THE EQUIVALENCE OF THREE SOCIAL DECISION FUNCTIONS

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Abstract

This paper demonstrates that three of the basic approaches to the solution of the social choice problem are in fact equivalent to one another. All will yield the same social decision functions--a winning set of permutations of the actions. The Combinatorial Optimization criterion of Blin and Whinston is shown to be monotonically related to the Kemeny function criterion proposed by Levenglick. The set covering formulation for the ℓ_1 norm case devised by Merchant and Rao is also shown to be equivalent to the other two. The geometrical aspect of the problem is also discussed and an example is provided.

Introduction

Recently several authors have proposed methods for determining a social ordering of a set of alternatives based on individual pairwise ordering of the set. In each case the author had a different motivation for developing the particular function, but in all cases it was shown that a relationship existed between majority voting and the resulting social order. In this paper we show that all these formulations of the problem lead to exactly the same social ordering of alternatives when the data on pairwise preference is identical.

Combinatorial Optimization Criterion

The Combinatorial Optimization criterion function of Blin and Whinston [1] seeks to determine a best ranking of actions such that the sum of vote proportions of each action over those lower ranking actions is maximized. This optimal assignment of actions to ranks is determined by a permutation on the original order of the actions $[a_1, a_2, \ldots, a_m]$ to get $[a_{p(1)}, a_{p(2)}, \ldots, a_{p(m)}]$, such that $p(i) = k$ implies that a_i has k-1 actions considered superior to it.

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Let $A = [a_{ij}]$ be the matrix of vote proportions such that:

$a_{ij} \equiv$ proportion of individuals who have $a_i > a_j$

$a_{ji} \equiv$ proportion of individuals who have $a_j < a_i$

$a_{ii} \equiv 0$ (rather than the alternative value 1/2)

The case of individuals who have $a_i \sim a_j$ (non-resoluteness) is discussed in [5] and can be resolved by either method described there.

Define $Q = [q_{ij}]$ as the appropriate summation matrix with:

$$q_{ij} = \begin{cases} 0 & \text{if} \quad i \geq j \\ 1 & \text{if} \quad i < j \end{cases}$$

Then,

$$\phi(\bar{p}) = \underset{p}{\text{Max}}\ A_p \cdot Q \tag{1}$$

is the desired criterion function where

$$A_p = [a_{p(i)p(j)}] \quad .$$

Here $a_{p(i)p(j)}q_{ij} > 0$ implies that the permutation p has ranked action i over action j, and the dot product of two matrices of the same dimension is defined as

$$A_p \cdot Q = \sum_{\forall (i,j) \varepsilon Q} a_{p(i)p(j)}q_{ij} \quad .$$

Since p is a permutation, we can define the corresponding permutation matrix $P_p = [p_{kj}]$ as

$$p_{kj} = \begin{cases} 1 & \text{if } a_j \text{ has rank } k \\ 0 & \text{otherwise} \end{cases} \quad .$$

Now $A_p = P_p A P_p'$, and (1) can be rewritten to emphasize its Quadratic Assignment nature

$$\phi(\bar{p}) = \underset{p}{\text{Max}}\ P_p A P_p' \cdot Q \quad . \tag{2}$$

This criterion function seeks to determine the optimal assignment of actions to ranks, $\bar{p}$, over all feasible m! permutations of the a_i.

Kemeny Function

The Kemeny function seeks to find those permutation points, P^p, that maximize a dot product with the translated election matrix E_d. Here a permutation point is defined as $P^p = P_p' X P_p$ where P_p is the same as before and

$$X = [x_{ij}] = \begin{cases} +1 & \text{if} \quad i < j \\ 0 & \text{if} \quad i = j \\ -1 & \text{if} \quad i > j \end{cases} \quad .$$

The translated election matrix is defined to be in skew-symmetric form and is related to the previous election matrix A by:

$$E_d = A - A' \quad . \tag{3}$$

As Levenglick has demonstrated[1], the Kemeny function equivalently seeks to determine the permutation point of minimum Euclidean distance from E_d, thus maximizing the total amount of agreement between P^p and E_d. Equivalently, one can view the problem from the position of choosing the optimal permutation matrix in the following criterion function

$$H(p^*) = \underset{p}{\text{Max}}\ E_d \cdot P_p' X P_p \quad . \tag{4}$$

[1] [4], Theorem 4, p. 41.

