# A DYNAMIC CONSUMPTION MODEL AND OPTIMIZATION OF UTILITY FUNCTIONALS

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October 1975 WP-75-121

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## 1. Introduction

A vast amount of literature exists on consumer behavior in terms of multi-objective or utility function approaches. Most of the work done in that field concentrates on the static situation when utility of a given commodity is an instantaneous function of expenditures or, in other words, when the commodities purchased in the past have no influence on the present utility. This, however, is not the case when one considers consumers' expenditure for durable goods or government expenditures in the fields of education, health, welfare, environmental protection, etc. In that case, it is possible to introduce the dynamic consumption model using the dynamic utility function or, speaking more precisely, the utility functional.

The present paper has been motivated by the research concerned with the construction of a complex, long-range, national development model. The model--MRI--is being constructed at the Institute of Organization and Management of the Polish Academy of Sciences. It consists of three main submodels: production, consumption and environment. The consumption submodel is characterized by the utility function with parameters which are estimated on the basis of past statistical data. It takes into account the utility structure changes caused by GNP per capita and the change in price indices. The model is normative, in the sense that one can investigate the national growth path as a result of alternative development strategies in terms of productive investment and government expenditures in the fields of education, R & D, health service, pollution control, etc. For more details concerning the MRI methodology see references [3, 4, 5].

The optimum investment strategies for MRI have already been derived in reference [3, 5]. In the present paper, an effort is being made to derive the optimum consumer and government expenditure strategies which maximize the utility functional.

## 2. Dynamic consumption models

Consider a single (or aggregated) consumer having at his disposal in the time interval [0,T] the given amount of financial resources Z. The financial resources, generally speaking, consist of salaries, savings, etc. It is assumed also that the consumer can obtain a loan in order to realize the dynamic consumption strategy in [O,T] in case that strategy exceeds his salary in certain subsets of [O,T]. He is supposed, however, to pay back the loan (together with interest) before the end of T interval.

Assume also that the consumer's utility function $U(\underline{x})$, $\underline{x} \in \Omega$, where $\Omega$ is a given subset (usually the positive orthant) of N dimensional space $E^n$, be given. It is well known (see reference [1, 2] that $U(\underline{x})$ must be real-valued, order preserving vector function. However, to be more specific we shall deal with the widely used function * .

$$U(\underline{x}) = U_0 \prod_{i=1}^{N} X_i^{\beta_i} \quad , \tag{1}$$

where

$U_0$, $\beta_i$, i=1,2,...,N given positive constants,

and $$\sum_{i=1}^{N} \beta_i = \beta \leq 1. \tag{2}$$

Assume that $x_i$, i=1,...,N represents the consumer's utility levels which are related to the expenditure intensity $y_i(t)$, $t \in [0,T]$, i=1,...,N in an inertial and nonlinear fashion: ‡

$$X_i(t) = \int_0^t k_i(t,\tau)\ [y_i(\tau)]^{\alpha_i}\, d\tau \tag{3}$$

where

$k_i(t,\tau)$ = given non-negative function, $k_i(t,\tau) = 0$ for $t < \tau$;

$\alpha_i$ = given positive number less unity.

A typical example of $k_i(t,\tau)$ is the stationary (i.e. $k_i(t,\tau) = k_i(t-\tau)$) delayed exponential function, i.e.

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* That function has also been adopted in the first version MRI models. An extension of (1) to the C.E.S. function is also possible.

‡ For convenience, rather than general methodology, we shall deal with continuous instead of discrete time variables.

$$k_i(t) = A_i e^{-\delta_i(t-T_i)} \quad , \quad t > T_i,$$
$$= 0 \quad , \quad t < T_i.$$

In the last case the consumer expenditures create no utility before $t = T_i$. Such a situation happens, e.g. in the case of education, health, etc., expenditures. For example, in order to get a better salary the consumer must finish school with the tuition cost $y_i(t)$ and $T_i$ years of study.

It should be observed that due to the exponent $\exp(-\delta_i t)$ the utility level (i.e., the health or training level) decreases along with time if no additional expenditures are incurred.

