A NOTE ON THE ⊗-KERNEL FOR QUASIDIFFERENTIABLE FUNCTIONS

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FOREWORD

The continuity of the star-kernel of quasidifferentials of a quasidifferentiable function with a star-equivalent bounded quasidifferential subfamily are studied and parts of results are represented in this note. It also has been pointed out that the directional subderivative and superderivative of a function can be expressed as support functions of its star-kernel if the star-kernel can be generated by a quasidifferential of the function.

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In this short note the demonstrations of some propositions related to the upper semicontinuity of the ⊗-kernel for quasidifferentiable functions and some examples concerning the ⊗-kernel will be given, [1], [2] and [3].

Suppose that $f(x)$ is a quasidifferentiable function, defined on $S \subset \mathbf{R}^n$ where S is an open set, with a ⊗-equivalent bounded quasidifferential subfamily. The notations we will use can be found out in [3]. Their definitions will not be repeated here.

LEMMA 1

(a) $u \in \partial_{\otimes} f(x) \Longleftrightarrow \underline{\mathbf{W}}(x , u) \neq \phi$.

(b) $w \in \underline{\mathbf{W}}(u) \Longleftrightarrow \varphi(u \otimes d) \geq \langle w , d \rangle , \forall d \in \mathbf{R}^n$.

(c) $u \in \partial_{\otimes} f(x) \Longleftrightarrow u \in \underline{\mathbf{W}}(x , u)$.

(d) $u \in \partial_{\otimes} f(x) \Longleftrightarrow$

$$\underline{f}'(x ; d) \leq \max_{w \in \underline{\mathbf{W}}(x , u)} \langle w , d \rangle$$

$$= \delta(d \mid \partial_C \underline{\varphi}(u \otimes \bullet)(0)) ,$$

$$\forall d \in \mathbf{R}^n .$$

For the sake of convenience, assume that $\underline{\mathbf{W}}$ is a closed set.

LEMMA 2 If $\underline{\varphi}(x , u \otimes d)$ is upper semicontinuous in $(x , u) \in S \times \partial_{\otimes} f(x)$ for each $d \in \mathbf{R}^n$, then the mapping $\partial_{\otimes} f(\bullet)$ and $\underline{\mathbf{W}}(\bullet , \bullet)$ are closed, i.e., $u \in \partial_{\otimes} f(x)$ and $w \in \underline{\mathbf{W}}(x , u)$ (or $\partial_C \underline{\varphi} (u \otimes \bullet)(0)$) whenever $x_i \to x$, $u_i \to u$, $w_i \to w$, and $u_i \in \partial_{\otimes} f(x_i)$, $w_i \in \underline{\mathbf{W}}(x_i , u_i)$, $i \to \infty$.

PROOF Suppose $x_i \to x$,$u_i \to u$, $w_i \to w$ and $u_i \in \partial_{\otimes} f(x_i)$, $w_i \in \underline{\mathbf{W}}(x_i , u_i)$, $i \to \infty$. According to the *LEM 1 (b)*, one has

$$\underline{\varphi}(x_i , u_i \otimes d) \geq \langle w_i , d \rangle$$

for each $d \in \mathbf{R}^n$. Since $\underline{\varphi}(x , u \otimes d)$ is upper semicontinuous in (x , u) for each $d \in \mathbf{R}$, one has

$$\underline{\varphi}(x , u \otimes d) \geq \langle w , d \rangle .$$

In other words, $w \in \underline{\mathbf{W}}(x , u)$. It follows from the *LEM 1 (a)* that

$$u \in \partial_{\otimes} f(x) .$$

The proof is completed. □

THEOREM 3. Suppose $D_M f(x)$ is bounded uniformly in a neighborhood of x , $N_x(\delta)$, where δ is a positive number. If $\underline{\varphi}(x , u \otimes d)$ is upper semicontinuous in $(x , u) \in S \times \partial_{\otimes} f(x)$ for each $d \in \mathbf{R}^n$, then $\partial_{\otimes} f(\bullet)$ is upper semicontinuous.

PROOF By contradiction, suppose $\partial_{\otimes} f(\bullet)$ is not upper semicontinuous. Then there exists an open set O, an $\varepsilon > 0$, and there exists sequences $\{x_i\}$ and $\{u_i\}$ such that

$$x_i \to x , i \to \infty$$

$\{x_i\} \subset B(0, \varepsilon)$,

$u_i \in \partial_{\otimes} f(x_i)$, $\forall i$

$\partial_{\otimes} f(x) \subset O$, $\forall x \in B(0, \varepsilon)$

$\{u_i\} \subset O^c$.