As Levenglick has shown[2], (4) is extremely attractive in that it "is symmetric, faithful, equitable; Condorcet, consistent and continuous for all $m \geq 2$," and its consistent extension to the set of rationals on $\binom{m}{2}$ space is the <u>only</u> function that satisfies all the above properties.

<u>Theorem 1</u>. The criterion function (4) is an equivalent representation of (2), and hence the Combinatorial Optimization criterion also has the above fairness properties.

<u>Proof</u>. By equivalence is meant that if $\phi(p_1) > \phi(p_2)$ then $H(p_1) > H(p_2)$ and vice versa. Thus equivalence implies $\{\bar{p}\} = \{p^*\}$; the same ranking of actions optimizes both criteria. To establish the equivalence we shall introduce the column vector of m ones labelled e. Furthermore, define

$$Z = [z_{ij}] = ee' - I = \begin{cases} 0 & \text{if } i = j \\ 1 & \text{otherwise} \end{cases} .$$

Now, A was defined so that

$$A + A' = Z \quad . \tag{5}$$

Conditions (3) and (5) together imply

$$E_d = 2A - Z \quad . \tag{6}$$

Similarly, $X = Q - Q'$ and $Z = Q + Q'$ imply

$$X = 2Q - Z \quad . \tag{7}$$

It is easily seen that

$$P_p'ZP_p = P_pZP_p' = Z \quad . \tag{8}$$

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[2] [4], Theorems 5 and 7, pp. 41-44.

The dot product operation in (4) is invariant to any translation T( ) of E_d and $P_p' X P_p$ that preserves the matchup of their respective matrix entries. That is

$$E_d \cdot P_p' X P_p = T(E_d) \cdot T(P_p' X P_p)$$

for all such valid T.

T( ) can be considered to be the set of all possible permutations of the elements of matrix ( ).

$$T(\ ) = P_p(\ )P_p' \tag{9}$$

is a valid form-preserving translation.

Now, (4), (6), (7), (8), and (9) combine to yield:

$$\begin{aligned} E_d \cdot P_p' X P_p &= P_p E_d P_p' \cdot P_p P_p' X P_p P_p' \\ &= (2P_p A P_p' - Z) \cdot (2Q - Z) \qquad (10) \\ &= 4P_p A P_p' \cdot Q - 2P_p A P_p' \cdot Z - 2Z \cdot Q + Z \cdot Z \quad . \end{aligned}$$

Since

$$2P_p A P_p' \cdot Z = 2Z \cdot Q = Z \cdot Z = m(m-1) \quad , \tag{11}$$

then

$$E_d \cdot P_p' X P_p = 4P_p A P_p' \cdot Q - m(m-1) \quad ; \tag{12}$$

hence (4) and (2) are equivalent. ||

Geometrical Interpretation

As Blin and Whinston showed[3], whenever majority voting yields a transitive social ordering of the a_i, the associated permutation matrix will be optimal for (2). Correspondingly,

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[3] [1]

the permutation point P^{p*} would be a matrix of +1's, -1's, and 0's with:

$$p^*_{ij} = \begin{cases} 0 & \text{if } i = j \\ +1 & \text{if } a_i > a_j \text{ in the optimal social order} \\ -1 & \text{if } a_j > a_i \text{ in the optimal social order} \end{cases}$$

p* would be optimal for

$$H(p^*) = \underset{p}{\text{Max}}\ E_d \cdot P^p \quad . \qquad (4)'$$

Let us define F as the set of all permutation points and $\text{sgn}(E_d)$ as a matrix of +1's, -1's, and 0's, whose entries correspond to the sign of the entries of E_d. Then it is clear that if $\text{sgn}(E_d) \in F$, then $\text{sgn}(E_d) = P^{p*}$ and majority voting has yielded a transitive social ordering.

Thus, we can rewrite (4)' as the following equivalent problem:

$$\underset{p}{\text{Min}}\ E_d \cdot [\text{sgn}(E_d) - P^p] \quad . \qquad (13)$$

Since $Z \cdot P^p = 0$, (13) is equivalent to (14):

$$J(p^*) = \underset{p}{\text{min}}\ A \cdot [\text{sgn}(E_d) - P^p] \quad . \qquad (14)$$

Thus, if $\text{sgn}(E_d) \notin F$, we seek a permutation point that will entail a least cost for moving from an intransitive majority solution $\text{sgn}(E_d)$ to a transitive social ordering given by P^{p*}. In a geometric sense, $\text{sgn}(E_d)$ and P^p are vertices of a hypercube in $\binom{m}{2}$ space, centered about the origin, with edge length of two. The problem is to choose the closest vertex to $\text{sgn}(E_d)$ that belongs to F, where the measurement of distance is conditional upon A.