In the case where no dynamic effects are present, (i.e. where the change in expenditure results in an immediate change of utility) one can write formally $K_i(t) = A_i\ \delta(t)$ (*) (where $\delta(t)$ is the Dirac's unitary pulse) and get

$$X_i(t) = A_i[y_i(t)]^{\alpha_i}$$

It should be observed also that due to $0 < \alpha_i < 1$ there is a "decreasing return to scale" effect in (3). It corresponds to saturation of utility level (e.g., the training or health level) with respect to increase of expenditures.

Since generally $X_i$, $i=1,\ldots,N$ are functions of time, it is necessary to deal with the time-averaged utility, i.e.

$$\overline{U}(\underline{y}) = \int_0^T w(t)\ U(\underline{x})\ dt \quad , \tag{4}$$

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(*) A more elegant notation based on distribution theory can also be used here. In that case $\delta(t)$ can be regarded as a linear functional.

where

$W(t)$ = given discount function, such as $W(t) = (1+r)^{-t}$;

$r$ = discount rate.

The $\bar{U}(\underline{y})$ is a nonlinear function with respect to the vector-function $\underline{y}(t) = \{y_1(t), y_2(t), \ldots, y_N(t)\}$, $t \in [0,T]$.

In the model being discussed we do not take into account the effects caused by past expenditures, i.e. the expenditures for $t < 0$. One possible way of taking these expenditures into account is to replace (3) by

$$X_i(t) = \bar{X}_i(t) + \int_0^t k_i(t,\tau)\ [y_i(\tau)]^{\alpha_i}\, d\tau \quad , \qquad (5)$$

where

$$\bar{X}_i(t) = \int_{-\infty}^{0} k_i(t,\tau)\ [y_i(\tau)]^{\alpha_i}\, d\tau \ ,$$

represents the utility of commodities purchased in the past.

In order to derive the optimum consumption strategy one should maximize (4) subject to the monetary constraints:

$$\sum_{i=1}^{N} \int_0^T W_i(t)\ y_i(t)\ dt \leq Z \quad , \qquad (6)$$

$$y_i(t) \geq 0 \ , \quad t \in [0,T], \quad i=1,\ldots,N \quad , \qquad (7)$$

where

$W_i(t)$ = weights or interest functions (when the loans are used one can assume $W_i(t) = (1 + E_i)^{T-t}$, $E_i$ = interest rate);

$Z$ = total consumer financial resources.

The present model can be used mainly for investigation of the behavior of single (or aggregated) consumer only.

In the case of macro-modeling of centrally planned economies, one can use the following extension of the present model.

Let there be n different consumer classes each described by the utility

$$U_i(\underline{y}_i) = U_{oi} \prod_{\nu=1}^{N} \left\{ \int_0^t k_{\nu i}(t-\tau) \, [y_{\nu i}(\tau)]^{\alpha_\nu} \, d\tau \right\}^{\beta\nu} , \qquad (8)$$

where

$U_{oi} > 0$, $\alpha_\nu \in (0,1)$, $\sum_{\nu=1}^{N} \beta_\nu = 1$, $i=1,\ldots,n$,

$k_{\nu i}(t)$ - non-negative functions, $\nu=1,\ldots,N$.

The expenditure intensities $y_{\nu i}(t)$, $\nu=1,\ldots,N$, $i=1,\ldots,n$, should satisfy the following contraints:

$$\sum_{i=1}^{n} Z_{\nu i} \le Z_\nu, \; Z_{\nu i} = \int_0^T W_\nu(t) \; Y_{\nu i} \; (t) \; dt, \; \nu=1,\ldots,N \qquad (9)$$

$$y_{\nu i}(t) \ge 0, \; t \in [0,T], \; i=1,\ldots,n, \; \nu=1,\ldots,n \qquad (10)$$

where

$Z_{\nu i}$ = the expenditure of i-th consumer class for $\nu$-th commodity.

In the present model $Z_1$ may represent the aggregated consumer private expenditures (out of his salary) while $Z_2,\ldots,Z_N$ = the government expenditures in the fields of education, health, social care, environment, etc. The expenditures $Z_2,\ldots,Z_N$ can be regarded as government contribution to the social welfare. The government recognizes here that the utility functionals are different for different classes of consumers and it tries to allocate the financial resources (by means of differentiating salaries, medical and social care, etc.) in such a way that the resulting utility is maximum. For example, the government may recompensate the hard work of miners by improving their medical and social care, etc.

