PRICES ON INFORMATION AND STOCHASTIC INSURANCE MODELS

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FOREWORD

This paper is closely connected with the studies on decision making under uncertainty, particularly with the stochastic optimization problems that are investigated in the Adaptation and Optimization Project of the System and Decision Sciences Program.

The paper deals with some economic models in which it appears possible to formalize the notion of the price on information concerning the problem parameters. Insurance models under uncertainty are studied here with more detail.

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The aim of this work is to study the information constraints in economic problems of decision making under uncertainty.

The information constraints, as well as the resources constraints, play an important role in an economic system. However, until recently, only the constraints of the latter type have been systematically studied.

It is well known that the Lagrangian multipliers which remove the resources constraints in an economic optimization problem can be regarded as prices of the resources. It turns out that the Lagrangian multipliers associated with the information constraints can also be interpreted as prices, namely, prices which characterize the effectiveness of information. Moreover, it can be shown that there are relations between the prices under consideration and such an important economic phenomenon as insurance.

This work was stimulated by a series of papers by R.T. Rockafellar and R.J.B. Wets (see, e.g., [1,2]) devoted to a profound mathematical investigation of stochastic extremum problems. An essential role was also played by some comments of economic nature that were made in the course of the discussion of R.T. Rockafellar's lecture in the Central Economic and Mathematical Institute (Moscow, 1974).

We emphasize that no purely mathematical aims are pursued in this work. On the contrary, examples are considered that are most simple from the mathematical point of view, and the main attention is paid to the clarification of the economic sense of mathematical ideas.

We start with a description of the model. Let $f(s,x)$ be a function of a random parameter $s \in S$ (S is a finite set) and of a vector $x \in X$ (X is a subset of R^n). Suppose that $f(s,x)$ is continuous and concave in x, and the set X is convex and compact with $\text{int } X \neq \emptyset$.

Problem A. Find a plan (decision) x in the set X such that the mathematical expectation $Ef(s,x)$ is a maximum. In symbols,

$$E\ f(s,x) \to \max \tag{1}$$

$$x \in X\ . \tag{2}$$

Let us consider together with the problem A the following problem B.

Problem B.

$$E\ f(s,x(s)) \to \max \tag{3}$$

$$x(s) \in X,\ s \in S\ . \tag{4}$$

In problem B one has to find a function (strategy) $x(\cdot)$ which maximizes the functional (3) under the constraint (4). Note that in the problem A the maximum is taken over the set of deterministic vectors (rather than vector functions $x(s)$ of the random parameter s).

The problem A can be obtained from the problem B by adding the following constraint:

$$x(s) \text{ does not depend on } s, \tag{5}$$

which can be also rewritten as

$$x(s) - Ex(\cdot) = 0\ . \tag{6}$$

This form of the information constaint has been studied (in a much more general setting) by Rockafellar and Wets [1,2] (see also [3]).

Thus the information constraint (5) is represented in the form

$$Ax(\cdot) = 0 ,$$

where

$$Ax(\cdot) = x(\cdot) - Ex(\cdot)$$

is a linear operator. By applying an appropriate variant of the Kuhn - Tucker theorem, we obtain that the constraint (5) can be removed by a Lagrangian multiplier $p(\cdot)$. Namely, there exists a function $p(s)$ such that

$$Ef(s,x(s)) + Ep(s)[x(s) - Ex(\cdot)] \leq E\ f(s,\bar{x}) \qquad (7)$$

$(x(s) \in X,\ s \in S)$, where $\bar{x}$ is a solution of the problem A. It is established by a standard argument that (7) is equivalent to the following inequality

$$f(s,x) - \tilde{p}x + p(s)x \leq f(s,\bar{x}) - \tilde{p}\bar{x} + p(s)\bar{x} \quad (s \in S,\ x \in X) \qquad (8)$$

where

$$\tilde{p} = Ep(\cdot) . \qquad (9)$$

It has been demonstrated in [4] that $\tilde{p}$ and $p(s)$ satisfying (8) and (9) can be interpreted as insurance prices ($p(s)x$ is the compensation and $\tilde{p}x$ is the premium). In the present note we briefly sketch this interpretation.