The following lemma gives necessary and sufficient conditions for determining whether or not $\mathrm{sgn}(E_d) \in F$.

Lemma: Let $\mathrm{sgn}(E_d) = [\sigma_{ij}]$. If there exist distinct i,j,k such that $\mathrm{sgn}(\sigma_{ij}) = \mathrm{sgn}(\sigma_{jk}) \neq \mathrm{sgn}(\sigma_{ik})$ then $\mathrm{sgn}(E_d) \notin F$. If no such i,j,k exist then $\mathrm{sgn}(E_d) \in F$.

Proof. If $\mathrm{sgn}(\sigma_{ij}) = \mathrm{sgn}(\sigma_{jk}) \neq \mathrm{sgn}(\sigma_{ik})$, then either $\sigma_{ij} = \sigma_{jk} = +1$ or -1. In the former case, $\sigma_{ij} = +1$ implies $a_{ij} - a_{ji} > 0$, which in turn means that $a_i > a_j$ by majority voting. Thus, $\sigma_{ij} = \sigma_{jk} = +1$ implies

$$a_i > a_j \quad , \quad a_j > a_k \quad \text{and} \quad a_i < a_k$$

by majority voting, which is an intransitive ordering. Thus, $\mathrm{sgn}(E_d) \notin F$. In the latter case, the proof follows analogously.

The above establishes sufficiency; necessity is now shown. Assume that no such i,j,k exist that provide intransitivity. Since all higher order intransitivities[4] require an intransitivity of triplets [2], there must exist a transitive ordering of the a_i provided by $\mathrm{sgn}(E_d)$; hence majority voting is transitive and optimal, and $\mathrm{sgn}(E_d) \in F$. ||

In graph-theoretic terms, if one places a directed arc from node i to j to indicate $a_i > a_j$, then $\mathrm{sgn}(E_d) \in F$ if and only if there are no directed cycles within the graph. Furthermore, if there are directed cycles, an attempt to eliminate them by determining whether vertices of the hypercube adjacent to $\mathrm{sgn}(E_d)$ belong to C will incur a cost of $\Delta_{ij} = a_{ij} - a_{ji}$, where Δ_{ij} means that the adjacent vertex only differs in that component (i,j) is now -1 instead of $+1$.

[4] A t^{th} order intransitivity is characterized as $a_{i_1} > a_{i_2}$, $a_{i_2} > a_{i_3}, \ldots, a_{i_{t-1}} > a_{it}$, but $a_{it} > a_{i_1}$.

Set Covering Criterion

The optimization problem formulated by Merchant and Rao is as follows [5]:

$$\text{Min} \sum_{(i,j)\varepsilon C}{}' (\beta_{ij} + \beta_{ji})(a_{ij} - a_{ji})y_{ij} \tag{15}$$

s.t.

$$\sum_{(i,j)\varepsilon C_k} y_{ij} \geq 1 \qquad k = 1,2,\ldots,r$$

$$y_{ij} = 0 \text{ or } 1 \qquad (i,j)\ \varepsilon\ C$$

$$C = \bigcup_{i=1}^{r} C_i \text{ and } C_k \text{ is a directed cycle.}$$

The β's are just weighting coefficients and as long as they are constant (say 1/2 for simplicity), the following theorem holds. Call (15) with constant β's (15)'.

Theorem 2. (15) is an equivalent representation of (4) for β_{ij} constant $\forall$ $(i,j)\ \varepsilon\ C$.

Proof. (4) is equivalent to (14); hence, it is sufficient to argue that (14) and (15)' are equivalent. As Merchant and Rao demonstrated[5], the constraints of (15)' ensure that all old cycles will be eliminated and no new ones created. From the lemma of the previous section, this is equivalent to moving from $\text{sgn}(E_d)$ to P^p. $y_{ij} = 1$ is equivalent to moving from $\text{sgn}(E_d)$ to an adjacent vertex as before, and the criterion function is the same as Δ_{ij}. Finally, since the costs of moving to a P^p that is not adjacent to $\text{sgn}(E_d)$ are simply additive (ℓ_1 norm used), the criterion function of (15)' follows. ||

[5] [5], Theorem 1, p.8

Discussion

Although the three criteria are all equivalent, each formulation of the social choice problem has unique characteristics. (2) and (4) are similar in that they are both "primal" approaches to the problem; both search over the feasible set of $m!$ permutation matrices, and at each stage of the problem, a feasible solution is known. In contrast, (15) is a "dual" approach; until the y_{ij} are discovered, no feasible solution to the problem is known.