The optimization of consumption strategy problem consists of finding $y_{\nu i}(t) = \hat{y}_{\nu i}(t)$, $i=1,\ldots,n$, $\nu=1,\ldots,N$, such that the functional

$$U(\underline{y}) = \int_0^T \sum_{i=1}^{n} U_i(y_i)\, W(t)\, dt, \qquad (11)$$

attains maximum subject to the constraints (9) and (10).

It should be observed that in the consumption model being discussed a decentralized system of consumption strategies has been adopted. According to that system the government is concerned with the best allocation of Z among N different spheres of activity while each individual consumer is concerned with the best allocation of his salary (represented e.g. by $z_{1i}$).

## 3. Solution of Optimization Problem

In order to solve the problem introduce the following notation

$$z_{\nu i}(t) = [y_{\nu i}(t)]^{\alpha_\nu}, \quad \nu = 1,\ldots,N, \quad i=1,\ldots,n .$$

It is convenient to consider first of all the single consumer model (1) ÷ (6).

In the present case we can drop the $i$ index in the formulae (8)-(11) and our problem boils down to finding the non-negative strategies $z_\nu(t) = \hat{z}_\nu(t)$, $\nu=1,\ldots,N$, which maximize the functional

$$Y = \int_0^T \prod_{\nu=1}^{N} f_\nu(z_\nu)\ dt\ , \tag{12}$$

where

$$f_\nu(z_\nu) = \left\{ \int_0^t k_\nu(t-\tau) z_\nu(\tau)\ d\tau \right\}^{\beta_\nu}\ , \quad \sum_{\nu=1}^{N} \beta_\nu = 1\ ,$$

subject to

$$\int_0^T w_i(\tau)\ [z_i(\tau)]^{1/\alpha_i}\ d\tau \leq Z_i,\ i=1,\ldots,N\ . \tag{13}$$

In order to solve the present problem one can apply the generalized Hölder inequality

$$Y = \int_0^T \prod_{\nu=1}^{N} f_\nu(\tau)\ d\tau \leq \prod_{\nu=1}^{N} \left\{ \int_0^T f_\nu^{1/\beta_\nu}(\tau) d\tau \right\}^{\beta_\nu}$$

which becomes an equality if and only if (almost everywhere)

$$c_1 f_1^{1/\beta_1}(\tau) = c_i f_i^{1/\beta_i}(\tau),\ \tau \in [O,T],\ i=1,\ldots,N, \tag{14}$$

$c_i$ = const., $i=1,\ldots,N$. In that case, one obtains

$$Y \leq \prod_{i=1}^{N} (c_1/c_i)^{\beta_i} \int_0^T dt \int_0^t k_1(t-\tau)\ z_1(\tau) d\tau$$

$$= \prod_{i=1}^{N} (c_1/c_i)^{\beta_i} \int_0^T z_1(\tau) d\tau \int_\tau^T k_1(t-\tau) d\tau\ . \tag{15}$$

Applying again the Hölder inequality we get

$$Y \leq \prod_{i=1}^{N} (c_1/c_1)^{\beta_i} \left\{ \int_0^T w_1(\tau)\ [z_1(\tau)]^{1/\alpha_1}\ d\tau \right\}^{\alpha_1}$$

$$\cdot \left\{ \int_0^T \left[ w_1(t)^{-\alpha_1} \cdot \int_\tau^T k_1(t-\tau)\ dt \right]^{\frac{1}{1-\alpha_1}} d\tau \right\}^{1-\alpha_1}\ .$$

where the equality appears if and only if (almost everywhere)

$$\hat{z}_1(t) = c\,\bar{z}_1(t),\quad \bar{z}_1(t) = \left[w_1^{-1}(\tau)\int_\tau^T k_1(t-\tau)dt\right]^{\frac{\alpha_1}{1-\alpha_1}}. \tag{16}$$

The value of $c$ can be derived by (13) yielding

$$c = \left\{\frac{Z_1}{\int_0^T w_1(\tau)\,[\bar{z}_1(\tau)]^{1/\alpha_1}\,d\tau}\right\}^{\alpha_1}.$$