Since $\bigcup_{x \in B(0, \varepsilon)} D_M f(x)$ is bounded, there exists a subsequence $\{u_{i_k}\}$ convergent to u. The u belongs to O^c because of O being an open set. Obviously, $u \not\in \partial_{\otimes} f(x)$. However, in terms of the *LEM. 2*, we have

$$u \in \partial_{\otimes} f(x),$$

since the mapping $\partial_{\otimes} f(\cdot)$ is closed. This contradicts the fact that $u \not\in \partial_{\otimes} f(x)$. Therefore, $\partial_{\otimes} f(\cdot)$ is an upper semi-continuous mapping. The theorem is proved. □

LEMMA 4. Suppose $\underline{f}'(x ; d)$ is lower semi-continuous in $x \in S$ for each $d \in \mathbf{R}^n$ and the mapping $\underline{W}(x, u)$ is upper semicontinuous in $(x, u) \in S \times \partial_{\otimes} f(x)$. Then the mapping $\partial_{\otimes} f(\cdot)$ is closed.

PROOF Suppose that

$$x_i \to x, \; i \to \infty$$

$$u_i \to u, \; i \to \infty$$

$$u_i \in \partial_{\otimes} f(x_i).$$

Since for any $u \in \partial_{\otimes} f(x)$ and for each $d \in \mathbf{R}^n$,

$$\underline{f}'(x, d) \le \max_{w \in \underline{W}(x, u)} \langle w, d \rangle,$$

one has that for $\{x_i\}$ and $\{u_i\}$ there exists a sequence $\{w_i\}$ such that

$$\underline{f}'(x_i, d_i) \le \max_{w \in \underline{W}(x_i, u)} \langle w, d \rangle,$$

$$= \langle w_i, d \rangle.$$

According to the assumption of the upper semi-continuity of the mapping $\underline{W}(\cdot, \cdot)$ there exists a subsequence, $\{w_{i_k}\}$, convergent to w such that

$$w \in \underline{W}(x, u).$$

On the other hand, we have

$$\underline{f}'(x, d) \le \langle w, d \rangle,$$

because of the lower semicontinuity of the function $\underline{f}'(\cdot, d)$. From this, one has

$$\underline{f}'(x, d) \le \max_{w \in \underline{W}(x, u)} \langle w, d \rangle.$$

Hence,

$$u \in \partial_{\otimes} f(x).$$

from **LEM. 1 (d).** The lemma has been proved. □

THEOREM 5. Suppose $\underline{f}'(x, d)$ is lower semi-continuous in $x \in S$ for each $d \in \mathbf{R}^n$, and $D_M f(x)$ is bounded uniformly in a neighbourhood of x, $N_x(\delta)$, and $\underline{W}(x, u)$ is upper semicontinuous in $(x, u) \in S \times \partial_{\otimes} f(x)$. Then $\partial_{\otimes} f(\cdot)$ is a upper semicontinuous mapping.

THEOREM 6. If there exists a quasidifferential

$$[\underline{\partial}_0 f(x),\ \overline{\partial}_0 f(x)] \in D_M f(x)$$

such that

$$\underline{\partial}_0 f(x) + \overline{\partial}_0 f(x) = \bigcap_{\substack{[\underline{\partial} f(x),\ \overline{\partial} f(x)] \\ \in D_M f(x)}} (\underline{\partial} f(x) + \overline{\partial} f(x)) \tag{1}$$

and

$$\overline{\partial}_0 f(x) - \overline{\partial}_0 f(x) = \bigcap_{\substack{[\underline{\partial}_0 f(x),\ \overline{\partial}_0 f(x)] \\ \in D_M f(x)}} (\overline{\partial} f(x) - \overline{\partial} f(x)), \tag{2}$$

then

$$\partial_{\otimes} f(x) = \underline{\partial}_0 f(x) + \overline{\partial}_0 f(x) \tag{3}$$

and

$$\partial^{\otimes} f(x) = \overline{\partial}_0 f(x) - \overline{\partial}_0 f(x), \tag{4}$$

and

$$\underline{f}'(x\ ;\ d) = \max_{u\ \in\ \partial_{\otimes} f(x)} \langle u\ ,\ d \rangle,\ \forall d \in \mathbf{R}^n \tag{5}$$

and

$$\bar{f}'(x\ ;\ d) = \max_{u\ \in\ \partial^{\otimes} f(x)} \langle u\ ,\ d \rangle,\ \forall d \in \mathbf{R}^n\ . \tag{6}$$

