^n \times \mathbb{R}^n$. Consider the continuous linear operator A from the Banach space $W^{1,1}(0,T)$ into the finite dimensional space $\mathbb{R}^n$ defined by

$$A(x) = x(T) \text{ for all } x \in W^{1,1}(0,T) \quad .$$

Theorem 1.1 then states that if x_o denotes the constant trajectory $x_o(\cdot) \equiv \xi$ and $\{w(T) : w \in C_{S_T(\xi)}(x_o)\} = \mathbb{R}^n$ then the relation (5) holds true.

Let B denote the closed unit ball in $\mathbb{R}^n$. We say that a set-valued map F is Lipschitzian (in the Hausdorff metric) on an open neighborhood V of ξ if for a constant $L \geq 0$ and all $x,y \in V$

$$F(x) \subset F(y) + L\|x-y\|B \quad .$$

Thanks to this property we can compute a subset of $C_{S_T(\xi)}(x_o)$:

Theorem 2.1. Assume that F has a closed graph and is Lipschitzian around the equilibrium ξ. Then every solution of the differential inclusion

$$(6)\qquad \begin{cases} w'(t) \in CF(\xi,0)\ w(t) & \text{a.e. in } [0,T] \\ w\ (0) = 0 \end{cases}$$

belongs to $C_{S_T(\xi)}(x_o)$. ▲

The proof of the last result is based on a Filippov Theorem [1967].

We say that the inclusion (6) is *controllable* if its reachable set at some time $T > 0$ is equal to the whole space.

Theorems 1.1 and 2.1 together imply

Theorem 2.2. Assume that F has a closed graph and is Lipschitzian around the equilibrium ξ. The inclusion (3) is locally controllable around ξ if the inclusion (6) is controllable. ▲

Remark. Actually the idea of the proof of Theorem 1.1 allows us to prove a stronger result: We denote by co $F(\xi)$ the closed convex hull of the set $F(\xi)$.

Theorem 2.3. Assume that F has a closed graph and is Lipschitzian around the equilibrium ξ. The inclusion (3) is locally controllable around ξ if the inclusion

$$(7)\qquad \begin{cases} w' \in \text{cl}\ [CF(\xi,0)w + C_{\text{co}F(\xi)}(0)] \\ w(0) = 0 \end{cases}$$

is controllable. ▲

The proof requires a very careful calculation of variations of solutions (see Frankowska [1984]).

A necessary condition for the controllability of the inclusions (6), (7) is

$$\text{Dom } CF\ (\xi,0) := \{w \in \mathbb{R}^n : CF\ (\xi,0)w \neq \emptyset\} = \mathbb{R}^n \quad .$$

Whenever it holds true the right-hand sides of (6), (7) are set-valued maps whose graphs are *closed convex cones*. Such maps, called "closed convex processes", are set-valued analogues of linear operators. The controllability of such differential inclusions is the subject of the next section.

First, we provide the following

Example. Using Theorem 2.3 one can obtain a classical result on local controllability of control system (4) without assuming too much regularity. Let U be a compact set in $\mathbb{R}^m$ and let $f : \mathbb{R}^n \times U \to \mathbb{R}^n$ be a continuous function. Assume that for some $(\xi,\bar{u}) \in \mathbb{R}^n \times U$, $f(\xi,\bar{u}) = 0$ and for some $\beta > 0$, $L > 0$ and all $u \in U$; $x,y \in \xi + \beta B$

$$\begin{cases} \|f(x,u) - f(y,u)\| \leq L\|x-y\| \\ \frac{\partial f}{\partial x}(\cdot,\bar{u}) \text{ is continuous on } \xi + \beta B \end{cases} .$$

Theorem 2.4. If the sublinearized differential inclusion

$$\begin{cases} w' \in \frac{\partial f}{\partial x}(\xi,\bar{u})w + C_{co\ f(\xi,U)}(0) \\ w(0) = 0 \end{cases}$$

is controllable, then the system (4) is locally controllable around ξ. ▲

3. Controllability of Convex Processes

A *convex process* A from $\mathbb{R}^n$ to itself is a set-valued map satisfying

$$\forall x, y \in \text{Dom } A \ , \ \lambda,\mu \geq 0 \ , \quad \lambda A(x) + \mu A(y) \subset A(\lambda x + \mu y)$$

or, equivalently, a set-valued map whose graph is a convex cone. Convex processes are the set-valued analogues of linear operators. We shall say that a convex process is *closed* if its graph is closed and that it is *strict* if its domain is the whole space. Convex processes were introduced and studied in

Rockafellar [1967], [1970], [1974] (see also Aubin-Ekeland [1984]). We associate with a strict closed convex process A the Cauchy problem for the differential inclusion

(8) $$\begin{cases} x'(t) \in A(x(t)) & \text{a.e.} \\ x(0) = 0 & . \end{cases}$$

We say that the differential inclusion (8) is controllable if the *reachable set*

$$R := \{x(t) : x \in W^{1,1}(0,t) \text{ is a solution of (8)}, \ t \geq 0\}$$

is equal to the whole space $\mathbb{R}^n$.

A particular case of (8) is a linear control system

(9) $$\begin{cases} x' = Fx + GU & u \in U \\ x(0) = 0 \end{cases}$$

where U is an m-dimensional space and $F \in L(\mathbb{R}^n, \mathbb{R}^n)$, $G \in L(\mathbb{R}^m, \mathbb{R}^n)$ are linear operators.