Then

$$Y(\hat{z}) = \prod_{i=1}^{N}(c_1/c_i)^{\beta_i}\,Z_1^{\alpha_1}\left\{\int_\tau^T w_1(\tau)\right.$$

$$\left.\cdot\left[w_1^{-1}(\tau)\int_\tau^T k_1(t-\tau)dt\right]^{\frac{1}{1-\alpha_1}}d\tau\right\}^{1-\alpha_1}$$

$$= \prod_{i-1}^{N}\left(c\,\frac{c_1}{c_i}\right)^{\beta_i}\int_0^T w_1(\tau)\,[\bar{z}_1(\tau)]^{1/\alpha_1}\,d\tau\,. \tag{17}$$

The optimum strategies $\hat{z}_1(t)$, $i\geq 2,\ldots,N$, can be derived by (14)* which can be transformed to an equivalent form, assuming the Laplace transformations

$$K_i(p) = L\{k_i(t)\},\quad Z_i(p) = L\{z_i(t)\},\quad Z_1(p) = c\,K_1^*(p),$$

$$K_1^*(p) = L\left\{w_1^{-1}(\tau)\int_\tau^T k_1(t-\tau)dt\right\}^{\frac{1}{1-\alpha_1}}$$

exist.

Then

$$c_1\,K_1(p)\,\hat{Z}_1(p) = c_1 c\,K_1(p)\,K_1^*(p) = c_i\,K_i(p)\,\hat{Z}_i(p),$$

and

$$\hat{Z}_i(p) = \frac{cc_1\,K_1(p)\,K_1^*(p)}{c_i\,K_i(p)},\quad i=2,\ldots,N,$$

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* It can be shown that in order to solve (14) the index $i = 1$ should be assigned to the most inertial production factor specified by the set of $\{K_i(p)\}$.

or

$$\hat{z}_i(t) = c\,\frac{c_1}{c_i}\,\bar{z}_i(t), \text{ where } \bar{z}_i(t) = L^{-1}\left\{\frac{K_1^*(p)\ K_1(p)}{K_i(p)}\right\}.$$

Using the conditions (13) we get

$$\hat{z}_i(t) = \left\{\frac{Z_i}{\int_0^T w_i(t)\ [\bar{z}_i(t)]^{1/\alpha_i}\,dt}\right\}^{\alpha_i} \bar{z}_i(t),\ i=1,\ldots,N, \tag{18}$$

and--instead of (17)--we obtain

$$Y(\hat{\underline{z}}) = G^q \prod_{\nu=1}^{N} Z_\nu^{\alpha_\nu\beta_\nu},\ q = 1 - \sum_{\nu=1}^{N} \alpha_\nu\beta_\nu \tag{19}$$

where

$$G = \left\{\int_0^T w_1(\tau)\ [\bar{z}_1(\tau)]^{1/\alpha_1}\,d\tau\right\}^{1/q} \cdot \prod_{\nu=1}^{N}\left\{\int_0^T w_\nu(\tau)\ [\bar{z}_\nu(\tau)]^{1/\alpha_\nu}\,d\tau\right\}^{-\alpha_\nu\beta_\nu/q}.$$

The solution obtained can also be used for the utility functions with noninertial factors. In the simplest case $N = 2$, when

$$Y = \left\{\int_0^t k_1(t-\tau)\ z_1(\tau)\ d\tau\right\}^{\beta} [z_2(t)]^{1-\beta}$$

one can use formulae (18) setting $k_2(t) = \delta(t)$, $[K_2(p) = 1]$. The form of the solution in the present case coincides with the result obtain in Reference [5].