In (2) an optimal assignment of actions to ranks is determined, while in (4) a search for the optimal matching of a permutation point to a translated election matrix is undertaken. Thus, while both procedures are quadratic assignment problems of a set of "objects" to "positions," the procedures differ by reversing the meanings of objects and positions.

Example

In order to illustrate the three approaches we consider the following example:

Individual Ranking	Number with Preference
$(a_1\ \ a_2\ \ a_3)$	23
$(a_2\ \ a_3\ \ a_1)$	17
$(a_2\ \ a_1\ \ a_3)$	2
$(a_3\ \ a_2\ \ a_1)$	8
$(a_3\ \ a_1\ \ a_2)$	10

where $a_1\ \ a_2\ \ a_3$ means that alternative a_1 is preferred to a_2 and a_2 is preferred to a_3. Majority voting leads to an intransitive solution. In order to analyze the problem we construct the following:

$$A = \begin{bmatrix} 0 & \frac{33}{60} & \frac{25}{60} \\ \frac{27}{60} & 0 & \frac{42}{60} \\ \frac{35}{60} & \frac{18}{60} & 0 \end{bmatrix} \qquad E_d = \begin{bmatrix} 0 & \frac{6}{60} & -\frac{10}{60} \\ -\frac{6}{60} & 0 & \frac{24}{60} \\ \frac{10}{60} & -\frac{24}{60} & 0 \end{bmatrix}$$

$$\operatorname{sgn}(E_d) = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

Since $\sigma_{12} = \sigma_{23} \neq \sigma_{13}$, $\operatorname{sgn}(E_d) \notin F$.

Noting that

$$X = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

we obtain the permutation point:

$$p^{p^*} = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

which is optimal, implying the social order $(a_2 \;\; a_3 \;\; a_1)$.

The optimal permutation matrix is:

$$P_{p^*} = P_{\overline{p}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

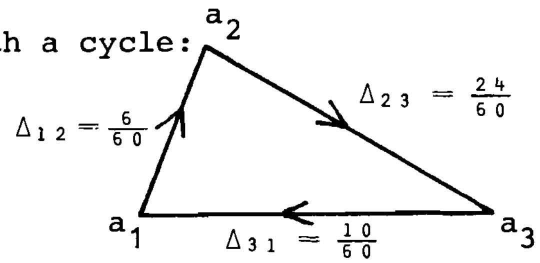
$$A_{\overline{p}} = \begin{bmatrix} 0 & \frac{42}{60} & \frac{27}{60} \\ \frac{18}{60} & 0 & \frac{35}{60} \\ \frac{33}{60} & \frac{25}{60} & 0 \end{bmatrix} .$$

Computing the solutions we have:

$$\phi(\overline{p}) = \frac{42 + 27 + 35}{60} = \frac{104}{60}$$

$$H(p^*) = 2\,\frac{(-6 + 10 + 24)}{60} = \frac{56}{60} = 4\phi(\overline{p}) - n(n-1) .$$

From the majority voting solution we obtain the following graph with a cycle:



We obtain the set covering problem:

$$\text{Min} \sum_{(i,j)\in C} (a_{ij} - a_{ji}) y_{ij} \qquad C = \{(23),(31),(12)\}$$

s.t.

$$y_{23} + y_{31} + y_{12} \geq 1$$

$$y_{ij} = 0 \text{ or } 1 \qquad \forall (i,j) \in C$$

The optimal solution is

$$\hat{y}_{12} = 1$$
$$\hat{y}_{23} = \hat{y}_{31} = 0$$

which is again equivalent.

References

[1] Blin, J.M. and Whinston, A.B. "Discriminant Functions and Majority Voting" Management Science Vol. 21, No. 5, 1975, pp. 557-566.

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[5] Merchant, D.K. and Rao, M.R. "Majority Decisions and Transitivity: Some Special Cases," (forthcoming).