The inequality (8) means that, by paying $\tilde{p}x$ (premium) and getting $p(s)x$ (compensation), we guarantee, that the plan $\bar{x}$ becomes optimal in each random situation. In this sense, the insurance system based on the prices p and $p(s)$ makes it possible to eliminate the future uncertainty. The relation (9) reflects the fact that the premium should be (approximately) equal to the expected value of the compensation.*)

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*) Variants of the above model are considered, in which the premium price $\tilde{p}$ is greater, than $Ep(s)$.

Let us illustrate the above idea by the following example. Consider a model for insurance of a good transported by a ship.

A ship transports x units of a good from one port to another. There are two possibilities: successful transportation and catastrophe. In this example, the random parameter s takes two values: $s = s_1$ (catastrophe) and $s = s_2$ (success). The capacity of the ship equals x_o. Thus, the set X of possible plans is as follows: $X = \{x: 0 \leq x \leq x_o\}$.

Suppose that the income (profit) obtained from a successful transportation equals $f(s_2,x) = q_2x$, and losses which we have in case of a catastrophe equal $f(s_1,x) = -q_1x$. Denote by λ_i the probability $P\{s = s_i\}$ $(i = 1,2)$ and assume that $\lambda_1q_1 < \lambda_2q_2$. Then the optimal plan $\bar{x}$ coincides with x_o.

Let us consider the following task. Find all the prices $\tilde{p}$ and p(s) possessing the properties (8), (9) and the additional property $p(s_2) = 0$. The last equation means that, in case of success, the compensation equals zero.

It can be shown that the solution of the problem is given by the formulae:

$$p(s_1) = \frac{\tilde{p}}{\lambda_1}; \quad \frac{\lambda_1}{\lambda_2} q_1 \leq \tilde{p} \leq q_2 \; . \tag{10}$$

Thus, there are two bounds for $\tilde{p}$, and $p(s_1)$ is a linear function of $\tilde{p}$.

The inequalities in (10) have an obvious economic sense. The inequality $\frac{\lambda_1}{\lambda_2} q_1 \leq \tilde{p}$ is equivalent to the following one

$$p(s_1)x \geq \tilde{p}x + q_1x \qquad (x \in X)$$

which means that the compensation is not less than the sum of the losses and of the premium.

Some variants of the above model are considered in which the profit $f(s_2,x) = q_2(x)$ is a nonlinear function (diminishing returns to scale). It turns out that the price $\tilde{p}$ is unique in these models and $\tilde{p}$ coincides with the left bound in (10).

Let us now discuss the relation between the prices considered and the problem of effectiveness of information.

It is natural to expect that the Lagrangian multiplier which removes the information constraint gives an economic evaluation of information just as the Lagrangian multipliers which remove the resources constraints evaluate these resources.

In order to discuss this idea in rigorous terms, we have to be able to measure the quantity of information. It is well known that an important role is played in the technology and discrete mathematics by Shannon's method of information measuring. Simple argumentation shows that this method is not quite appropriate for our aims.*)

In order to outline an alternative approach, let us imagine the following situation (which is formalized in our model). At the beginning of the planning period, we have to make a decision x without any information about s. Generally speaking, the value of s is observed only at the end of the planning period. Then only, we learn, whether the initial decision x is good or bad. However, if it is possible to make some special efforts which result in learning s earlier, then we can make a correction, i.e. replace x by $x + h(s)$ ($h(\cdot)$ is the correction).