PROOF. Since $\forall [\underline{\partial} f(x),\ \overline{\partial} f(x)] \in D_M f(x)$

$$\delta(\cdot \mid \underline{\partial} f(x) + \overline{\partial} f(x)) \geq \delta(\cdot \mid \underline{\partial}_0 f(x) + \overline{\partial}_0 f(x))$$

one has

$$\underline{f}'(x\ ;\ \cdot) \leq \delta(\cdot \mid \underline{\partial}_0 f(x) + \overline{\partial}_0 f(x))$$

$$\leq \inf_{\substack{[\underline{\partial} f(x),\ \overline{\partial} f(x)] \\ \in D_M f(x)}} \delta(\cdot \mid \underline{\partial}_0 f(x) + \overline{\partial}_0 f(x))\ .$$

In other words,

$$\underline{f}'(x\ ;\ \cdot) = \delta(\cdot \mid \underline{\partial}_0 f(x) + \overline{\partial}_0 f(x))\ .$$

Therefore,

$$\underline{\partial}_0 f(x) + \overline{\partial}_0 f(x) \subset \partial_{\otimes} f(x)\ .$$

According to the definition of $\underline{U}_0$, [3], we have

$$\underline{U}_0(x) \subset \underline{\partial}_0 f(x) + \overline{\partial}_0 f(x)\ ,$$

i.e.,

$$\partial_{\otimes} f(x) \subset \underline{\partial}_0 f(x) + \overline{\partial}_0 f(x)\ .$$

So the equality (3) holds. Similarly, (4) can be proved in the same way. In this case where the conditions (1) and (2) are satisfied one has that for any $u \in \partial_{\otimes} f(x)$ and for any $d \in \mathbb{R}^n$ the relations

$$\underline{\varphi}(u \otimes d) = \max_{u \in \partial_{\otimes} f(x)} <u , d>$$

and

$$\overline{\varphi}(u \otimes d) = \max_{u \in \partial^{\otimes} f(x)} <u , d>$$

are true. Hence, the equalities (5) and (6) are true. The theorem has been proved. □

EXAMPLE 7 Let $f \in C^1(\mathbb{R}^n)$. Then

$$\partial_{\otimes} f(x) = \{\nabla f(x)\} , \partial^{\otimes} f(x) = \{0\} .$$

EXAMPLE 8 Let f be a convex function defined in $\mathbb{R}^n$. Suppose

$$D_0 f(x) = [\underline{\partial}_0 f(x) , \{0\}]$$

$$= [\partial f(x) , \{0\}] .$$

$$Df(x) = [\underline{\partial} f(x) , \overline{\partial} f(x)] ,$$

where $\partial f(x)$ is the subdifferential of f at x in the convex sense.

Since

$$\partial f(x) - \overline{\partial} f(x) = \underline{\partial} f(x) ,$$

one has

$$\underline{\partial} f(x) + \overline{\partial} f(x) = \partial f(x) + (\overline{\partial} f(x) - \overline{\partial} f(x)) \tag{6}$$
$$=(\underline{\partial}_0 f(x) + \overline{\partial}_0 f(x)) + (\overline{\partial} f(x) - \overline{\partial} f(x))$$

and

$$\overline{\partial} f(x) - \overline{\partial} f(x) = (\overline{\partial}_0 f(x) - \overline{\partial}_0 f(x)) + (\overline{\partial} f(x) - \overline{\partial} f(x)) . \tag{7}$$

So

$$\partial f(x) = (\underline{\partial}_0 f(x) + \overline{\partial}_0 f(x)) = \bigcap_{\substack{[\underline{\partial} f(x) , \overline{\partial} f(x)] \\ \in D_M f(x)}} (\underline{\partial} f(x) + \overline{\partial} f(x))$$

and

$$\{0\} = \overline{\partial}_0 f(x) - \overline{\partial}_0 f(x) = \bigcap_{\substack{[\underline{\partial} f(x) , \overline{\partial} f(x)] \\ \in D_M f(x)}} (\overline{\partial} f(x) - \overline{\partial} f(x)) .$$

From Th. 6, one has

$$\partial_{\otimes} f(x) = \partial f(x) , \partial^{\otimes} f(x) = \{0\} .$$

EXAMPLE 9. Let f be a concave function defined in $\mathbb{R}^n$ and

$$D_0 f(x) = [\{0\} , \overline{\partial}_0 f(x)]$$

$$=[\{0\} , -\partial(-f)(x)]$$

and

$$D\, f(x) = [\underline{\partial}\, f(x)\,,\ \overline{\partial}\, f(x)]\,.$$

Then

$$\begin{aligned} \partial_{\otimes} f(x) &= \overline{\partial}_0\, f(x) \\ &= -\partial(-f)(x) \end{aligned}$$

and

$$\partial^{\otimes} f(x) = \overline{\partial}_0\, f(x) - \overline{\partial}_0\, f(x)\,.$$