We observe that the reachable set $R(T,0)$ of (8) at time T is convex. Since $0 \in A(0)$ the family $\{R(T,0)\}_{T>0}$ is increasing. Moreover, $R = \bigcup_{T>0} R(T,0)$. Hence (8) is controllable if and only if it is controllable at some time $T > 0$, i.e. $\exists\, T > 0$ such that

$$R(T,0) = \mathbb{R}^n \quad .$$

a) <u>The rank condition</u>

Let A be a strict closed convex process. Set $A^1(0) = A(0)$ and for all integer $i \geq 2$ set

$$A^i(0) = A(A^{i-1}(0)) \quad .$$

<u>Theorem 3.1</u>. The differential inclusion (8) is controllable if and only if

$$\text{for some } m \geq 1 \quad A^m(0) = (-A)^m(0) = \mathbb{R}^n \quad .$$

▲

In the case of system (9) for all $x \in \mathbb{R}^n$ $Ax = Fx + \operatorname{Im} G$. Thus

$$A^m(0) = (-A)^m(0) = \operatorname{Im} G + F(\operatorname{Im} G) + \ldots + F^{m-1}(\operatorname{Im} G) \quad .$$

The Cayley-Hamilton theorem then implies the *Kalman rank condition* for the controllability of the linear system (9):

$$\operatorname{rk}[G, FG, \ldots, F^{n-1}G] = n \quad .$$

Theorem 3.1 is a consequence of the following

b) <u>"Eigenvalue" criterion for controllability</u>

We say that a subspace P of $\mathbb{R}^n$ is *invariant* under a strict closed convex process A if $A(P) \subset P$.

A real number λ is called an eigenvalue of A if $\operatorname{Im}(A-\lambda I) \neq \mathbb{R}^n$, where I denotes the identity operator.

<u>Theorem 3.2</u>. The differential inclusion (8) is controllable if and only if A has neither a proper invariant subspace nor eigenvalues. ▲

It is more convenient to write the above criterion in a *"dual form":*

c) <u>"Eigenvector" criterion for controllability</u>

The convex processes can be transposed as linear operators. Let A be a convex process; we define its *transpose* A^* by

$$p \in A^*(q) \Leftrightarrow \forall (x,y) \in \text{Graph } A, \quad \langle p,x \rangle \leq \langle q,y \rangle \quad .$$

It can easily be shown that λ is an eigenvalue of A if and only if for some $q \in \operatorname{Im}(A-\lambda I)^{\perp}$, $q \neq 0$

$$\lambda q \in A^* q \quad .$$

We call such a vector $q \neq 0$ an eigenvector of A^*. Theorem 3.2 is then equivalent to

Theorem 3.3. The differential inclusion (8) is controllable if and only if A^* has neither a proper invariant subspace nor eigenvectors. ▲

The proof of Theorem 3.3 is based on a separation theorem and the KY-FAN coincidence theorem [1972]. (See Aubin-Frankowska-Olech [1985]).

Examples: a) Let F be a linear operator from $\mathbb{R}^n$ to itself, L be a closed convex cone of controls and A be the strict closed convex process defined by

$$A(x) := Fx + L \quad .$$

Then its transpose is equal to

$$A^*(q) = \begin{cases} F^*q & \text{if } q \in L^+ \\ \emptyset & \text{if } q \notin L^+ \end{cases} \quad .$$

When $L = \{0\}$, i.e., when $A = F$, we deduce that $A^* = F^*$, so that transposition of convex processes is a legitimate extension of transposition of linear operators.

Consider the control system

$$(10) \qquad \begin{cases} x' = Ax + u, & u \in L \\ x(0) = 0 \end{cases} \quad .$$

Corollary 3.4.

The following conditions are equivalent.

a) the system (10) is controllable

b) For some $m \geq 1$ $L + F(L) + \ldots + F^m(L) = L - F(L) + \ldots + (-1)^m F^m(L) = \mathbb{R}^n$ (see Korobov [1980]).

c) F has neither a proper invariant subspace containing L nor an eigenvalue λ satisfying $\mathcal{I}m(F-\lambda I) + L \neq \mathbb{R}^n$.

d) F^* has neither a proper invariant subspace contained in L^+ nor an eigenvector in L^+.

e) the subspace spanned by $L, F(L), \ldots, F^{n-1}(L)$ is equal to $\mathbb{R}^n$ and F^* has no eigenvector in L^+ (see Brammer [1972]) ▲

b) Consider the control system with feedback in $\mathbb{R}^2$:

$$(11)\quad \begin{cases} x' = xv + y + u + xu & u,w \in U = [0,1] \\ y' = -x + w & v \in V(x) = \begin{cases} +1 & x \geq 0 \\ -1 & x < 0 \end{cases} \\ x(0) = y(0) = 0 & \end{cases} .$$

Set $F(x,y) = \{(xv + y + u + xu, -x + w) : (u,w,v) \in U \times U \times V(x)\}$.

Then $0 \in F(0)$, i.e. zero is a point of equilibrium. Direct computation gives

$$CF(0,0)(x,y) = (|x| + y + \mathbb{R}_+ , -x + \mathbb{R}_+) .$$

Set $A(x,y) = CF(0,0)(x,y)$. Then

$$A(0) = \mathbb{R}_+ \times \mathbb{R}_+ ; \; -A(0) = \mathbb{R}_- \times \mathbb{R}_-$$

$$A^2(0) = \mathbb{R}_+ \times \mathbb{R} ; \; (-A)^2(0) = \mathbb{R} \times \mathbb{R}_-$$

$$A^3(0) = \mathbb{R}^2 \; ; \; (-A)^3(0) = \mathbb{R}^2 .$$

Thus by Theorem 2.2 and 3.1 the control system (11) is locally controllable around zero.

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