The results obtained so far can easily be extended to the general (8) ÷ (11) model. We shall assume that the whole amount of expenditures in $\nu$-th, sphere of activity ($Z_\nu$) is given and it should be allocated among consumers (so that $Z_{\nu i}$ represents the amount of $\nu$-th expenditure for i-th consumer in such a way that the following relation holds

$$\sum_{i=1}^{n} z_{\nu i} \leq Z_{\nu} , \qquad \nu=1,\ldots,N. \tag{20}$$

Taking into account (19) the consumer utility functions can be written as

$$Y_i = G_i^q \prod_{\nu=1}^{N} z_{\nu i}^{\alpha_\nu \beta_\nu} \tag{21}$$

where

$$q = 1 - \sum_{\nu=1}^{N} \alpha_\nu \beta_\nu , \quad i=1,\ldots,n,$$

$$G_i = \left\{ \int_0^T w_1(\tau) \; [\bar{z}_{1i}(\tau)]^{1/\alpha_1} \, d\tau \right\}^{1/q}$$

$$\cdot \prod_{\nu=1}^{N} \left\{ \int_0^T w_\nu(\tau) \; [\bar{z}_{\nu i}(\tau)]^{1/\alpha_\nu} \, d\tau \right\}^{-\alpha_\nu \beta_\nu / q} .$$

Then the problem of optimum allocation of $Z_\nu$, $\nu=1,\ldots,N$, among the n-consumers can be formulated as follows. Find the non-negative $z_{\nu i} = \hat{z}_{\nu i}$, $\nu=1,\ldots,N$, $i=1,\ldots,n$, such that

$$\bar{Y} = \sum_{i=1}^{n} G_i^q \prod_{\nu=1}^{N} z_{\nu i}^{\alpha_\nu \beta_\nu}$$

attains maximum subject to (20).

The solution of that problem (see Reference 3) is unique and assumes the following form

$$\hat{z}_{\nu i} = (G_i/G) Z_\nu , \quad \nu=1,\ldots,N, \; i=1,\ldots,n, \tag{22}$$

where

$$G = \sum_{i=1}^{n} G_i \text{ and}$$

$$\bar{Y}(\hat{z}_{\nu i}) = G_i^q \sum_{\nu=1}^{N} Z_\nu^{\alpha_\nu \beta_\nu} . \tag{23}$$

The consumer's dynamic strategies $\hat{z}_{\nu i}(t)$, $\nu=1,\dots,N$, $i=1,\dots,n$, can be derived by (18) i.e.

$$\hat{z}_{\nu i}(t) = \left\{ \frac{\hat{Z}_{\nu i}}{\int_0^T \cdot w_i(t) \; [\bar{z}_{\nu i}(t)]^{1/\alpha_i} \, dt} \right\}^{\alpha_i} \bar{z}_{\nu i}(t),$$

$$\nu=1,\dots,N, \quad i=1,\dots,n \tag{24}$$

where

$$\bar{z}_{\nu i}(t) = L^{-1}\left\{ \frac{K^{*}_{1i}(p) \; K_{1i}(p)}{K_{\nu i}(p)} \right\}$$

$$K_{\nu i}(p) = L\{k_{\nu i}(t)\}, \quad K_{1i}(p) = L\{k_{1i}(t)\}$$

$$K^{*}_{1i}(p) = L\left\{ w_1^{-1}(\tau) \int_\tau^T k_{1i}(t-\tau)\,dt \right\}^{\frac{\alpha_1}{1-\alpha_1}} .$$

Now it is also possible to solve explicitly the problem of optimum allocation of total consumption Z generated by the economy among the different spheres of activity represented by the expenditures $Z_\nu$, $\nu=1,\dots,N$. In other words, one would like to find the non-negative values $Z_\nu = \hat{Z}_\nu$, $\nu=1,\dots,N$, such that the aggregated utility described by (23) attains maximum subject to the constraint $\sum_{\nu=1}^{N} Z_\nu \leq Y^{*}$ .

The unique solution of that problem becomes

$$Z_\nu = \gamma_\nu \, Y^{*} = \frac{\alpha_\nu \beta_\nu}{\sum_{\nu=1}^{N} \alpha_\nu \beta_\nu} \, Y^{*} , \quad \nu=1,\dots,N. \tag{25}$$

It can be shown that the strategy (25) also maximizes the resulting utility function

$$U = \bar{U} \prod_{\nu=1}^{N} Z_\nu^{\gamma_\nu} , \quad \bar{U} = \text{const.}$$

The main result of the present paper can be formulated as follows:

In the decentralized consumption model described by (1)-(11) the unique optimum strategy of allocation of expenditures exists and it can be derived explicitely by (22), (24), (25).

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