EXAMPLE 10. Let f_1 be a convex function and f_2 be a concave function defined in $\mathbb{R}^n$, and $f = f_1 + f_2$. Then

$$\begin{aligned} D_0\, f(x) &= [\underline{\partial}_0\, f(x)\,,\ \overline{\partial}_0\, f(x)] \\ &= [\partial\, f_1(x)\,,\ -\partial\,(-f_2)(x)] \\ &= [\underline{\partial}_0\, f_1(x)\,,\ \overline{\partial}_0\, f_2\,(x)]\,. \end{aligned}$$

For any $[\underline{\partial}\, f(x)\,,\ \overline{\partial}\, f(x)] \in D_M\, f(x)$ one has from (6) and Ex.9 that

$$\begin{aligned} \underline{\partial} f(x) + \overline{\partial} f(x) &= (\underline{\partial}\, f_1(x) + \overline{\partial}\, f_1(x)) + (\underline{\partial}\, f_2(x) + \overline{\partial}\, f_2(x)) \\ &= \partial f_1(x) + (\overline{\partial}\, f_1(x) - \overline{\partial} f(x)) \\ &\quad + (\underline{\partial}\, f_2(x) - \underline{\partial} f_2(x)) - \partial(-f_2)(x) \\ &= (\overline{\partial} f_1(x) - \overline{\partial} f_1(x)) + (\underline{\partial} f_2(x) - \underline{\partial} f_2(x)) \\ &\quad + (\partial f_1(x) - \partial(-f_2)(x)), \end{aligned} \tag{8}$$

where

$$D f_1(x) = [\underline{\partial} f_1(x)\,,\ \overline{\partial} f_1(x)]$$

and

$$D f_2(x) = [\underline{\partial} f_2(x)\,,\ \overline{\partial} f_2(x)]\,.$$

It follows from (8) that

$$\underline{\partial} f(x) + \overline{\partial} f(x) \supset \underline{\partial}_0 f_1(x) + \overline{\partial}_0 f_2(x) = \partial f_1(x) - \partial(-f_2)(x)\,. \tag{9}$$

On the other hand, since

$$\overline{\partial} f(x) - \overline{\partial} f(x) = (\overline{\partial} f_1(x) - \overline{\partial} f_1(x)) + (\overline{\partial} f_2(x) - \overline{\partial} f_2(x))\,,$$

$$\overline{\partial} f_1(x) - \overline{\partial} f_1(x) = (\overline{\partial}_0 f_1(x) - \overline{\partial}_0 f_1(x)) + (\overline{\partial} f_1(x) - \overline{\partial} f_1(x))$$

and

$$\overline{\partial} f_2(x) - \overline{\partial} f_2(x) = (\overline{\partial}_0 f_2(x) - \overline{\partial} f_2(x)) - (\overline{\partial}_0 f_2(x) - \underline{\partial}_0 f_2(x))\,,$$

one has

$$\overline{\partial} f(x) - \overline{\partial} f(x) = (\overline{\partial} f_1(x) - \overline{\partial} f_1(x)) + (\underline{\partial} f_2(x) - \underline{\partial} f_2(x)) + (\overline{\partial}_0 f_2(x) - \overline{\partial}_0 f_2(x))\,.$$

Hence

$$\overline{\partial} f(x) - \overline{\partial} f(x) \supset \overline{\partial}_0 f_2(x) - \overline{\partial}_0 f_2(x)) . \qquad (10)$$

From (9), (10) and Th. 6 we now have

$$\partial_{\otimes} f(x) = \partial f_1(x) - \partial f_2(x)$$
$$= \underline{\partial}_0 f_1(x) + \overline{\partial}_0 f_2(x)$$

and

$$\partial^{\otimes} f(x) = \overline{\partial}_0 f_2(x) - \overline{\partial}_0 f_2(x)) .$$

EXAMPLE 11 Let f_1 and f_2 be convex, and $f = f_1 - f_2$.

Then we have

$$\partial_{\otimes} f(x) = \partial f_1(x) - \partial f_2(x)$$

and

$$\partial^{\otimes} f(x) = \partial f_2(x) - \partial f_2(x) .$$

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