For example, suppose that, basing on uncertain data, we have decided to work out a project. Suppose further that after some time we have got a reliable prediction that the project is doomed to failure. Then this information makes it possible to stop payments (or supply) and thus spend the sum of money

$$x - u \qquad (u \geq o),$$

rather than the sum x initially planned. The strategy used here is as follows:

$$x(s_1) = x - u \qquad \text{(failure)},$$

$$x(s_2) = x \qquad \text{(success)}.$$

The earlier we get the information about failure, the more essential is the correction, and, consequently, the more flexible

*) However, some connections can be found between the Shannon theory and the point of view on information which is considered here (see the last pages of the present paper).

strategy is used. (The flexibility of a strategy means here the difference between our actions in the cases $s = s_1$ and $s = s_2$.)

Thus, the earlier we get the information about s, the more freedom for corrections we have. It follows that the value of information (in the example considered) depends on the time when the information is obtained. The value equals zero if the information comes so late that it is impossible to make any correction of the decision initially made. The value is maximal, if we get the information so early that the initial program can be completely revised.

The above argumentation shows that the central role in our problem is played by the class H of the corrections which can be carried out given the information about s. The class H characterises the useful, effective information contained in the communication of s. In other words, the property of information that is essential here is its property to improve the adaptivity of economic system, i.e. the ability of the system to react in a flexible way to a changing situation. It turns out that the approach outlined here makes it possible to regard $m(s) = \tilde{p} - p(s)$ as an information price.

Indeed, let us fix a function h(s) (correction) and compare the maximal value of the objective functional on the set of strategies of the form x+h(s)

$$\phi(h(\cdot)) = \max_{x(s) = x+h(s)} Ef(s,x(s))$$

with the maximal value of the objective functional on the set of strategies x independent of s

$$\phi(0) = \max_{x} E\, f(s,x).$$

We have

$$\phi(h(\cdot)) - \phi(0) = E\, m(s)h(s) + o(\|h\|) \qquad (11)$$

where $m(s) = \tilde{p} - p(s)$ and $\|\cdot\|$ is an arbitrary norm in the (finite-dimensional) space of functions $h(\cdot)$. The formula (11) is deduced from the following relation

$$\phi(h(\cdot)) - \phi(0) \leq E\, m(s)h(s),$$

which, in turn, is a consequence of (7). In order to make this argumentation rigorous, it is sufficient to assume that ϕ is differentiable at 0 and that 0 belongs to the interior of the domain of Φ.

Thus, we have established the (marginal) property (11) of the price m(s) which shows that the number E m(s)h(s) gives an evaluation of the economic effectiveness of information about s used in the correction h(·). The correction here is specified by a function h(·) or, equivalently (since S is a finite set), by a finite-dimensional vector. Hence, the "quantity of information" is a vector. This is the approach that fits the structure of our problem.

On the other hand, it is more convenient to use scalar-valued (rather than vector-valued) characteristics for measuring the effectiveness of information. One of the possible ways to find such characteristics is based on the concept of flexibility (adaptivity) of the correction h(·) (or of the strategy x(·) = x + h(·)).

Let us compare the problems A and B. In the first problem, we use strategies of the form x(s) = const, i.e. strategies that do not react to a possible difference in the values of s. The second problem corresponds to the other extreme case: we can employ arbitrary strategies x(·). In the latter case, the reaction to a change in the situation s is maximally flexible.

Our aim is now as follows. We would like to define a number which measures the flexibility (adaptivity) of a strategy and makes it possible to consider problems that are "intermediate" between A and B. This number will at the same time characterize the "quantity of information" about s used in the strategy x(s). Indeed, the flexibility of the strategy x(s) (the degree of dependence of the function x(s) on s) reflects also the degree of utilization of the information about s in the process of making the decision x(s).

Assume that such a characteristic of the flexibility of a strategy (= of the quantity of information) is established. Then we consider the class K_r of the strategies x(·) with the flexibility not greater than a real number r. The zero value of the flexibility and the class K_o correspond to the problem A. The maximal value of the flexibility r^* and the class $K_r{}^*$ containing all strategies correspond to the problem B.

Let us consider the maximal value of the objective functional on the class K_r

$$\ell(r) = \max_{x(\cdot) \in K_r} E\, f(s,x(s)).$$

Denote by βr the principal linear part of the function $\ell(r)$ at the point $r = o$. Then the coefficient x can be regarded as a (shadow) price which gives an evaluation of the flexibility of a strategy, and thus an evaluation of the amount of information used in this strategy. Roughly speaking, it is worth paying βr in order to have a possibility to use strategies with the flexibility not greater than r. In other words, if we have an economic mechanism that makes it possible to use strategies with flexibility $\leq r$, then the (shadow) cost of such a mechanism is equal to βr

The simplest and the most common way to measure the scattering of values of a random variable is to consider its standard deviation. The standard deviation of x(s) is defined by

$$\sigma_{x(\cdot)} = (E(x(\cdot) - Ex(\cdot)^2)^{1/2} = ||x(\cdot) - Ex(\cdot)||_{L_2} \quad .$$

We shall use $\sigma_{x(\cdot)}$ in order to measure the flexibility of a strategy $x(\cdot)$.

Let us calculate the shadow price corresponding to the measure of flexibility just defined.

Consider the function

$$\ell(r) = \max_{x(\cdot) \in K_r} E\, f(s,x(s)),$$

where $K_r = \{X(\cdot)\colon \sigma_{x(\cdot)} \leq r\}$ is the class of strategies with flexibility $\leq r$. We shall find $\Phi'(o)$ and express this value in terms of $p(\cdot)$.

It is easily seen that

$$\ell(r) = \max_{\substack{x(s) = x + h(s) \in X \\ E\, h(s) = 0 \\ \sigma_{h(\cdot)} \leq r}} E\, f(s,x(s))$$

By virtue of (11), we have

$$\Phi(h(\ell)) = \Phi(0) + E\, m(s)h(s) + o(||h||) ,$$

where $||\cdot||$ is any norm in the finite-dimensional space of functions $h(\cdot)$, e.g., the norm $||\cdot||_{L_2}$. Consequently, for sufficiently small r's,

$$\ell(r) - \ell(0) = \max_{\substack{E\, h(\cdot) = 0 \\ \sigma_{h(\cdot)} \leq r}} [\Phi(h(\cdot)) - \Phi(0)] =$$

$$= \max_{\substack{E\, h(\cdot) = 0 \\ \sigma_{h(\cdot)} \leq r}} E\, m(s)h(s) + o(r) = \max_{\substack{E\, h(\cdot) = 0 \\ ||h(\cdot)||_{L_2} \leq r}} E\, m(s)h(s) + o(r) =$$

$$= \max_{\substack{E\, h(\cdot) = 0 \\ ||h(\cdot)||_{L_2} \leq r}} E\, (\tilde{p} - p(s))h(s) + o(r) = ||\tilde{p} - p(\cdot)||_{L_2} + o(r) .$$

The last inequality becomes obvious, if we regard $h(\cdot)$ and $\tilde{p} - p(\cdot)$ as elements of the Hilbert space

$$\{h(\cdot):\ E\, h(\cdot) = 0, \quad E\, ||h(\cdot)||^2 < \infty \}$$

with the usual scalar product. The above argumentation is based on the fact that $h(\cdot) = 0$ is an interior point of the domain of the functional $\Phi(h(\cdot))$.

Thus, the following result is obtained:

$$\ell(r) - \ell(0) = \sigma_{p(\cdot)} \cdot r + o(r) .$$

Consequently, if the quantity of information used in the strategy $x(\cdot)$ is measured by the standard deviation $\sigma_{x(\cdot)}$, then the price of information equals the standard deviation $\sigma_{p(\cdot)}$ of the function $p(s)$. This means that the difference between $\ell(r)$ [the maximal income for the strategies using the amount of information $\leq r$] and $\ell(0)$ [the maximal income for the strategies using the zero amount of information] is approximately equal to $\sigma_{p(\cdot)} \cdot r$. In other words, it is worth paying approximately $\sigma_{p(\cdot)} \cdot r$ units of money for a small amount r of information.

Let us return for a minute to the insurance model. The above result shows that the information price is equal to the standard deviation $\sigma_{p(\cdot)}$ of the insurance price p(s). Thus, the larger is the dispersion of values of p(s), the more important is the information. Consequently, the more essential is the difference between the failure and success, the larger is the information price $\sigma_{p(\cdot)}$.

We outline now the relations between the above approach and the Shannon information theory.

Let us first recall the definition of Shannon's information. Assume that there is a random variable A which takes m values $A_1, \ldots, A_m$ with the probabilities $\Pi_1, \ldots, \Pi_m$. The entropy of A is defined by

$$H(A) = - \sum_i \Pi_i \log \Pi_i \equiv \sum_i \eta(\Pi_i) ,$$

where $\eta(a) = -a \log a$.

Suppose that B is another random variable which takes the values $B_1, \ldots, B_n$. Denote by Π_{ij} the conditional probability

$$P \{B = B_i \mid A = A_j\} .$$

The number

$$H_A(B) = \sum_i \Pi_i \sum_j \eta(\Pi_{ij})$$

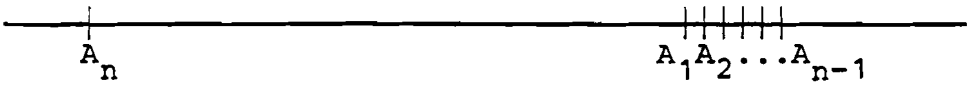
is called the conditional entropy of B given A ("the average uncertainty of the experiment B given the result of the experiment A"). The Shannon information is defined by the formula

$$I(B, A) = H(B) - H_A(B) .$$

This difference shows to what extent the knowledge of the result of experiment A reduces the uncertainty of the experiment B.

The Shannon information theory plays an important rôle in various fields of applied mathematics. It works especially good, when there is no measure of proximity between the different outcome of the experiment, i.e., all the outcomes are in some sense equivalent.

However, this approach is not always convenient. Imagine a random variable A that takes n real values with equal probabilities. Assume that n-1 values $A_1, \ldots, A_{n-1}$ are very close to each other, and the n-th value A_n differs essentially from $A_1, \ldots, A_{n-1}$:



If we neglect the difference between the values $A_1, \ldots, A_{n-1}$, then the uncertainty of the random variable A should be approximately equal to the uncertainty of the random variable A' taking two values A_1 and A_n with probabilities $\frac{1}{n}$ and $\frac{n-1}{n}$. On the other hand, the entropy of A is equal, e.g., to the entropy of a random variable A" taking n values 1,2,..., n with probability $\frac{1}{n}$, which is "much more uncertain".

The point is that Shannon's definition is purely discrete; it does not take into account, e.g., the linear structure of the space. This is one of the reasons, why the Shannon's theory as it is, cannot be applied to our problem, where the linearity and concavity play central roles.

We modify the above definition by introducing a "measure of indifference" between the outcomes of the experiment (i.e. between the values of the random variable). It is assumed that this measure of indifference is proportional to the distance between the values of the random variable. If the random variables takes two values x_1 and x_2 ($x_i \in R$), then, roughly speaking, we mix x_1 and x_2 up with probability $\frac{1}{2} + q$, where $q = |x_1 - x_2|$ (for small q's).

This idea is formulated in strict terms as follows:

Given a strategy

$$x(\cdot) = \{x(s_1), x(s_2)\} = \{x_1, x_2\}$$

with $q = |x_1 - x_2|$ sufficiently small, we introduce an auxiliary random variable $\hat{x}$ which is defined by the following rule. If $s = s_1$, then $\hat{x} = x_1$ with probability $1 - q'$ and $\hat{x} = x_2$ with probability q, where

$$q = \frac{1}{2} + |x_1 - x_2| .$$

If $s = s_2$, then $\hat{x} = x_2$ with probability $1 - q$ and $\hat{x} = x_1$ with probability q.

Denote by I(q) the Shannon information about s given $\hat{x}$. We have

$$\begin{aligned} I(q) &= - \lambda_1 \log \lambda_1 - \lambda_2 \log \lambda_2 + \\ &+ \lambda_1(1 - q)\log \left[\frac{\lambda_1(1-q)}{\lambda_1(1-q) + \lambda_2 q}\right] + \\ &+ \lambda_2 q \log\left[\frac{\lambda_2 q}{\lambda_1(1-q) + \lambda_2 q}\right] + \\ &+ \lambda_1 q \log\left[\frac{\lambda_1 q}{\lambda_1 q + \lambda_2(1-q)}\right] + \\ &+ \lambda_2(1-q) \log\left[\frac{\lambda_2(1-q)}{\lambda_1 q + \lambda_2(1-q)}\right] . \end{aligned}$$

A simple computation shows that

$$I(q) = \frac{16}{\ln 2} \lambda_1 \lambda_2 q^2 + o(q^2) \tag{12}$$

Consequently,

$$I''(0) = \frac{32}{\ln 2} \lambda_1 \lambda_2 .$$

Let us measure the amount of information used in the strategy $x(\cdot)$ by the quantity

$$\sqrt{I(|x_1 - x_2|)} = \sqrt{I(\lambda_1, \lambda_2, |x_1 - x_2|)} \equiv J(x(\cdot)) .$$

Then we have

$$\psi(r) \equiv \max_{J(x(\cdot)) \leq r} E f(s,x(s)) =$$

$$= \max_{\substack{x(\cdot) \\ \sqrt{I(|x_1 - x_2|)} \leq r}} E f(s,x(s)) = \max_{\substack{x(\cdot) \\ j(\sigma_{x(\cdot)}) \leq r}} E f(s,x(s)) ,$$

where

$$j(\sigma) = \sqrt{I(\frac{\sigma}{\sqrt{\lambda_1\lambda_2}})}.$$

Indeed,

$$j(\sigma_{x(\cdot)}) = \sqrt{I(\frac{\sigma_{x(\cdot)}}{\sqrt{\lambda_1\lambda_2}})} = \sqrt{I(|x_2 - x_2|)} \quad ,$$

since

$$\sigma_{x(\cdot)} = |x_1 - x_2| \sqrt{\lambda_1\lambda_2} \quad .$$

Furthermore,

$$\ell(r) = \max_{\sigma_{x(\cdot)} \leq i(r)} E f(s,x(s)) = \ell(i(r)) \quad ,$$

where $i(r) = j^{-1}(r)$. Thus,

$$\psi(r) - \psi(0) = \ell'(0) \cdot i'(0) \quad r + o(r) \quad .$$

Since $i'(0) = 1/j'(0)$, it remains to compute $j'(0)$. This can be done as follows

$$j'(0) = \lim_{\sigma \to 0} \frac{j(0)}{\sigma} = \lim \frac{1}{\sigma}\sqrt{\frac{32}{\ln 2}\sigma^2 + o(\sigma^2)} =$$
$$= 4 \cdot \sqrt{\frac{2}{\ln 2}}$$

(see (12)). Hence,

$$\psi(r) - \psi(0) = \sqrt{\frac{\ln 2}{2}} \cdot \frac{1}{4}\sigma_{p(\cdot)} \cdot r + o(r) \quad ,$$

i.e., the information price (corresponding to information measure $\sqrt{I(|x_1 - x_2|)}$ equals

$$\frac{1}{4}\sqrt{\frac{\ln 2}{2}}\sigma_{p(\cdot)} \quad ,$$

where $\sigma_{p(\cdot)}$ is the standard deviation of the function $p(\cdot)$.

We note that the constant $\frac{1}{4}\sqrt{\frac{\ln 2}{2}}$ is universal: it does not depend on the probability distribution $\{\lambda_1, \lambda_2\}$.

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